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OCCUPATION TIME PROBLEM OF CERTAIN SELF-SIMILAR PROCESSES RELATED TO THE FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper, we prove some limit theorem for occupation time problem of certain self-similar processes related to the fractional Brownian motion, namely the bifractional Brownian motion, the subfractional Brownian motion and the weighted fractional Brownian motion. The key ingredients to prove our results is the well known Potter's Theorem involving slowly varying functions. We give also the L^p-estimate version of strong approximation of our limit theorem.

1. Introduction

Throughout this paper, we use the same symbol $Y^\tau := (Y^\tau_t \ , \ t \geq 0)$ to denote each of the Gaussian τ-self-similar processes: the fractional Brownian motion ($\tau = H$, fBm for short), the bifractional Brownian motion ($\tau = HK$, bfBm for short), the subfractional Brownian motion ($\tau = \frac{\alpha}{2}$, sfBm for short) and the weighted fractional Brownian motion ($\tau = \frac{1+b}{2}$, wfBm for short) and we denote $L^\tau := (L^\tau(t,x) \ , \ t \geq 0 \ , \ x \in \mathbb{R})$ its local time, (see the definitions below). For any fixed $t_0 \geq 0$ and any $\rho > 0$, we define the tangent process related to Y^τ as follows:

$$Y^{\rho,\tau} := \left(Y^{\rho,\tau}_t = \frac{Y^\tau_{t_0+\rho t} - Y^\tau_{t_0}}{\rho^\tau} \ , \ t \geq 0 \right),$$

and we denote $L^{\rho,\tau} := (L^{\rho,\tau}(t,x) \ , \ t \geq 0 \ , \ x \in \mathbb{R})$ its local time.

It is proved recently in [1], and in [18] that when ρ goes to zero, the process $(Y^{\rho,\tau}_t \ , \ t \geq 0)$ converges, in the sense of the finite dimensional distributions, to the fBm of Hurst parameter τ, (up to a multiplicative constant). Notice that in the fBm case we have: $(Y^{\rho,H} \ , \ t \geq 0) \ \underline{\underline{d}} \ (Y^H \ , \ t \geq 0)$, where $\underline{\underline{d}}$ denotes the equalities of the finite dimensional distributions.

The aim of the present paper is to obtain a limit theorem for normalized occupation time integrals of the form:

$$\frac{1}{n^{1-\tau(1+\gamma)}} \int_0^{nt} f(Y^{\rho,\tau}_s) ds, \tag{1.1}$$

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where $f = K_{\pm}^{l,\gamma} g$, and $K_{\pm}^{l,\gamma}$ is the generalized fractional derivative of order $\gamma > 0$ generated by a slowly varying function l, (see the definitions below), and $g \in \mathcal{C}^{\beta} \cap L^1(\mathbb{R})$ with compact support. $\mathcal{C}^{\beta}$ is the space of functions satisfying a Hölder condition of order some $\beta > 0$. The process in (1.1) was studied in the case of the classical fractional derivative $D_{\pm}^{\gamma}$ where $l \equiv 1$, and we refer to Yamada [24],[25] for Brownian motion case ($\tau = \frac{1}{2}$) and to Shieh [21] for fBm case ($\tau = H$). Notice that even if f is not a fractional derivative of some function g, the limiting process in (1.1) is a fractional derivative of local time.

We end this section by the definitions of the Gaussian τ-self-similar processes studied in this paper. The first process is the bfBm with parameters $H \in (0, 1)$ and $K \in (0, 1]$ introduced in [14]. It is a ($\tau = HK$)-self-similar Gaussian process, centered, starting from zero, with covariance function:

$$R^{H,K}(t, s) = \frac{1}{2^K} \left[\left(t^{2H} + s^{2H}\right)^K - |t - s|^{2HK} \right].$$

The case $K = 1$ corresponds to the fBm [16] of Hurst parameter $\tau = H \in (0, 1)$. The second process is the sfBm with parameter $\alpha \in (0, 2)$. It is an extension of Brownian motion ($H = \frac{1}{2}$) or ($\alpha = 1$), which preserves many properties of fBm but not the stationarity of the increments. It was introduced by Bojdecki et *al.* [7]. It is a ($\tau = \frac{\alpha}{2}$)-self-similar Gaussian process, centered, starting from zero, with covariance function:

$$R^{H}(t, s) = t^{\alpha} + s^{\alpha} - \frac{1}{2}[(t + s)^{\alpha} + |t - s|^{\alpha}].$$

The third process is the wfBm with parameters a and b introduced in [8]. It is a ($\tau = \frac{1+b}{2}$)-self-similar Gaussian process, centered, starting from zero, with covariance function:

$$R^{a,b}(t, s) = \int_0^{s \wedge t} u^a \left[(t - u)^b + (s - u)^b \right] du,$$

where $a > -1$, $-1 < b < 1$, and $|b| < 1 + a$. Clearly, if $a = 0$, the process coincides with the fBm with Hurst parameter $\frac{1}{2}(1 + b)$, (up to a multiplicative constant).

The remainder of this paper is organized as follows: In the next section, we present some basic facts about local time and the generalized fractional derivative. In section 3, we give the proof of our limit theorem. Finally, in the last section, we state and prove strong approximation version of our limit theorem, more precisely, we show the L^p-estimate version.

Notice that most of the estimates in this paper contain unspecified finite positive constants. We use the same symbol C to denote these constants, even when they vary from one line to the next.

2. Local Time and the Generalized Fractional Derivatives

We begin this section by a briefly survey on local time and we refer to [12].

