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STABILITY OF X_d^*-FRAMES IN BANACH SPACES

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Abstract

X_d-frames with respect to some Banch space of scalar valued sequences associated with a Banach space X, were introduced and studied by Casazza et al [1]. We further study X_d^*-frames in conjugate Banach spaces and a necessary and sufficient condition for the stability of X_d^*-frame have been given. Also, a sufficient condition for a sequence to be X_d^*-frame in terms of X_d^*-Bessel sequences has been obtained.

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Key Words: X_d*-frame,* X_d^**-Bessel sequences,* X_d^**-frame.*

1. INTRODUCTION

Fourier transform has been a major tool in analysis for over a century. It has a lacking for signal analysis in which it hides in its phases information concerning the moment of emission and duration of a signal. In 1946 Dennis Gabor [8], filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals. On the basis of this development, in 1952 the notion of frame was determined by Duffin and Schaeffer [6] in Hilbert spaces in the following way: Let X be a Separable Hilbert space, the system of non-zero elements $\{x_n\}_{n\in\mathbb{N}} \subset X$ be called a frame in X if there exist the constants $0 < A \le B < \infty$ such that for each $x \in X$, it is valid

$$A \| x \|_X^2 \le \sum_{n=1}^{\infty} | (x, x_n) |^2 \le B \| x \|_X^2 \tag{1.1}$$

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where $\| \bullet \|_X$ and $(\bullet, \bullet)$ is a norm and scalar product in X respectively. The constants A and B in (1.1) are called lower and upper frame bounds, the number $K = A/B$ is called condition coefficient of the frame $\{x_n\}_{n\in\mathbb{N}}$. In the case, when $K = 1$, $\{x_n\}_{n\in\mathbb{N}}$ is a tight frame. Development in theory of frames in Hilbert space reduced to obtaining the analogues of the known results for the Banach case. By the theory of frames [3,6,7,9] we have their Banach extensions in [1,2,4 ,5,10]. Frames have many properties of bases but lacks a very important one, namely, uniqueness. This property of frames make them very useful in the study of function spaces, signal and image processing, filter banks, wireless and communications etc.

The notion of X_d-frame generalizing the notion of p-frame studied in [4] was introduced in [1]. In the present paper, We study X_d^*-frames in conjugate Banach spaces . A necessary and sufficient condition and also a sufficient condition for the stability of X_d^*-frame have been given. Also, a sufficient condition for a sequence to be X_d^*-frame in terms of X_d^*-Bessel sequences has been obtained.

2. PRELIMINARIES

Throughout this paper X will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let X^* and X^{**} denote the first and second conjugate space of X respectively, $L(X, X)$ Banach space of all continuous linear mapping from X into X. Let X_d and X_d^* be an associated Banach spaces of scalar valued sequences index by $\mathbb{N}$ of X and X^* respectively. Let $[x_n]$ the closed linear span of $\{x_n\}$. A sequence $\{x_n\} \subset X$ is said to be complete if $[x_n] = X$.

Definition 2.2 ([1]): Let X be a Banach space and X^* be its conjugate space. Let X_d^* be a Banach space of scalar valued sequences associated with X^* and $\{x_i\}_{i=1}^{\infty} \subset X$.

The sequence $\{x_i\}_{i=1}^{\infty}$ is called a X_d^*-frame for X^* with lower bound A and upper bound B if $0 < A \le B < \infty$ and for every $f \in X^*$ one has

$$\{f(x_i)\}_{i=1}^{\infty} \in X_d^*;$$

$$A \| f \|_{X^*} \le \| \{f(x_i)\}_{i=1}^{\infty} \|_{X_d^*} \le B \| f \|_{X^*}. \qquad (2.1)$$

When (i) and the upper inequality in (ii) hold for every $f \in X^*$, $\{x_i\}_{i=1}^{\infty}$ is called a X_d^*-Bessel sequence for X^* with bound B. The positive constants A and B respectively, are called lower and upper frame bounds of the X_d^*-frame. The inequality (2.1) is called the frame inequality. It is easy to observe that frame bounds need not be unique. Further, the X_d^*-frame is called tight frame if it is possible to choose A and B satisfying (2.1) with $A = B$ and normalized tight if $A = B = 1$. If removal of one x_n renders the collection $\{x_n\} \subset X$ no longer a X_d^*-frame for X^*, then $\{x_n\}$ is called can an exact X_d^*-frame.

