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STABILITY OF JUNGCK-NOOR ITERATION IN B-METRIC SPACE

Bhagwati Prasad and Komal Goyal

Abstract: The convergence of Jungck-Noor iteration procedure is studied in the setting of *b*-metric space for the maps satisfying some general conditions. Some well known results are obtained as special cases. The convergence rate of Jungck-Mann and Jungck-Noor iterative schemes is also discussed with multiple examples.

Key Words: *Jungck -Noor iteration, Jungck -Ishikawa iteration, stability of iteration, Fixed point iteration.*

I. INTRODUCTION AND PRILIMINIRIES

Let (X,d) be a complete metric space and $T : X \to X$. Let $F_T = \{x \in X : Tx = x\}$ be the set of fixed points of T in X. The approximation of these fixed points can be achieved through several iterative methods. Some of them are discussed below.

Let $\{x_n\}_0^\infty \subset X$ be a sequence constructed by the following rule:

$$x_{n+1} = f(T, x_n) \tag{1.1}$$

If

$$f(T, x_n) = Tx_n, \tag{1.2}$$

Then it is called Picard iteration process. It is well known that the Picard iteration is stable corresponding to a strict type contraction but it is not stable in case of a non-expansive mapping.

The concept of stability was initiated by Urabe [29]. But a formal definition of stability of general iterative procedure was given by Harder and Hick [6-7].

Definition 1.1 [6-7]. For the functional equations (1.1) and (1.2), Suppose $\{x_n\}$ converges to a fixed point u of T. Let $\{y_n\}$ be an arbitrary sequence in X, and define

$$\varepsilon_n = d(y_{n+1}, f(T, y_n)), \; n = 0, 1, 2, \dots .$$

If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = u$, then the iteration procedure is said to be T-stable or stable with respect to T.

First stability result on T-stable mapping was due to Ostrowski [16] for the stability of Picard iteration using Banach contraction condition.

If $f(T, x_n) = (1-\alpha_n)x_n + \alpha_n Tx_n$ with $\{\alpha_n\}_{n=0}^\infty \subset [0,1]$, then (1.1) becomes,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n = 0, 1, \dots \tag{1.3}$$

Department of Mathematics, Jaypee Institute of Information Technology, Noida, India
Emails: b_prasad10@yahoo.com, komal.goyal0988@gmail.com

the scheme (1.3) is called Mann iteration process [11]. If T is continuous and Mann iteration converges, then it converges to a fixed point of T. But If T is not continuous, then there is no guarantee for its convergence in the space.
Ishikawa [9] defined a two step iteration procedure as follows.
For $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tz_n,$$
$$z_n = (1-\beta_n)x_n + \beta_n Tx_n, n = 0,1,..., \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are the real sequences in $[0,1]$.
Observe that if $\alpha_n = 1$ in (1.3), it reduces to (1.2). Similarly Noor [12] defined the following three step iteration scheme.

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n Tz_n,$$
$$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \ n = 0,1,..., \tag{1.5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are the real sequences in $[0,1]$. If we put $\gamma_n = 0$ for each 'n' in (1.5), it reduces to (1.4).
Singh et al. [28] defined Junck-Mann iteration process as follows:
Let Y be an arbitrary non empty set and (X,d) be a metric space and $S,T : Y \to X$ with $T(Y) \subset S(Y)$ for some $x_0 \in Y$, consider the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n, n = 0,1,2..., \tag{1.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$.
If $\alpha_n = 1$ in (1.6), we get Jungck [10] iterative scheme.

$$Sx_{n+1} = Tx_n, n = 0,1,2... \tag{1.7}$$

Singh et al. [28] obtained the following important result for (S,T) stability in metric space.

Definition 1.2 [28]. Let $S,T : Y \to X, T(Y) \subset S(Y)$ and 'z' a coincidence point of T and S that is $Sz = Tz = p$ (say), for any $x_0 \in Y$. Let the sequence $\{Sx_n\}$, generated by iterative procedure, $Sx_{n+1} = f(T,x_n), n = 0,1,2...$ converges to 'p'. Let $\{Sy_n\} \subset X$ be an arbitrary sequence, and set $\varepsilon_n = d(Sy_{n+1}, f(T,y_n)), n = 0,1,2...$ then the iterative procedure $f(T,x_n)$ will be called (S,T) stable if and only if $\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} Sy_n = p$.