Let $X := (X_t\ ,\ t \geq 0)$ be a real-valued separable random process with Borel sample functions. For any Borel set $B \subset \mathbb{R}^+$, the occupation measure of X on B is defined as:

$$\mu_B(A) = \lambda\{s \in B\ ;\ X_s \in A\}, \qquad \forall A \in \mathcal{B}(\mathbb{R}),$$

where λ is the one-dimensional Lebesgue measure on $\mathbb{R}^+$. If μ_B is absolutely continuous with respect to λ, we say that X has a local time on B denoted by $L(B,.)$. Moreover, the local time satisfies the occupation density formula: for every Borel set $B \subset \mathbb{R}^+$ and every measurable function $f : \mathbb{R} \to \mathbb{R}^+$, we have

$$\int_B f(X_t)dt = \int_{\mathbb{R}} f(x)L(B,x)dx,$$

and we have the following representation of local time:

$$L(t,x) := L([0,t],x) = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_0^t e^{iu(X_s-x)}ds\right)du. \tag{2.1}$$

This representation due to the Fourier analysis for local time, have played a central role to study the regularities properties of local time of our processes. Tudor and Xiao [22] have proved, by using Lamperti's transform and the concept of strong local nondeterminism introduced by Berman [5], the existence and the joint continuities of local time of bfBm. The case of fBm was given by Xiao [23]. Mendy [17] have studied the local time of sfBm for any $\alpha \in (0,1)$, by using a decomposition in law of sfBm given in [4]. Notice that the same arguments used in [17] with a decomposition in law of bfBm given in [15] give easily the Hölder regularities of bfBm. The case of wfBm for any $a \geq 0$ and $-1 < b < 1$ was given in [18]. Finally and more precisely, we have the following Hölder regularities of the local time L^τ where $\tau = HK \in (0,1)$ for the bfBm, $\tau = H \in (0,1)$ for the fBm, $\tau = \frac{\alpha}{2} \in (0,\frac{1}{2})$ for the sfBm and $\tau = \frac{1+b}{2} \in (0,1)$ for the wfBm.

Theorem 2.1. *For any integer $p \geq 1$, there exists a constant $\delta > 0$ and $C > 0$ such that for any $t \geq 0$, any $h \in (0,\delta)$, any $x,y \in \mathbb{R}$ and any $0 < \xi < \frac{1-\tau}{2\tau}$, there hold:*

$$\|L^\tau(t+h,x) - L^\tau(t,x)\|_{2p} \leq Ch^{1-\tau}, \tag{2.2}$$

$$\|L^\tau(t+h,y) - L^\tau(t,y) - L^\tau(t+h,x) + L^\tau(t,x)\|_{2p} \leq C|y-x|^\xi h^{1-\tau(1+\xi)}, \tag{2.3}$$

where $\|.\|_{2p} = (\mathbb{E}|.|^{2p})^{\frac{1}{2p}}$.

Remark 2.2. Following the same arguments used in [1] to prove Theorems 3.1 and 3.2, and the motivation in [9]: page 862, it is easy to see that the tangent process $Y^{\rho,\tau}$ has the local time:

$$L^{\rho,\tau}(t,x) = \frac{L^\tau(t_0+\rho t, \rho^\tau x + Y^\tau_{t_0}) - L^\tau(t_0, \rho^\tau x + Y^\tau_{t_0})}{\rho^{1-\tau}}.$$

In fact, by virtue of (2.1) and a changes of variables: $v = \frac{u}{\rho^\tau}$ and $z = t_0 + \rho s$, we have

$$\begin{aligned}
&\frac{L^\tau(t_0+\rho t, \rho^\tau x + Y^\tau_{t_0}) - L^\tau(t_0, \rho^\tau x + Y^\tau_{t_0})}{\rho^{1-\tau}}\\
&= \frac{1}{2\pi\rho^{1-\tau}}\left(\int_{\mathbb{R}}\int_0^{t_0+\rho t} e^{iv\left(Y^\tau_z - (\rho^\tau x + Y^\tau_{t_0})\right)}dzdv - \int_{\mathbb{R}}\int_0^{t_0} e^{iv\left(Y^\tau_z - (\rho^\tau x + Y^\tau_{t_0})\right)}dzdv\right)\\
&= \frac{1}{2\pi\rho^{1-\tau}}\int_{\mathbb{R}}\int_{t_0}^{t_0+\rho t} e^{iv\left(Y^\tau_z - (\rho^\tau x + Y^\tau_{t_0})\right)}dzdv
\end{aligned}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_0^t e^{iu(Y_s^{\rho,\tau} - x)} ds \right) du = L^{\rho,\tau}(t,x).$$

An immediate consequence of Theorem 2.1 is the following result concerning the regularities properties of the local time $L^{\rho,\tau}$.

Theorem 2.3. *For any integer $p \geq 1$, there exists a constant $C > 0$ such that for any $s,t \geq 0$, any $x,y \in \mathbb{R}$, any $0 < \xi < \frac{1-\tau}{2\tau}$ and any $\rho > 0$ sufficiently small, there hold:*

$$\|L^{\rho,\tau}(t,x) - L^{\rho,\tau}(s,x)\|_{2p} \leq C \mid t-s \mid^{1-\tau} \tag{2.4}$$

$$\|L^{\rho,\tau}(t,y) - L^{\rho,\tau}(s,y) - L^{\rho,\tau}(t,x) + L^{\rho,\tau}(s,x)\|_{2p} \leq C|y-x|^{\xi} \mid t-s \mid^{1-\tau(1+\xi)} \tag{2.5}$$

Proof. The proofs of (2.4) and (2.5) are similar. Let us deal for exemple with (2.5). By virtue of (2.3), we have

$$\begin{aligned} &\|L^{\rho,\tau}(t,y) - L^{\rho,\tau}(s,y) - L^{\rho,\tau}(t,x) + L^{\rho,\tau}(s,x)\|_{2p} \\ &= \frac{\|L^{\tau}(t_0+\rho t, \rho^{\tau} y + Y_{t_0}^{\tau}) - L^{\tau}(t_0+\rho s, \rho^{\tau} y + Y_{t_0}^{\tau})}{\rho^{1-\tau}} \\ &\quad + \frac{-L^{\tau}(t_0+\rho t, \rho^{\tau} x + Y_{t_0}^{\tau}) + L^{\tau}(t_0+\rho s, \rho^{\tau} x + Y_{t_0}^{\tau})\|_{2p}}{\rho^{1-\tau}} \\ &\leq C|y-x|^{\xi} \mid t-s \mid^{1-\tau(1+\xi)} . \end{aligned}$$

This gives the desired estimate. □

Remark 2.4. The passage through the tangent process allowed us to obtain regularities without the condition $\mid t-s \mid< \delta$, it was the motivation for which we chose the tangent process $Y^{\rho,\tau}$ instead of choosing the process Y^{τ} in (1.1).