Let $\{x_i\}_{i=1}^{\infty} \subset X$. The operators U and T given by $Uf = \{f(x_i)\}$, $f \in X^*$, and $T\{d_i\} = \sum_{i=0}^{\infty} d_i x_i$, are called the analysis operator for $\{x_i\}_{i=1}^{\infty}$ and the synthesis operator for $\{x_i\}_{i=1}^{\infty}$ respectively.

3. MAIN RESULTS

Motivated by Jain and Kaushik [11,12], we prove that following result regarding stability of X_d^*-frames.

Theorem 3.2: If $\{x_n\} \subset X$ be a X_d^*-frames for X^* and let $\{y_n\} \subset X$ be such that $\{f(y_n)\} \in X_d^*$, for all $f \in X^*$. Then, $\{y_n\}$ is a X_d^*-frames for X^* if and only if there exist a constant $K > 0$, such that

$$\| \{f(x_n - y_n)\} \|_{X_d^*} \leq K \min\{\| \{f(x_n)\} \|_{X_d^*}, \| \{f(y_n)\} \|_{X_d^*}\}, \; f \in X^*.$$

Proof: Let $\{x_n\}$ and $\{y_n\}$ are X_d^*-frames with frame bounds are A_x, B_x; A_y, B_y, respectively. Then by applying frame inequalities for these frames, we get

$$\| \{f(x_n - y_n)\} \|_{X_d^*} \leq \left(1 + \frac{B_y}{A_x}\right) \| \{f(x_n)\} \|_{X_d^*}, \text{ for all } f \in X^*. \quad (3.1)$$

Similarly, we obtain

$$\| \{f(x_n - y_n)\} \|_{X_d^*} \leq \left(1 + \frac{B_x}{A_y}\right) \| \{f(y_n)\} \|_{X_d^*}, \text{ for all } f \in X^*. \quad (3.2)$$

Choosing $K = \left(1+\frac{B_y}{A_x}\right)$ or $\left(1+\frac{B_x}{A_y}\right)$ according as

$min\{\| \{f(x_n)\} \|_{X_d^*}, \| \{f(y_n)\} \|_{X_d^*}\}$ is $\| \{f(x_n)\} \|_{X_d^*}$ or $\| \{f(y_n)\} \|_{X_d^*}$.

Conversely, let C_x and D_x are bounds for the X_d^*-frame $\{x_n\}$ in X. Then, for all $f \in X^*$, we have

$$\begin{aligned} C_x \| f \|_{X^*} &\leq \| \{f(x_n)\} \|_{X_d^*} \\ &\leq \| \{f(x_n - y_n)\} \|_{X_d^*} + \| \{f(y_n)\} \|_{X_d^*} \\ &\leq (1+K) \| \{f(y_n)\} \|_{X_d^*} \\ &\leq (1+K)(\| \{f(x_n - y_n)\} \|_{X_d^*} + \| \{f(x_n)\} \|_{X_d^*}) . \\ &\leq (1+K)^2 \| \{f(x_n)\} \|_{X_d^*} \\ &\leq (1+K)^2 D_x \| f \|_{X^*} \end{aligned}$$

Therefore, $\frac{C_x}{(1+K)} \| f \|_{X^*} \leq \|\{f(y_n)\}\|_{X_d^*} \leq (1+K) D_x \| f \|_{X^*}$ for all $f \in X^*$.

Hence, $\{y_n\}$ is a X_d^*-frame for X^* with bounds $\frac{C_x}{(1+K)}$ and $D_x(1+K)$.

In the following theorem, we give a sufficient condition for a sequence to be a X_d^*-frame for X.

Theorem 3.3: If $\{x_n\} \subset X$ be a X_d^*-frames for X^* with frame bounds A and B. Let $\{y_n\} \subset X$ be such that $\{f(x_n)\} \in X_d^*$, for all $f \in X^*$. If there exists non-negative constants α, β, γ and δ such that

$$\frac{\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}{(1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}})} < A, \text{ where } \max\{\alpha,\beta,\gamma,\delta\} < 1.$$

$$\left\|\{f(x_n - y_n)\}\right\|^2_{X_d^*} \leq \alpha \left\|\{f(x_n)\}\right\|^2_{X_d^*} + 2\beta \left\|\{f(x_n)\}\right\|_{X_d^*} \left\|\{f(y_n)\}\right\|_{X_d^*} + \gamma \left\|\{f(y_n)\}\right\|^2_{X_d^*}$$

$$+\delta \left\| f \right\|^2_{X^*}, \quad f \in X^*.$$

Then, $\{y_n\}$ is a X_d^* -frame for X^* with frame bounds $\dfrac{A-\sqrt{\max\{\alpha,\beta,\gamma,\delta,\}}(1+A)}{1+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}$ and $\dfrac{B+\sqrt{\max\{\alpha,\beta,\gamma,\delta,\}}(1+B)}{1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}$.