On this line, Olatinwo and Imoru [13] defined $\{Sx_n\}_{n=0}^{\infty}$ in the following manner.

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n,$$
$$Sz_n = (1-\beta_n)Sx_n + \beta_n Tx_n, n = 0,1,..., \tag{1.8}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are the real sequences in $[0,1]$, is called Jungck-Ishikawa iteration scheme.
When $Y = X$ and $S = id$, the identity map on X, (1.8) becomes (1.4).

Further, Olatinwo [14] extended condition (1.8) for three step iteration procedure as follows.
Definition 1.3 [14]. Let $S : X \to X$ and $T(X) \subseteq S(X)$. Define

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n,$$
$$Sz_n = (1-\beta_n)Sx_n + \beta_n Tr_n,$$
$$Sr_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \quad (1.9)$$

where $n = 0,1,...$ and $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\alpha_0 = 1$ (ii) $0 \le \alpha_n, \beta_n, \gamma_n \le 1, n > 0$

(iii) $\sum \alpha_n = \infty$ (iv) $\sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i)$ converges.

This scheme is called Jungck-Noor iteration scheme

Definition 1.4 [2]. Suppose $\{a_n\}$ and $\{b_n\}$ are two real convergent sequence with limits 'a' and 'b' respectively.

Then $\{a_n\}$ is said to converge faster than $\{b_n\}$ if $\lim_{n\to\infty} \left|\frac{a_n - a}{b_n - b}\right| = 0$.

Harder and Hick [7] established stability results for Picard and Mann both iterative schemes using Banach contraction as well as Zamfirescu contraction conditions. Rhoades [25, 26] extended these results to the following contractive conditions:

There exist $c \in [0,1)$ such that

$$d(Tx,Ty) \le c \max\{d(x,y), d(x,Ty), d(y,Tx)\} \ \forall x, y \in X. \quad (1.10)$$

and

$$d(Tx,Ty) \le c \max\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, d(x,Ty), d(y,Tx)\} \ \forall x, y \in X. \quad (1.11)$$

Osilike and Udomene [15] studied the stability of a map satisfying a more general contractive condition that is, there exits $a \in [0,1]$ such that

$$d(Tx,Ty) \le Ld(x,Tx) + ad(x,y) \ \forall x, y \in X. \quad (1.12)$$

Imoru and Olantinwo [8] generalized some stability results of Rhoades [25-26]. Singh et al. [28] obtained stability results for Jungck and Jungck-Mann iterative procedures in metric space using the following contractive condition:

$$d(Tx,Ty) \le Ld(Sx,Tx) + ad(Sx,Sy) \ \forall x, y \in X. \quad (1.13)$$

It is to be noticed that the condition (1.13) is more general than that of Osilike and Udomene [15] but independent of Imoru and Olantinwo [8]. Thereafter many researchers studied the stability of various iterations for a variety of maps in different settings, see for instance Prasad [17], Prasad and Katiyar [18], Prasad et al. [19] Prasad and Sahni [20-24] and several references therein.

The following metric space is widely studied in the literature due to its generality (see [5], [27] and several references thereof).

Definition 1.5 [5]. Let X be a nonempty set and $r \ge 1$ be a given real number. A function $d : X \times X \to R_+$ is said to be a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x,y) = 0$ iff $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \le r[d(x,y) + d(y,z)]$

A pair (X,d) is called a b-metric space.

We shall require the following Lemma of [3] in the sequel.

Lemma 1.1 [3]. If δ is a real number such that $0 \le \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive number such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \le \delta u_n + \varepsilon_n, n = 0,1,2...$ we have $\lim_{n\to\infty} u_n = 0$.

In this paper, we study the stability of Junck-Noor iterative procedure for the maps satisfying condition(1.13) in the setting of b-metric spaces.