Now, we give the definition of the generalized fractional derivatives and we refer to [11] and the references therein. For this, we collects some basic facts about slowly varying function and we refer for example to Bingham et *al.* [6] and Seneta [20].

Definition 2.5. A measurable function $l : \mathbb{R}^+ \to \mathbb{R}^+$ is *slowly varying at infinity* (in Karamata's sense), if for all t positive, we have

$$\lim_{x \to +\infty} \frac{U(tx)}{U(x)} = 1.$$

We are interested in the behavior of l at $+\infty$, then in what follows, we assume that l is bounded on each interval of the form $[0,a], (a > 0)$. This assumption is provided by Lemma 1.3.2 in [6]. For $\gamma > 0$, let k_γ the function defined by:

$$k_\gamma(y) := \begin{cases} \frac{l(y)}{y^{1+\gamma}}, & \text{if } y > 0, \\ 0, & \text{if } y \leq 0, \end{cases}$$

where l is slowly varying function at $+\infty$, continuously differentiable on $[a, +\infty[$, $(a > 0)$, (this property is given by Theorem 1.3.3 in [6]), and $l(x) > 0$ for all $x > 0$ and $l(0^+) = 1$.

For any $\gamma \in]0, \beta[$ and $g \in \mathcal{C}^\beta \cap L^1(\mathbb{R})$, we define:

$$K_\pm^{l,\gamma} g(x) := \frac{1}{\Gamma(-\gamma)} \int_0^{+\infty} k_\gamma(y) \left[g(x \pm y) - g(x) \right] dy,$$

and we put:

$$K^{l,\gamma} := K_+^{l,\gamma} - K_-^{l,\gamma}.$$

The following theorem called Potter's Theorem, (see Theorem 1.5.6 in [6]), has played a central role in the proof of our main results.

Theorem 2.6. *1) If l is slowly varying function, then for any chosen constants $A > 1$ and $\delta > 0$, there exists $X = X(A, \delta)$ such that:*

$$\frac{l(y)}{l(x)} \leq A \max \left\{ \left(\frac{y}{x}\right)^\delta, \left(\frac{y}{x}\right)^{-\delta} \right\} \qquad (x \geq X, y \geq X).$$

2) If further, l is bounded away from 0 and ∞ on every compact subset of $[0, +\infty[$, then for every $\delta > 0$, there exists $A = A(\delta) > 1$ such that:

$$\frac{l(y)}{l(x)} \leq A \max \left\{ \left(\frac{y}{x}\right)^\delta, \left(\frac{y}{x}\right)^{-\delta} \right\} \qquad (x > 0, y > 0).$$

Remark 2.7. 1) $K_+^{l,\gamma}$ and $K_-^{l,\gamma}$ satisfy the switching identity:

$$\int_{\mathbb{R}} f(x) K_-^{l,\gamma} g(x) dx = \int_{\mathbb{R}} g(x) K_+^{l,\gamma} f(x) dx, \tag{2.6}$$

for any $f, g \in \mathcal{C}^\beta \cap L^1(\mathbb{R})$ and $\gamma \in]0, \beta[$.
2) For $h : \mathbb{R} \to \mathbb{R}$ and $a > 0$, we denote by h_a the function $x \to h(ax)$. Then,

$$K_\pm^{l,\gamma}(h_a) = a^\gamma (K_\pm^{l(\frac{\cdot}{a}),\gamma})_a, \quad \forall\, \gamma > 0, \quad \forall\, a > 0, \tag{2.7}$$

where $l(\frac{\cdot}{a}) : x \longmapsto l(\frac{x}{a})$.
3) If we take $l \equiv 1$, we recover the definition of the classical fractional derivative denoted by: D^γ, (see [19], [24] and the references therein), where

$$k_\gamma(y) := \begin{cases} \frac{1}{y^{1+\gamma}}, & \text{if } y > 0, \\ 0, & \text{if } y \leq 0, \end{cases}$$

$$D_\pm^\gamma g(x) := \frac{1}{\Gamma(-\gamma)} \int_0^{+\infty} \frac{g(x \pm y) - g(x)}{y^{1+\gamma}} dy,$$

and

$$D^\gamma := D_+^\gamma - D_-^\gamma.$$

Now we are ready to state and prove the main results of this section.