Proof: Let $\zeta = \max\{\alpha,\beta,\gamma,\delta\}$. Then (ii) may be reproduced as:

$$\|\{f(x_n-y_n)\}\|_{X_d^*} \le \sqrt{\zeta}\{\|\{f(x_n)\}\|_{X_d^*} + \|\{f(y_n)\}\|_{X_d^*} + \|f\|_{X^*}\}, \quad f \in X^*. \tag{3.3}$$

Now

$$\begin{aligned}\|\{f(y_n)\}\|_{X_d^*} &\le \|\{f(x_n)\}\|_{X_d^*} + \|\{f(x_n-y_n)\}\|_{X_d^*} \\ &\le \|\{f(x_n)\}\|_{X_d^*} + \sqrt{\zeta}\{\|\{f(x_n)\}\|_{X_d^*} \\ &\quad + \|\{f(y_n)\}\|_{X_d^*} + \|f\|_{X^*}\} \text{ (using (3.3))}\end{aligned}$$

This gives

$$\begin{aligned}\left(1-\sqrt{\zeta}\right)\left\|\{f(y_n)\}\right\|_{X_d^*} &\le \left(1+\sqrt{\zeta}\right)\left\|\{f(x_n)\}\right\|_{X_d^*} + \sqrt{\zeta}\|f\|_{X^*} \\ &\le \left(1+\sqrt{\zeta}\right)B\|f\|_{X^*} + \sqrt{\zeta}\|f\|_{X^*} \\ &\le \left(B+\sqrt{\zeta}(1+B)\right)\|f\|_{X^*}.\end{aligned}$$

Therefore

$$\left\|\{f(y_n)\}\right\|_{X_d^*} \le \left(\frac{B+\sqrt{\zeta}(1+B)}{1-\sqrt{\zeta}}\right)\|f\|_{X^*}, \text{ for all } f \in X^*. \tag{3.4}$$

Also

$$\begin{aligned}\left\|\{f(y_n)\}\right\|_{X_d^*} &= \left\|\{f(x_n)\} - \{f(x_n-y_n)\}\right\|_{X_d^*} \\ &\ge \left\|\{f(x_n)\}\right\|_{X_d^*} - \|\{f(x_n-y_n)\}\|_{X_d^*} \\ &\ge \left\|\{f(x_n)\}\right\|_{X_d^*} - \sqrt{\zeta}\left\{\left\|\{f(x_n)\}\right\|_{X_d^*} + \left\|\{f(y_n)\}\right\|_{X_d^*} + \|f\|_{X^*}\right\},\end{aligned}$$

(using (3.3)).

This gives

$$\left(1+\sqrt{\zeta}\right)\left\|\left\{f\left(y_n\right)\right\}\right\|_{X_d^*} \geq \left(1-\sqrt{\zeta}\right)\left\|\left\{f\left(x_n\right)\right\}\right\|_{X_d^*} - \sqrt{\zeta}\,\|f\|_{X^*}$$

$$\geq \left(1-\sqrt{\zeta}\right)A\|f\|_{X^*} - \sqrt{\zeta}\,\|f\|_{X^*}$$

$$\geq \left(A-\sqrt{\zeta}\left(1+A\right)\right)\|f\|_{X^*}.$$

Therefore

$$\left\|\left\{f\left(y_n\right)\right\}\right\|_{X_d^*} \geq \left(\frac{A-\sqrt{\zeta}\left(1+A\right)}{1+\sqrt{\zeta}}\right)\|f\|_{X^*}, \text{ for all } f \in X^*. \qquad (3.5)$$