II. MAIN RESULT

Theorem 2.1. Let (X,d) be a b-metric space and $S,T : Y \to X$ such that $T(Y) \subseteq S(Y)$, and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, that is $Sz = Tz = p$. Let $x_0 \in Y$ and the sequence $\{Sx_n\}$, generated by (1.9) converges to p. Let $\{Sy_n\} \subset X$ and define

$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n), \ \ n \ge 0$

If the pair (S,T) satisfies (1.13). Then,

(I) $$d(p, Sy_{n+1}) \le rd(p, Sx_{n+1}) + r^2 \prod_{i=0}^{n} (1-\alpha_i + a\alpha_i) d(Sx_0, Sy_0) + Lr^2a^2 \sum_{j=0}^{n} r^{n-i} \alpha_j \beta_j \gamma_j \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i) d(Sx_j, Tx_j)$$
$$+Lr^2 a \sum_{j=0}^{n} r^{n-i} \alpha_j \beta_j \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i) d(Sr_j, Tr_j) + Lr^2 \sum_{j=0}^{n} r^{n-i} \alpha_j \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i) d(Sz_j, Tz_j)$$
$$+\sum_{j=0}^{n} r^{j+2} \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i) \varepsilon_j$$

(II) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof: By triangle inequality,

$$\begin{aligned} d(p, Sy_{n+1}) &\le r[d(p, Sx_{n+1}) + d(Sx_{n+1}, Sy_{n+1})] \\ &\le rd(p, Sx_{n+1}) + r[d((1-\alpha_n)Sx_n + \alpha_n Tz_n, Sy_{n+1})] \\ &\le rd(p, Sx_{n+1}) + r^2[d((1-\alpha_n)Sx_n + \alpha_n Tz_n, (1-\alpha_n)Sy_n + \alpha_n Ts_n) + d((1-\alpha_n)Sx_n + \alpha_n Ts_n, Sy_{n+1})] \\ &\le rd(p, Sx_{n+1}) + r^2(1-\alpha_n) d(Sx_n, Sy_n) + r^2\alpha_n d(Tz_n, Ts_n) + r^2\varepsilon_n \\ &\le rd(p, Sx_{n+1}) + r^2(1-\alpha_n) d(Sx_n, Sy_n) + r^2\alpha_n [ad(Sz_n, Ss_n) + Ld(Sz_n, Tz_n)] + r^2\varepsilon_n. \end{aligned} \quad (2.1)$$

But,

$$\begin{aligned} d(Sz_n, Ss_n) &= d((1-\beta_n)Sx_n + \beta_n Tr_n, (1-\beta_n)Sy_n + \beta_n Tq_n) \\ &= (1-\beta_n)d(Sx_n, Sy_n) + \beta_n d(Tr_n, Tq_n) \\ &\le (1-\beta_n)d(Sx_n, Sy_n) + \beta_n[ad(Sr_n, Sq_n) + Ld(Sr_n, Tr_n)] \\ &\le (1-\beta_n)d(Sx_n, Sy_n) + a\beta_n d((1-\gamma_n)Sx_n + \gamma_n Tx_n, (1-\gamma_n)Sy_n + \gamma_n Ty_n) + \beta_n Ld(Sr_n, Tr_n) \\ &\le (1-\beta_n)d(Sx_n, Sy_n) + a\beta_n(1-\gamma_n)d(Sx_n, Sy_n) + a\beta_n\gamma_n d(Tx_n, Ty_n) + \beta_n Ld(Sr_n, Tr_n) \\ &\le (1-\beta_n + a\beta_n(1-\gamma_n))d(Sx_n, Sy_n) + a\beta_n\gamma_n d(Tx_n, Ty_n) + \beta_n Ld(Sr_n, Tr_n) \\ &\le (1-\beta_n + a\beta_n(1-\gamma_n))d(Sx_n, Sy_n) + a\beta_n\gamma_n[ad(Sx_n, Sy_n) + Ld(Sx_n, Tx_n)] + \beta_n Ld(Sr_n, Tr_n) \\ &\le (1-\beta_n + a\beta_n(1-\gamma_n) + a^2\beta_n\gamma_n)d(Sx_n, Sy_n) + a\beta_n\gamma_n Ld(Sx_n, Tx_n) + \beta_n Ld(Sr_n, Tr_n) \\ &\le d(Sx_n, Sy_n) + \beta_n Ld(Sr_n, Tr_n) + a\beta_n\gamma_n Ld(Sx_n, Tx_n). \end{aligned} \tag{2.2}$$