Theorem 2.8. *Let $0 < \gamma < \xi$ and $K \in \{K_\pm^{l,\gamma}, K^{l,\gamma}\}$. For any integer $p \geq 1$, there exists a constant $C > 0$, such that for any $t, s \geq 0$ and any $x \in \mathbb{R}$ and any $\rho > 0$ sufficiently small, there hold:*

$$\|KL^{\rho,\tau}(t,.)(x) - KL^{\rho,\tau}(s,.)(x)\|_{2p} \leq C \mid t - s \mid^{1-\tau(1+\gamma)} .$$

Proof. We treat only the case $K = K_+^{l,\gamma}$, the other cases are similar. Let $b = |t-s|^\tau$. By the definition of $K_+^{l,\gamma}$, we have

$$\begin{aligned}
&\|K_+^{l,\gamma} L^{\rho,\tau}(t,.)(x) - K_+^{l,\gamma} L^{\rho,\tau}(s,.)(x)\|_{2p} \\
&\leq \frac{1}{|\Gamma(-\gamma)|}\int_0^{+\infty} l(u)\frac{\|L^{\rho,\tau}(t,x+u) - L^{\rho,\tau}(t,x) - L^{\rho,\tau}(s,x+u) + L^{\rho,\tau}(s,x)\|_{2p}}{u^{1+\gamma}}du \\
&\leq I_1 + I_2.
\end{aligned}$$

where

$$\begin{aligned}
I_1 := \frac{1}{|\Gamma(-\gamma)|}\int_0^b l(u) \\
\times \frac{\|L^{\rho,\tau}(t,x+u) - L^{\rho,\tau}(t,x) - L^{\rho,\tau}(s,x+u) + L^{\rho,\tau}(s,x)\|_{2p}}{u^{1+\gamma}}du
\end{aligned}$$

and

$$\begin{aligned}
I_2 := \frac{1}{|\Gamma(-\gamma)|}\int_b^{+\infty} l(u) \\
\times \frac{\|L^{\rho,\tau}(t,x+u) - L^{\rho,\tau}(t,x) - L^{\rho,\tau}(s,x+u) + L^{\rho,\tau}(s,x)\|_{2p}}{u^{1+\gamma}}du.
\end{aligned}$$

We estimate I_1 and I_2 separately. Since l is bounded on each compact in $\mathbb{R}^+$, it follows from (2.5) that:

$$\begin{aligned}
I_1 &\leq C \mid t-s \mid^{1-\tau(1+\xi)} b^{\xi-\gamma} \\
&\leq C \mid t-s \mid^{1-\tau(1+\gamma)} .
\end{aligned}$$

Potter's Theorem with $0 < \xi < \gamma$ implies the existence of $A(\xi) > 1$ such that:

$$l(u) \leq A(\xi) l(b)\Big(\frac{u}{b}\Big)^{\xi}.$$

Combining this fact with (2.4), we obtain:

$$I_2 \leq C \mid t-s \mid^{1-\tau(1+\gamma)} .$$

The proof of Theorem 2.8 is done. □

We end this section by the following result. It will be useful in the sequel to prove the tightness in our limit theorem.

Corollary 2.9. *Let $0 < \gamma < \xi$ and $K \in \{K_\pm^{l,\gamma}, K^{l,\gamma}\}$. For any integer $p \geq 1$, there exists a constant $C > 0$, such that for any $t, s \geq 0$, any $x \in \mathbb{R}$, any $\rho > 0$ sufficiently small and any n sufficiently large, there holds:*

$$\begin{aligned}
&[l(n^\tau)]^{-1}\left\| K^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma} L^{\rho,\tau}(t,.)\Big(\frac{x}{n^\tau}\Big) - K^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma} L^{\rho,\tau}(s,.)\Big(\frac{x}{n^\tau}\Big)\right\|_{2p} \\
&\leq C \mid t-s \mid^{1-\tau(1+\gamma)},
\end{aligned}$$

where $l\Big(\frac{\cdot}{n^{-\tau}}\Big) : x \longmapsto l\Big(\frac{x}{n^{-\tau}}\Big)$.

Proof. We treat only the case $K_+^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma}$, the other cases are similar. Let $b = |t-s|^\tau$. By the definition of $K_+^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma}$, we have

$$[l(n^\tau)]^{-1}\left\|K_+^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma}L^{\rho,\tau}(t,.)\left(\frac{x}{n^\tau}\right) - K_+^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma}L^{\rho,\tau}(s,.)\left(\frac{x}{n^\tau}\right)\right\|_{2p}$$

$$\leq \frac{1}{|\Gamma(-\gamma)|}\int_0^b \frac{l(n^\tau u)}{l(n^\tau)} \times \frac{\left\|L^{\rho,\tau}(t,\frac{x}{n^\tau}+u) - L^{\rho,\tau}(s,\frac{x}{n^\tau}+u) - L^{\rho,\tau}(t,\frac{x}{n^\tau}) + L^{\rho,\tau}(s,\frac{x}{n^\tau})\right\|_{2p}}{u^{1+\gamma}}du$$

$$+ \frac{1}{|\Gamma(-\gamma)|}\int_b^{+\infty} \frac{l(n^\tau u)}{l(n^\tau)} \times \frac{\left\|L^{\rho,\tau}(t,\frac{x}{n^\tau}+u) - L^{\rho,\tau}(s,\frac{x}{n^\tau}+u) - L^{\rho,\tau}(t,\frac{x}{n^\tau}) + L^{\rho,\tau}(s,\frac{x}{n^\tau})\right\|_{2p}}{u^{1+\gamma}}du$$

$$:= J_1 + J_2.$$

We estimate J_1 and J_2 separately. It follows from (2.5) that:

$$J_1 \leq C\sup_{u\in\mathbb{R}^+}\frac{l(n^\tau u)}{l(n^\tau)} \mid t-s\mid^{1-\tau(1+\delta)} b^{\delta-\gamma}$$
$$\leq C\sup_{u\in\mathbb{R}^+}\frac{l(n^\tau u)}{l(n^\tau)} \mid t-s\mid^{1-\tau(1+\gamma)}.$$

Potter's Theorem with $0<\xi<\gamma$ implies the existence of $A(\xi)>1$ such that:

$$l(n^\tau u) \leq A(\xi)l(n^\tau b)(\frac{u}{b})^\xi.$$

Combining this fact with (2.4), we obtain:

$$J_2 \leq C\frac{l(n^\tau b)}{l(n^\tau)} \mid t-s\mid^{1-\tau(1+\gamma)}.$$

Finally, by using the fact that:

$$\lim_{n\to+\infty}\frac{l(n^\tau u)}{l(n^\tau)} = 1,$$

we complete the proof of Corollary 2.9. □

3. Limit Theorems

The main result of this section is the following result.