By inequality (3.4) and (3.5), we obtain

$$\left(\frac{A-\sqrt{\zeta}\left(1+A\right)}{1+\sqrt{\zeta}}\right)\|f\|_{X^*} \leq \left\|\left\{f\left(y_n\right)\right\}\right\|_{X_d^*}$$

$$\leq \left(\frac{B+\sqrt{\zeta}\left(1+B\right)}{1-\sqrt{\zeta}}\right)\|f\|_{X^*}, \forall f \in X^*.$$

Or

$$\left(\frac{A-\sqrt{\max\left\{\alpha,\beta,\gamma,\delta\right\}}\left(1+A\right)}{1+\sqrt{\max\left\{\alpha,\beta,\gamma,\delta\right\}}}\right)\|f\|_{X^*} \leq \left\|\left\{f\left(y_n\right)\right\}\right\|_{X_d^*}$$

$$\leq \left(\frac{B+\sqrt{\max\left\{\alpha,\beta,\gamma,\delta\right\}}\left(1+B\right)}{1-\sqrt{\max\left\{\alpha,\beta,\gamma,\delta\right\}}}\right)\|f\|_{X^*}, \forall f \in X^*.$$

Hence, $\left\{y_n\right\}$ is a X_d^*-frame for X^* with respect to X_d^* and with the frame bounds $\frac{A-\sqrt{\max\{\alpha,\beta,\gamma,\delta,\}}(1+A)}{1+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}$ and $\frac{B+\sqrt{\max\{\alpha,\beta,\gamma,\delta,\}}(1+B)}{1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}$.

The next two theorems also gives a sufficient condition for a sequence to be a X_d^*-frame for X^*.

Theorem 3.4: Let $\{x_n\} \subset X$ be a X_d^*-frame for X^* with frame bounds A and B and let $\{y_n\} \subset X$ be a X_d^*-Bessel sequence for X^* with bounds $K < A$, then, $\left\{f_n \pm g_n\right\}$ is a X_d^*-frame for X^*.

Proof: Suppose that R_T, R_Q are analysis operators of X_d^*-Bessel sequences $\left\{x_n\right\}, \left\{y_n\right\}$ for X^* respectively. For any $f \in X^*$, we have

$$\left\|\{f(x_n \pm y_n)\}\right\|_{X_d^*} = \left\|R_T(f) + R_Q(f)\right\|_{X_d^*}$$
$$\leq \left\|\{R_T(f)\}\right\|_{X_d^*} + \left\|\{R_Q(f)\}\right\|_{X_d^*}$$
$$\leq (B+K)\|f\|_{X^*}\ ,\ f \in X^*.$$

Thus, $\{x_n \pm y_n\}$ is Bessel sequence for X^*. We also have

$$\left\|\{f(x_n \pm y_n)\}\right\|_{X_d^*} = \left\|R_T(f) + R_Q(f)\right\|_{X_d^*}$$
$$\geq \left\|\{R_T(f)\}\right\|_{X_d^*} - \left\|\{R_Q(f)\}\right\|_{X_d^*}$$
$$\geq (A-K)\|f\|_{X^*}\ ,\ f \in X^*.$$

Hence, $\{x_n \pm y_n\}$ is a X_d^*-frame for X^* with respect to X_d^*.

Theorem 3.5: Let $\{x_n\} \subset X$ be a X_d^*-frame for X^* with frame bounds A and B.

Let $\{y_n\} \subset X$ be such that $\{f(y_n)\} \in X_d^*$, for all $f \in X^*$ and let $\{x_n + y_n\}$ be a X_d^*-Bessel sequence for X^* and with bounds $K < A$. Then $\{y_n\}$ is a X_d^*-frame for X^* with bounds $A-K$ and $B+K$.

Proof: In the light of the frame inequality for the X_d^*-frame $\{x_n\}$ and the fact that K is a X_d^*-Bessel bounds for the X_d^*-Bessel sequence $\{x_n + y_n\}$, we have

$$(A-K)\| f \|_{X^*} \leq \| \{f(x_n)\} \|_{X_d^*} - \| \{f(x_n + y_n)\} \|_{X_d^*}$$
$$\leq \| \{f(y_n)\} \|_{X_d^*}$$
$$\leq \| \{f(x_n)\} \|_{X_d^*} + \| \{f(x_n + y_n)\} \|_{X_d^*}$$
$$\leq (B+K)\| f \|_{X^*},\ \ f \in X^*$$

Hence , $\{y_n\}$ is a X_d^*-frame for X^* with the required frame bounds $A-K$ and $B+K$.

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