Therefore,

$$\begin{aligned} d(p, Sy_{n+1}) &\le rd(p, Sx_{n+1}) + r^2(1-\alpha_n)d(Sx_n, Sy_n) + ar^2\alpha_n[d(Sx_n, Sy_n) + \beta_n Ld(Sr_n, Tr_n) + a\beta_n\gamma_n Ld(Sx_n, Tx_n)] \\ &\quad + r^2\alpha_n Ld(Sz_n, Tz_n) + r^2\varepsilon_n. \end{aligned} \tag{2.3}$$

$$\begin{aligned} d(Sx_n, Sy_n) &= d((1-\alpha_{n-1})Sx_{n-1} + \alpha_{n-1}Tz_{n-1}, Sy_n) \\ &\le r[d((1-\alpha_{n-1})Sx_{n-1} + \alpha_{n-1}Tz_{n-1}, (1-\alpha_{n-1})Sy_{n-1} + \alpha_{n-1}Ts_{n-1}) + d((1-\alpha_{n-1})Sy_{n-1} + \alpha_{n-1}Ts_{n-1}, Sy_n)] \\ &\le r(1-\alpha_{n-1})d(Sx_{n-1}, Sy_{n-1}) + r\alpha_{n-1}d(Tz_{n-1}, Ts_{n-1}) + r\varepsilon_{n-1} \\ &\le r(1-\alpha_{n-1})d(Sx_{n-1}, Sy_{n-1}) + r\alpha_{n-1}[ad(Sz_{n-1}, Ss_{n-1}) + Ld(Sz_{n-1}, Tz_{n-1})] + r\varepsilon_{n-1} \\ &\le r(1-\alpha_{n-1})d(Sx_{n-1}, Sy_{n-1}) + ar\alpha_{n-1}d((1-\beta_{n-1})Sx_{n-1} + \beta_{n-1}Tr_{n-1}, (1-\beta_{n-1})Sy_{n-1} + \beta_{n-1}Tq_{n-1}) \\ &\quad + r\alpha_{n-1}Ld(Sz_{n-1}, Tz_{n-1}) + r\varepsilon_{n-1} \\ &\le r(1-\alpha_{n-1})d(Sx_{n-1}, Sy_{n-1}) + ar\alpha_{n-1}(1-\beta_{n-1})d(Sx_{n-1}, Sy_{n-1}) + ar\alpha_{n-1}\beta_{n-1}d(Tr_{n-1}, Tq_{n-1}) \\ &\quad + r\alpha_{n-1}Ld(Sz_{n-1}, Tz_{n-1}) + r\varepsilon_{n-1} \\ &\le r(1-\alpha_{n-1}(1-ar(1-\beta_{n-1})))d(Sx_{n-1}, Sy_{n-1}) + ar\alpha_{n-1}\beta_{n-1}[ad(Sr_{n-1}, Sq_{n-1}) + Ld(Sr_{n-1}, Tr_{n-1})] \\ &\quad + r\alpha_{n-1}Ld(Sz_{n-1}, Tz_{n-1}) + r\varepsilon_{n-1} \\ &\le d(Sx_{n-1}, Sy_{n-1}) + ar\alpha_{n-1}\beta_{n-1}[ad(Sr_{n-1}, Sq_{n-1}) + Ld(Sr_{n-1}, Tr_{n-1})] + r\alpha_{n-1}Ld(Sz_{n-1}, Tz_{n-1}) + r\varepsilon_{n-1}. \end{aligned} \tag{2.4}$$