Theorem 3.1. *Let $0<\gamma<\xi<\frac{1-\tau}{2\tau}$. Suppose that $f = K_\pm^{l,\gamma}g$ where $g\in\mathcal{C}^\beta\cap L^1(\mathbb{R})$ with compact support for some $\gamma<\beta$. Then when $n\to+\infty$ and $\rho\to 0$, the sequence of processes:*

$$\left([n^{1-\tau(1+\gamma)}l(n^\tau)]^{-1}\int_0^{nt} f(Y_s^{\rho,\tau})ds\right)_{t\geq 0},$$

converges in law to the process:

$$\Big(I(g)c^{-1-\gamma}D^{\gamma}_{\mp}L^{\tau}(t,.)(0)\Big)_{t\geq 0},$$

where $I(g)=\int_{\mathbb{R}} g(x)dx$, L^{τ} *is the local time of the fBm* B^{τ} *of Hurst parameter* τ *and the constant* c *is given by:*

$$c=1 \ \ \textit{(for sfBm)}, \quad c=2^{\frac{1-K}{2}} \ \ \textit{(for bfBm)} \quad \textit{and} \quad c=\frac{\sqrt{2}t_0^{\frac{\alpha}{2}}}{\sqrt{1+b}} \ \ \textit{(for wfBm).}$$

Remark 3.2.
1) Notice that even if f is not a fractional derivative of some function g, the limiting process is a fractional derivative of local time.
2) We recall that $\tau=HK\in(0,1)$ for bfBm, $\tau=H\in(0,1)$ for fBm, $\tau=\frac{\alpha}{2}\in(0,\frac{1}{2})$ for sfBm and $\tau=\frac{1+b}{2}\in(0,1)$ for wfBm.

Proof of Theorem 3.1. The convergence of the finite dimensional distributions follows easily by using the same arguments used in [9] to prove Proposition 5.2 in case of the well known multifractional Brownian motion, and Remark 3.18 in [11]. In fact, according to [1], section 2.3, the process: $(Y^{\rho,\tau}_t, t\geq 0)$ converges, in the sense of finite dimensional distributions, when $\rho\to 0$, to the process: $(c.B^{\tau}_t, t\geq 0)$, where c is the constant appeared in Theorem 3.1. Therefore by combining the fact that f is locally Riemann integrable and Theorem VI.4.2 in [13], we obtain

$$\int_0^{nt} f(Y^{\rho,\tau}_s)ds \longrightarrow \int_0^{nt} f(c.B^{\tau}_s)ds \qquad as \qquad \rho\to 0^+.$$

Using the occupation density formula, the scaling property of local time, (2.6) and (2.7), one can write:

$$\begin{aligned}
&c^{1+\gamma}[n^{1-\tau(1+\gamma)}l(n^{\tau})]^{-1}\int_0^{nt} f(c.B^{\tau}_s)ds\\
&=c^{\gamma}[n^{1-\tau(1+\gamma)}l(n^{\tau})]^{-1}\int_{\mathbb{R}} f(x)L^{\tau}\left(nt,\frac{x}{c}\right)dx\\
&=[l(n^{\tau})]^{-1}\,(cn^{\tau})^{\gamma}\int_{\mathbb{R}} f(x)L^{\tau}\left(t,\frac{x}{cn^{\tau}}\right)dx\\
&=[l(n^{\tau})]^{-1}\,(cn^{\tau})^{\gamma}\int_{\mathbb{R}} g(x)K^{\gamma}_{\mp}\left(L^{\tau}\left(t,\frac{.}{cn^{\tau}}\right)\right)(x)dx\\
&=\frac{l(cn^{\tau})}{l(n^{\tau})}[l(cn^{\tau})]^{-1}\,(cn^{\tau})^{\gamma}\int_{\mathbb{R}} g(x)K^{\gamma}_{\mp}\left(L^{\tau}\left(t,\frac{.}{cn^{\tau}}\right)\right)(x)dx.
\end{aligned}$$

According to [11], Remark 3.18, as $n\to\infty$, we have

$$[l(cn^{\tau})]^{-1}\,(cn^{\tau})^{\gamma}K^{l,\gamma}_{\mp}\left(L^{\tau}\left(t,\frac{.}{cn^{\tau}}\right)\right)(x)\longrightarrow D^{\gamma}_{\mp}L^{\tau}(t,.)(0).$$

By the definition of the slowly varying function l, we have

$$\lim_{n\to+\infty}\frac{l(cn^{\tau})}{l(n^{\tau})}=1.$$

Finally, we obtain the convergence in the sense of finite dimensional distributions. To end the proof of Theorem 3.1, we need only to show the tightness of the sequence:

$$A_t^n := [n^{1-\tau(1+\gamma)} l(n^\tau)]^{-1} \int_0^{nt} f(Y_s^{\rho,\tau}) ds.$$

By the occupation density formula, the scaling property of local time and (2.6), we have

$$\begin{aligned}
\|A_t^n - A_s^n\|_{2p} &= \left\| \frac{1}{l(n^\tau) n^{1-\tau(1+\gamma)}} \left(\int_0^{nt} f(Y_u^{\rho,\tau}) du - \int_0^{ns} f(Y_u^{\rho,\tau}) du \right) \right\|_{2p} \\
&= n^{\tau\gamma} [l(n^\tau)]^{-1} \left\| \int_{\mathbb{R}} f(x) L^{\rho,\tau}\Big(t, \frac{x}{n^\tau}\Big) dx - \int_{\mathbb{R}} f(x) L^{\rho,\tau}\Big(s, \frac{x}{n^\tau}\Big) dx \right\|_{2p} \\
&= n^{\tau\gamma} [l(n^\tau)]^{-1} \left\| \int_{\mathbb{R}} K_{\pm}^{l,\gamma} g(x) \Big[L^{\rho,\tau}\Big(t, \frac{x}{n^\tau}\Big) - L^{\rho,\tau}\Big(s, \frac{x}{n^\tau}\Big) \Big] dx \right\|_{2p} \\
&= n^{\tau\gamma} [l(n^\tau)]^{-1} \left\| \int_{\mathbb{R}} g(x) \Big[K_{\mp}^{l,\gamma} L^{\rho,\tau}\Big(t, \frac{\cdot}{n^\tau}\Big)(x) - K_{\mp}^{l,\gamma} L^{\rho,\tau}\Big(s, \frac{\cdot}{n^\tau}\Big)(x) \Big] dx \right\|_{2p}.
\end{aligned}$$

Therefore, it follows from (2.7), that:

$$\begin{aligned}
&\|A_t^n - A_s^n\|_{2p} \le C[l(n^\tau)]^{-1} \\
&\quad \times \int_S \left\| g(x) \Big(K_{\mp}^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma} L^{\rho,\tau}(t,.)\Big(\frac{x}{n^\tau}\Big) - K_{\mp}^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma} L^{\rho,\tau}(s,.)\Big(\frac{x}{n^\tau}\Big) \Big) \right\|_{2p} dx \\
&\quad \le C \int_S \|g\|_\infty [l(n^\tau)]^{-1} \\
&\qquad \times \left\| \Big(K_{\mp}^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma} L^{\rho,\tau}(t,.)\Big(\frac{x}{n^\tau}\Big) - K_{\mp}^{l\left(\frac{\cdot}{n^{-\tau}}\right),\gamma} L^{\rho,\tau}(s,.)\Big(\frac{x}{n^\tau}\Big) \Big) \right\|_{2p} dx,
\end{aligned}$$

where $S = \text{supp}(g)$.

Thanks to Corollary 2.9, for n sufficiently large, we have

$$\|A_t^n - A_s^n\|_{2p} \le C \mid t - s \mid^{1-\tau(1+\gamma)}.$$

Finally, we can take $p(1 - \tau(1+\gamma)) > 1$ and the tightness is proved.

4. Strong Approximation

In this section, we give a strong approximation of Theorem 3.1, more precisely the L^p-estimate. Our main result in this paragraph reads:

Theorem 4.1. *Let f be a Borel function on $\mathbb{R}$ satisfying:*

$$\int_{\mathbb{R}} |x|^k |f(x)| dx < \infty, \tag{4.1}$$

for some $k > 0$. Then, for any sufficiently small $\varepsilon > 0$ and $\rho > 0$, and any integer $p \ge 1$, when t goes to infinity, we have

$$\left\| \int_0^t K^{l,\gamma} f(Y_s^{\rho,\tau}) ds \right\|_{2p} = \frac{I(f)}{\Gamma(-\gamma)} \|D^\gamma L^\tau(t,.)(0)\|_{2p} + o(t^{1-\tau(1+\gamma)-\varepsilon}).$$

$0 < \gamma < \xi < \frac{1-\tau}{2\tau}$ and L^τ is the local time of the fBm B^τ of Hurst parameter τ.

In order to prove Theorem 4.1, we shall first state and prove some technical lemmas. The proofs are similar to that given by Ait Ouahra and Ouali [2] in case of fractional derivative and fBm.

Lemma 4.2. *Let $0 < \gamma < \xi < \frac{1-\tau}{2\tau}$. For any sufficiently small $\varepsilon > 0$ and $\rho > 0$, and any integer $p \geq 1$, when t goes to infinity, we have*

$$\sup_{x\in\mathbb{R}} \left\| K^{l,\gamma} L^{\rho,\tau}(t,.)(x) \right\|_{2p}^{2p} = o(t^{2p(1-\tau(1+\gamma))+\varepsilon}).$$

Proof. Using Theorem 2.8 for $s = 0$ and the fact that $K^{l,\gamma} L^{\rho,\tau}(0,.)(x) = 0$ a.s., we get:

$$\sup_{x\in\mathbb{R}} \left\| K^{l,\gamma} L^{\rho,\tau}(t,.)(x) \right\|_{2p}^{2p} \leq C t^{2p(1-\tau(1+\gamma))}.$$

The conclusion follows immediately. □

In the same way, using (2.5) for $s = 0$ and the fact that $L^{\rho,\tau}(0, x) = 0$ a.s., we get the following result.

Lemma 4.3. *Let $0 < \xi < \frac{1-\tau}{2\tau}$. For any sufficiently small $\varepsilon > 0$ and $\rho > 0$, and any integer $p \geq 1$, when t goes to infinity, we have*

$$\sup_{x\neq y} \frac{\|L^{\rho,\tau}(t,x) - L^{\rho,\tau}(t,y)\|_{2p}^{2p}}{|x-y|^{2p\xi}} = o(t^{2p(1-\tau(1+\xi))+\varepsilon}).$$

Lemma 4.4. *Let $0 < \gamma < \xi < \frac{1-\tau}{2\tau}$. For any sufficiently small $\varepsilon > 0$ and $\rho > 0$, and any integer $p \geq 1$, when t goes to infinity, we have*

$$\sup_{x\in\mathbb{R}} \left\| \int_0^1 l(y) \frac{L^{\rho,\tau}(t,x+y) - L^{\rho,\tau}(t,x-y)}{y^{1+\gamma}} dy \right\|_{2p}^{2p} = o(t^{2p(1-\tau(1+\xi))+\varepsilon}).$$