Now observe that

$$\begin{aligned} d(Sr_{n-1}, Sq_{n-1}) &= d((1-\gamma_{n-1})Sx_{n-1} + \gamma_{n-1}Tx_{n-1}, (1-\gamma_{n-1})Sy_{n-1} + \gamma_{n-1}Ty_{n-1}) \\ &\le (1-\gamma_{n-1})d(Sx_{n-1}, Sy_{n-1}) + \gamma_{n-1}d(Tx_{n-1}, Ty_{n-1}) \\ &\le (1-\gamma_{n-1})d(Sx_{n-1}, Sy_{n-1}) + \gamma_{n-1}[ad(Sx_{n-1}, Sy_{n-1}) + Ld(Sx_{n-1}, Tx_{n-1})] \\ &\le d(Sx_{n-1}, Sy_{n-1}) + \gamma_{n-1}Ld(Sx_{n-1}, Tx_{n-1}). \end{aligned} \tag{2.5}$$

So,

$$\begin{aligned} d(Sx_n, Sy_n) &\le d(Sx_{n-1}, Sy_{n-1}) + a^2r\alpha_{n-1}\beta_{n-1}[d(Sx_{n-1}, Sy_{n-1}) + \gamma_{n-1}Ld(Sx_{n-1}, Tx_{n-1})] + ar\alpha_{n-1}\beta_{n-1}Ld(Sr_{n-1}, Tr_{n-1}) \\ &\quad + r\alpha_{n-1}Ld(Sz_{n-1}, Tz_{n-1}) + r\varepsilon_{n-1} \\ &\le d(Sx_{n-1}, Sy_{n-1}) + a^2r\alpha_{n-1}\beta_{n-1}\gamma_{n-1}Ld(Sx_{n-1}, Tx_{n-1}) + ar\alpha_{n-1}\beta_{n-1}Ld(Sr_{n-1}, Tr_{n-1}) \\ &\quad + r\alpha_{n-1}Ld(Sz_{n-1}, Tz_{n-1}) + r\varepsilon_{n-1}. \end{aligned} \tag{2.6}$$

Thus from (2.3), we have

$$d(p,Sy_{n+1}) \le rd(p,Sx_{n+1}) + r^2(1-\alpha_n + a\alpha_n)[d(Sx_{n-1},Sy_{n-1}) + a^2 r\alpha_{n-1}\beta_{n-1}\gamma_{n-1}Ld(Sx_{n-1},Tx_{n-1})$$
$$+ar\alpha_{n-1}\beta_{n-1}Ld(Sr_{n-1},Tr_{n-1}) + r\alpha_{n-1}Ld(Sz_{n-1},Tz_{n-1}) + r\varepsilon_{n-1}] + ar^2\alpha_n[\beta_n Ld(Sr_n,Tr_n)$$
$$+a\beta_n\gamma_n Ld(Sx_n,Tx_n)] + r^2\alpha_n Ld(Sz_n,Tz_n) + r^2\varepsilon_n. \tag{2.7}$$

This process when repeated *(n-1)* times yields (I).

To prove (II), suppose that $\lim_{n\to\infty} Sy_n = p$. Then,

$$\begin{aligned}\varepsilon_n &= d(Sy_{n+1},(1-\alpha_n)Sy_n + \alpha_n Ts_n)\\ &\le r[d(Sy_{n+1},p) + d(p,(1-\alpha_n)Sy_n + \alpha_n Ts_n)]\\ &\le rd(Sy_{n+1},p) + r(1-\alpha_n)d(p,Sy_n) + r\alpha_n d(p,Ts_n)\\ &\le rd(Sy_{n+1},p) + r(1-\alpha_n)d(p,Sy_n) + r\alpha_n d(Tp,Ts_n)\\ &\le rd(Sy_{n+1},p) + r(1-\alpha_n)d(p,Sy_n) + r\alpha_n[ad(Sp,Ss_n) + Ld(Sp,Tp)] \to 0 \text{ as } n\to\infty.\end{aligned}$$

Now suppose that $\lim_{n\to\infty}\varepsilon_n = 0$. Let A denotes the lower triangular matrix with entries

$$\alpha_{nj} = \alpha_j\beta_j\gamma_j\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i).$$