Proof. We have

$$\sup_{x\in\mathbb{R}} \left\| \int_0^1 l(y) \frac{L^{\rho,\tau}(t,x+y) - L^{\rho,\tau}(t,x-y)}{y^{1+\gamma}} dy \right\|_{2p}^{2p}$$
$$\leq \sup_{x\in\mathbb{R}} \sup_{0<y\leq 1} \frac{\|L^{\rho,\tau}(t,x+y) - L^{\rho,\tau}(t,x-y)\|_{2p}^{2p}}{y^{2p\xi}} \left| \int_0^1 \frac{l(y)}{y^{1+\gamma-\xi}} dy \right|^{2p}$$

By virtue of Lemma 4.3 and the fact that l is bounded on $[0, 1]$, we deduce the lemma. □

Lemma 4.5. *Let $0 < \gamma < \xi < \frac{1-\tau}{2\tau}$. For any sufficiently small $\varepsilon > 0$ and $\rho > 0$, and any integer $p \geq 1$, when t goes to infinity, we have*

$$\sup_{|x|\leq t^a} \left\| \int_1^\infty l(y) \frac{L^{\rho,\tau}(t,x+y) - L^{\rho,\tau}(t,y)}{y^{1+\gamma}} dy \right\|_{2p}^{2p} = o(t^{2p(1-\tau(1+\xi))+2pa\xi+\varepsilon}),$$

for some $a > 0$.

Proof. We have

$$\sup_{|x|\leq t^a} \left\| \int_1^\infty l(y) \frac{L^{\rho,\tau}(t,x+y) - L^{\rho,\tau}(t,y)}{y^{1+\gamma}} dy \right\|_{2p}^{2p}$$

$$\leq \sup_{|x|\leq t^a} \sup_{y\in\mathbb{R}} \|L^{\rho,\tau}(t,x+y) - L^{\rho,\tau}(t,y)\|_{2p}^{2p} \left|\int_1^\infty \frac{l(y)}{y^{1+\gamma}}dy\right|^{2p}$$

$$\leq \sup_{|x|\leq t^a} |x|^{2p\xi} \sup_{y\in\mathbb{R}} \frac{\|L^{\rho,\tau}(t,x+y) - L^{\rho,\tau}(t,y)\|_{2p}^{2p}}{|x|^{2p\xi}} \left|\int_1^\infty \frac{l(y)}{y^{1+\gamma}}dy\right|^{2p}.$$

Using Potter's Theorem for $x = 1$, $y \geq 1$ and $0 < \delta < \gamma$, we obtain:

$$\int_1^{+\infty} \frac{l(y)}{y^{1+\gamma}}dy < \infty. \tag{4.2}$$

Finally, by virtue of Lemma 4.3, we deduce the desired estimate. □

Lemma 4.6. *Let f be a Borel function on $\mathbb{R}$ satisfying (4.1) for some $k > 0$. Then, for any sufficiently small $\varepsilon > 0$ and $\rho > 0$, and any integer $p \geq 1$, when t goes to infinity, we have:*

$$\left\| \int_0^t K^{l,\gamma} f(Y_s^{\rho,\tau})ds \right\|_{2p} = \frac{I(f)}{\Gamma(-\gamma)} \|K^{l,\gamma} L^{\rho,\tau}(t,.)(0)\|_{2p} + o(t^{1-\tau(1+\gamma)-\varepsilon}),$$

where $0 < \gamma < \xi < \frac{1-\tau}{2\tau}$.

Proof. By the occupation density, we have

$$\begin{aligned} I(t) &:= \left\| \int_0^t K^{l,\gamma} f(Y_s^{\rho,\tau})ds - \frac{I(f)}{\Gamma(-\gamma)} K^{l,\gamma} L^{\rho,\tau}(t,.)(0) \right\|_{2p}^{2p} \\ &= C \left\| \int_{\mathbb{R}} (K^{l,\gamma} L^{\rho,\tau}(t,.)(x) - K^{l,\gamma} L^{\rho,\tau}(t,.)(0)) f(x)dx \right\|_{2p}^{2p} \\ &\leq C(I_1(t) + I_2(t)), \end{aligned}$$

where

$$I_1(t) := \left\| \int_{|x|>t^a} (K^{l,\gamma} L^{\rho,\tau}(t,.)(x) - K^{l,\gamma} L^{\rho,\tau}(t,.)(0)) f(x)dx \right\|_{2p}^{2p},$$

and

$$I_2(t) := \left\| \int_{|x|\leq t^a} (K^{l,\gamma} L^{\rho,\tau}(t,.)(x) - K^{l,\gamma} L^{\rho,\tau}(t,.)(0)) f(x)dx \right\|_{2p}^{2p},$$

for some $0 < a \leq \tau$.

Let us deal with the first term $I_1(t)$. Lemma 4.3 and (4.1) imply that:

$$\begin{aligned} I_1(t) &\leq \sup_{|x|>t^a} \|K^{l,\gamma} L^{\rho,\tau}(t,.)(x) - K^{l,\gamma} L^{\rho,\tau}(t,.)(0)\|_{2p}^{2p} \left| \int_{|x|>t^a} |x|^{-k} |x|^k |f(x)|dx \right|^{2p} \\ &\leq t^{-2pak} \sup_{|x|>t^a} \|K^{l,\gamma} L^{\rho,\tau}(t,.)(x) - K^{l,\gamma} L^{\rho,\tau}(t,.)(0)\|_{2p}^{2p} \left| \int_{|x|>t^a} |x|^k |f(x)|dx \right|^{2p} \\ &= o(t^{2p(1-\tau(1+\gamma))-2pak+\varepsilon}). \end{aligned}$$