Then A is multiplicative, so that

$$\lim_{n\to\infty} Lr^2a^2\sum_{j=0}^{n} r^{n-i}\alpha_j\beta_j\gamma_j\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(Sx_j,Tx_j) = 0,$$

$$\lim_{n\to\infty} Lr^2a\sum_{j=0}^{n} r^{n-i}\alpha_j\beta_j\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(Sr_j,Tr_j) = 0$$

and
$$\lim_{n\to\infty} Lr^2\sum_{j=0}^{n} r^{n-i}\alpha_j\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(Sz_j,Tz_j) = 0.$$

Let B be the lower triangular matrix with entries

$$b_{ij} = \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i).$$

condition (iv) of iterative scheme implies that B is multiplicative and hence

$$\lim_{n\to\infty}\sum_{j=0}^{n} r^{j+2}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j = 0.$$

Finally, condition (iii) of iterative scheme implies

$$\lim_{n\to\infty}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i) = 0.$$

Hence, it follows from inequality that $\lim_{n\to\infty} Sy_n = p$.

This completes the proof.

On putting $Y = X$ and $S = id$, the identity map on X in Theorem 2.1, we obtain the following result.

Corollary 2.1. Let (X,d) be a b-metric space and $T: X\to X$. Let p be the fixed point such that $Tp = p$. Let $x_0 \in X$ and the sequence $\{Tx_n\}$, generated by,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tz_n,$$
$$z_n = (1-\beta_n)x_n + \beta_n Tr_n,$$
$$r_n = (1-\gamma_n)x_n + \gamma_n Tx_n$$

converges to *p*. Let $\{Ty_n\} \subset X$ and define $\varepsilon_n = d(y_{n+1}, (1-\alpha_n)y_n + \alpha_n Ts_n), \ \ n \geq 0$
If the mapping T satisfies

$$d(Tx,Ty) \leq ad(x,y) + Ld(x,Tx), \ a \in [0,1), \ L \geq 0 .$$

Then,

$$\text{(I) } d(p,y_{n+1}) \leq rd(p,x_{n+1}) + r^2 \prod_{i=0}^{n}(1-\alpha_i + a\alpha_i)d(x_0,y_0) + Lr^2a^2 \sum_{j=0}^{n} r^{n-i}\alpha_j\beta_j\gamma_j \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(x_j,Tx_j)$$
$$+Lr^2a\sum_{j=0}^{n} r^{n-i}\alpha_j\beta_j \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(r_j,Tr_j) + Lr^2 \sum_{j=0}^{n} r^{n-i}\alpha_j \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(z_j,Tz_j)$$
$$+\sum_{j=0}^{n} r^{j+2} \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j$$

if and only if $\lim_{n\to\infty} \varepsilon_n = 0.$ (II) $\lim_{n\to\infty} y_n = p$

If we put $r = s$ and $\gamma_n = 0$ for all $n \geq 0$ in Theorem 2.1, the following result of Prasad and Sahni [21] is derived.

Corollary 2.2 [21]. Let (X,d) be a *b*-metric space and *S*, *T* maps on an arbitrary set *Y* with values in *X* such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of *X*. Let *z* be a coincidence point of *T* and *S*, that is $Sz = Tz = p$. Let $x_0 \in Y$ and let the sequence $\{Sx_n\}$, generated by

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n,$$
$$Sz_n = (1-\beta_n)Sx_n + \beta_n Tx_n$$

converges to *p*. Let $\{Sy_n\} \subset X$ and define $Ss_n = (1-\beta_n)Sy_n + \beta_n Ty_n, n \geq 0$

$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n), \ \ n \geq 0$$

If the pair (S,T) is for all $x, y \in Y$,

$$d(Tx,Ty) \leq ad(Sx,Sy) + Ld(Sx,Tx), \ a \in [0,1), \ L \geq 0 .$$

$$\text{(I) } d(p,Sy_{n+1}) \leq sd(p,Sx_{n+1}) + s^2 \prod_{i=0}^{n}(1-\alpha_i + a\alpha_i)d(Sx_0,Sy_0) + Ls^2a\sum_{j=0}^{n} s^{n-i}\alpha_j\beta_j \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(Sx_j,Tx_j)$$

Then,

$$+Ls^2\sum_{j=0}^{n} s^{n-i}\alpha_j \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(Sz_j,Tz_j) + \sum_{j=0}^{n} s^{j+2} \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j.$$

(II) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0.$

If we put $\gamma_n = 0$ and $r = 1$ in Theorem 2.1, we obtain following result of Prasad and Sahni [24].