Now, we deal with $I_2(t)$. By the definition of $K^{l,\gamma}$ and the fact that f is integrable, we have:

$$I_2(t) \leq$$

$$\sup_{|x|\le t^a}\left\|\int_0^\infty l(y)\frac{L^{\rho,\tau}(t,x+y)-L^{\rho,\tau}(t,x-y)-L^{\rho,\tau}(t,y)+L^{\rho,\tau}(t,-y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p}$$

$$\times\left|\int_{|x|\le t^a}|f(x)|dx\right|^{2p}$$

$$\le C\sup_{|x|\le t^a}\left\|\int_0^1 l(y)\frac{L^{\rho,\tau}(t,x+y)-L^{\rho,\tau}(t,x-y)-L^{\rho,\tau}(t,y)+L^{\rho,\tau}(t,-y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p}$$

$$+$$

$$C\sup_{|x|\le t^a}\left\|\int_1^\infty l(y)\frac{[L^{\rho,\tau}(t,x+y)-L^{\rho,\tau}(t,y)]-[L^{\rho,\tau}(t,x-y)-L^{\rho,\tau}(t,-y)]}{y^{1+\gamma}}dy\right\|_{2p}^{2p}$$

which, in view of Lemmas 4.4 and 4.5, implies:

$$\begin{aligned}I_2(t)&=o(t^{2p(1-\tau(1+\xi))+\varepsilon})+o(t^{2p(1-\tau(1+\xi))+2pa\xi+\varepsilon})\\&=o(t^{2p(1-\tau(1+\xi))+2pa\xi+\varepsilon}).\end{aligned}$$

Then

$$I(t)=o(t^{2p(1-\tau(1+\gamma))-2pka+\varepsilon})+o(t^{2p(1-\tau(1+\xi))+2pa\xi+\varepsilon}).$$

Choosing:

$$a=\frac{\tau(\xi-\gamma)}{\xi+k}.$$

It is clear that $0<a\le\tau$. We finally get:

$$I(t)=o(t^{2pb+\varepsilon}),$$

with

$$b=\frac{\xi(1-\tau(1+\gamma))+k(1-\tau(1+\xi))}{k+\xi}.$$

Clearly $b<1-\tau(1+\gamma)$, because $\gamma<\xi$. Then for all sufficiently small $\varepsilon>0$, when t goes to infinity, we have

$$I(t)=o(t^{2p(1-\tau(1+\gamma))-\varepsilon}),$$

which gives the desired estimate. □

Now, to end the proof of Theorem 4.1, it suffices to establish the following result.

Lemma 4.7. *Let f be a Borel function on $\mathbb{R}$ satisfying (4.1) for some $k>0$. Then, for any sufficiently small $\varepsilon>0$ and $\rho>0$, and any integer $p\ge 1$, when t goes to infinity, we have:*

$$\frac{I(f)}{\Gamma(-\gamma)}\left|\|K^{l,\gamma}L^{\rho,\tau}(t,.)(0)\|_{2p}-\|D^\gamma L^\tau(t,.)(0)\|_{2p}\right|=o(t^{1-\tau(1+\gamma)-\varepsilon}),$$

where L^τ is the local time of the fBm of Hurst parameter τ.

Proof. Using (4.1), we get:

$$J(t) := \frac{I(f)}{\Gamma(-\gamma)} \|K^{l,\gamma} L^{\rho,\tau}(t,.)(0)\|_{2p} \leq C(J_1(t) + J_2(t)),$$

where

$$J_1(t) := \sup_{|x|>t^a} \|K^{l,\gamma} L^{\rho,\tau}(t,.)(0)\|_{2p} \int_{|x|>t^a} |x|^{-k}|x|^k|f(x)|dx,$$

and

$$J_2(t) := \sup_{|x|\leq t^a} \|K^{l,\gamma} L^{\rho,\tau}(t,.)(0)\|_{2p}.$$

The same arguments used in the proof of Lemma 4.6 implies that:

$$J_1(t) = o(t^{1-\tau(1+\gamma)-ka+\varepsilon}).$$

For $J_2(t)$, we have by Lemma 4.3:

$$J_2(t) = o(t^{1-\tau(1+\xi)+\varepsilon}),$$

therefore

$$J_2(t) = o(t^{1-\tau(1+\xi)+a\xi+\varepsilon}),$$

Consequently

$$J(t) = o(t^{1-\tau(1+\gamma)-\varepsilon}).$$

On the other hand, using Remark 2.2 and the fact that the fBm B^τ satisfies:

$$(B^{\rho,\tau}\ ,\ t \geq 0) \underline{\underline{d}} (B^\tau\ ,\ t \geq 0),$$

we get:

$$\Big(K^{l,\gamma} L^\tau(t,.)(0)\ ,\ t \geq 0\Big) \underline{\underline{d}} \Big(K^{l,\gamma} L^{\rho,\tau}(t,.)(0)\ ,\ t \geq 0\Big).$$

Therefore

$$\frac{I(f)}{\Gamma(-\gamma)} \|K^{l,\gamma} L^\tau(t,.)(0)\|_{2p} = \frac{I(f)}{\Gamma(-\gamma)} \|K^{l,\gamma} L^{\rho,\tau}(t,.)(0)\|_{2p} = o(t^{1-\tau(1+\gamma)-\varepsilon}).$$

In particular, if we take $l \equiv 1$, we get:

$$\frac{I(f)}{\Gamma(-\gamma)} \|D^\gamma L^\tau(t,.)(0)\|_{2p} = o(t^{-\tau(1+\gamma)-\varepsilon}).$$

The proof of Lemma 4.7 is done. □

Finally, combining Lemma 4.6 and Lemma 4.7, the proof of Theorem 4.1 is completed.

Remark 4.8. We should point out that in this paper we only study the L^p-estimate of our limit theorem. This is enough for the purpose of this study. We will study the *a.s.*, estimate in future work and apply this idea to study the law of the iterated logarithm of stochastic process of the form $\int_0^t K^{l,\gamma} f(Y_s^{\rho,\tau})ds$. For the *a.s.* estimate in case of Brownian motion and the symmetric stable process of index $\alpha \in (1, 2]$, we refer respectively to [10] and [3].

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