Corollary 2.3 [24]. Let (X,d) be a metric space and *S*, *T* maps on an arbitrary set *Y* with values in *X* such that $T(Y) \subseteq S(Y)$, and $S(Y)$ or $T(Y)$ is a complete subspace of *X*. Let *z* be a coincidence point of T and S, that is $Sz = Tz = p$. Let $x_0 \in Y$ and let the sequence $\{Sx_n\}$, generated by

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n,$$
$$Sz_n = (1-\beta_n)Sx_n + \beta_n Tx_n$$

converges to p. Let $\{Sy_n\} \subset X$ and define $Ss_n = (1-\beta_n)Sy_n + \beta_n Ty_n, n \geq 0$

$$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n), \quad n \geq 0$$

If the pair (S,T) is for all $x, y \in Y$,

$$d(Tx,Ty) \leq ad(Sx,Sy) + Ld(Sx,Tx), \; a \in [0,1), \; L \geq 0 .$$

Then,

(I) $d(p, Sy_{n+1}) \leq d(p, Sx_{n+1}) + \prod_{i=0}^{n}(1-\alpha_i + a\alpha_i)d(Sx_0, Sy_0) + La\sum_{j=0}^{n}\alpha_j\beta_j\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(Sx_j, Tx_j)$

$$+L\sum_{j=0}^{n}\alpha_j\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)d(Sz_j, Tz_j) + \sum_{j=0}^{n}\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j$$

(II) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0.$

If we put $\alpha_n = \beta_n = \gamma_n = 0$ for all $n \geq 0, \; q = a, \; s = r$ and $L = 0$ in Theorem 2.1, we obtain the following result of Singh and Prasad [27].

Corollary 2.4 [27]. Let (X,d) be a b-metric space and S, T be maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$, and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, that is $Sz = Tz = u$ (say). Let $x_0 \in Y$ and let the sequence $\{Sx_n\}$, generated by $Sx_{n+1} = Tx_n, n = 0,1,2...$ converges to u. Let $\{Sy_n\} \subset X$ and define $\varepsilon_n = d(Sy_{n+1}, Ty_n), \; n \geq 0$. If the pair (S,T) satisfies $d(Tx,Ty) \leq qd(Sx,Sy)$ for all $x, y \in Y$,$0 \leq q < 1$ and $k = sq < 1$. Then,

(I) $d(u, Sy_{n+1}) \leq sd(u, Sx_{n+1}) + sk^{n+1}d(Sx_0, Sy_0) + s^2\sum_{r=0}^{n}k^{n-r}\varepsilon_r.$

(II) $\lim_{n\to\infty} Sy_n = u$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0.$

If we put $\alpha_n = \beta_n = \gamma_n = 0$ for all $n \geq 0, \; q = a$ and $s = r$ in Theorem 2.1, we obtain the following result of Singh and Prasad [27].

Corollary 2.5 [27]. Let (X,d) be a b-metric space and S, T be maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$, and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, that is $Sz = Tz = u$ (say). Let $x_0 \in Y$ and let the sequence $\{Sx_n\}$, generated by $Sx_{n+1} = Tx_n, n = 0,1,2...$ converges to u. Let $\{Sy_n\} \subset X$ and define $\varepsilon_n = d(Sy_{n+1}, Ty_n), \; n \geq 0$. If the pair (S,T) satisfies $d(Tx,Ty) \leq qd(Sx,Sy) + Ld(Sx,Tx)$ for all $x, y \in Y$ where $0 \leq q < 1, s^2 q < 1$ and $L \geq 0$. Then,

(I) $d(u, Sy_{n+1}) \leq sd(u, Sx_{n+1}) + s(sq)^{n+1}d(Sx_0, Sy_0) + s^2L\sum_{r=0}^{n}(sq)^{n-r}d(Sx_r, Tx_r) + s^2\sum_{r=0}^{n}(sq)^{n-r}\varepsilon_r.$

(II) $\lim_{n\to\infty} Sy_n = u$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0.$

The following example illustrates that the convergence rate of Jungck-Noor iterative scheme is faster than Jungck-Mann iterative scheme.

Example 2.1. Let $T, S : [0,1] \to [0,1]$ be defined by $Tx = \frac{x}{4}, Sx = \frac{x}{2}$ and $\alpha_n = \beta_n = \gamma_n = 0, \; n = 1,2...15$

$\alpha_n = \beta_n = \gamma_n = \frac{4}{\sqrt{n}},\ n \geq 16$. It is clear that T satisfies (1.13) with a unique coincidence point. We see that α_n, β_n and γ_n atisfies all the condition of stability of coincidence point. Here, we show that Jungck-Noor scheme
converges faster than Jungck-Mann iterative scheme to the point $p = 0$.

Proof: Let $n \geq 16$. Then for Jungck-Mann iteration (1.8), we have

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n$$

$$=(1-\frac{4}{\sqrt{n}})\frac{x_n}{2} + \frac{4}{\sqrt{n}}\frac{x_n}{4} = (\frac{1}{2} - \frac{1}{\sqrt{n}})x_n$$

$$x_{n+1} = 2(\frac{1}{2} - \frac{1}{\sqrt{n}})x_n = (1-\frac{2}{\sqrt{n}})x_n.$$

Also, for Jungck-Noor iteration (1.9), we have

$$Sr_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n$$

$$=(1-\frac{4}{\sqrt{n}})\frac{x_n}{2} + \frac{4}{\sqrt{n}}\frac{x_n}{4} = (\frac{1}{2} - \frac{1}{\sqrt{n}})x_n$$

$$r_n = 2(\frac{1}{2} - \frac{1}{\sqrt{n}})x_n = (1-\frac{2}{\sqrt{n}})x_n,$$

$$Sz_n = (1-\beta_n)Sx_n + \beta_n Tr_n$$

$$=(1-\frac{4}{\sqrt{n}})\frac{x_n}{2} + \frac{4}{\sqrt{n}}(1-\frac{2}{\sqrt{n}})\frac{x_n}{4} = (\frac{1}{2} - \frac{2}{n})x_n$$

$$z_n = 2(\frac{1}{2} - \frac{2}{n})x_n = (1-\frac{4}{n})x_n,$$

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tz_n$$

$$=(1-\frac{4}{\sqrt{n}})\frac{x_n}{2} + \frac{4}{\sqrt{n}}(1-\frac{4}{n})\frac{x_n}{4} = (\frac{1}{2} - \frac{1}{\sqrt{n}} - \frac{4}{n\sqrt{n}})x_n,$$

$$x_{n+1} = 2(\frac{1}{2} - \frac{1}{\sqrt{n}} - \frac{4}{n\sqrt{n}})x_n = (1-\frac{2}{\sqrt{n}} - \frac{8}{n\sqrt{n}})x_n,$$

So,

$$\frac{x_{n+1}(JN)}{x_{n+1}(JM)} = \frac{\prod_{i=16}^{n}(1-\frac{2}{\sqrt{n}} - \frac{8}{n\sqrt{n}})x_0}{\prod_{i=16}^{n}(1-\frac{2}{\sqrt{n}})x_0} \leq \prod_{i=16}^{n}\left(1-\frac{\frac{8}{n\sqrt{n}}}{(1-\frac{2}{\sqrt{n}})}\right).$$

It is easy to see that

$$0 \leq \lim_{n\to\infty}\prod_{i=16}^{n}\left(1-\frac{\frac{8}{n\sqrt{n}}}{(1-\frac{2}{\sqrt{n}})}\right) \leq \lim_{n\to\infty}\prod_{i=16}^{n}(1-\frac{1}{i}) = \lim_{n\to\infty}\frac{15}{n} = 0.$$

Hence

$$\lim_{n\to\infty}\left|\frac{x_n(JN)-0}{x_n(JM)-0}\right| = 0.$$

Therefore, by definition 1.4, Jungck-Noor iterative scheme converges faster than Jungck-Mann scheme.

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