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ON COMPLEX VALUED METRIC SPACES

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Abstract

In this paper we study the complex valued metric space as a topological space and obtain some topological properties of this space. Here we prove that every complex valued metric space is first countable. We also prove that every complex valued metric space is T_3.

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1. INTRODUCTION AND PRELIMINARIES

We know, there are many generalizations of ordinary metric space like Semi metric space, Symmetric space, Cone metric space, Fuzzy metric space, Rectangular metric space etc. One such generalization is Complex valued metric space which was introduced by A. Azam, F. Brain and M. Khan [2]. Since then, many research articles ([1], [3], [4], [5], [8], [9]) have been published on this space. In every work, we see only fixed point results and its applications, but here we study this space as a topological space. Here we show that any complex valued metric space is a topological space under the topology induced by the complex valued metric. Now it is a natural query that how much the complex valued metric space is rich topologically like ordinary metric space. Like ordinary metric space, complex valued metric space has also many topological properties.

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

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(3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$ and

(4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of (2), (3) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1: We can easily check the followings:

(i) $a, b \in \mathbb{R}, a \le b \Rightarrow az \precsim bz, \forall z \in \mathbb{C}$.

(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

One can easily verify the following lemmas:

Lemma 1: The relation $\precsim$ is compatible with respect to addition i.e., $z_1 \precsim z_2 \Rightarrow z_1 + z \precsim z_2 + z, \forall z_1, z_2, z \in \mathbb{C}$.

Lemma 2: If $z_1, z_2 \in \mathbb{C}$ then $z_1 \prec z_2 \Rightarrow z_1 + z \prec z_2 + z, \forall z \in \mathbb{C}$.

Lemma 3: If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ then $z_1 \precsim z_2, z_3 \precsim z_4 \Rightarrow z_1 + z_3 \precsim z_2 + z_4$.

Lemma 4: If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ then $z_1 \precsim z_2, z_3 \prec z_4 \Rightarrow z_1 + z_3 \prec z_2 + z_4$.

Lemma 5: If $z_1, z_2, z_3 \in \mathbb{C}$ then $z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$ [(iii) of Remark 1].

Lemma 6: If $z_1, z_2, z_3 \in \mathbb{C}$ then $z_1 \prec z_2, z_2 \precsim z_3 \Rightarrow z_1 \prec z_3$.

The following lemma, similar as in real number system with ordinary order relation, holds also.

Lemma 7: If $a, b \in \mathbb{C}$ such that $a \prec b + \varepsilon$, for any $\varepsilon \in \mathbb{C}, 0 \prec \varepsilon$ then $a \precsim b$.

Proof: Let $\varepsilon > 0$ be given. Put $\varepsilon^* = \varepsilon + i\varepsilon$. Then $\varepsilon^* \in \mathbb{C}$ and $0 \prec \varepsilon^*$.

Thus by given condition

$$a \prec b + \varepsilon^*$$

$$\Rightarrow \qquad \operatorname{Re}(a) < \operatorname{Re}(b) + \varepsilon \text{ and } \operatorname{Im}(a) < \operatorname{Im}(b) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\operatorname{Re}(a) \le \operatorname{Re}(b) \text{ and } \operatorname{Im}(a) \le \operatorname{Im}(b)$$

Hence $a \precsim b$. ■

Azam et. al. [2] defined the complex valued metric space in the following way:

Definition 1 ([2]): Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

(C1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(C3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1: Define the mapping $d : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ by

$$d(x, y) = i\,|\,x - y\,|, \ \forall x, y \in \mathbb{R}.$$

One can easily verify that $(\mathbb{R}, d)$ is a complex valued metric space.

Definition 2: Let (X, d) be a complex valued metric space and $0 \prec r \in \mathbb{C}$, $a \in X$. Then we define the open ball B(a, r) as

$$B(a, r) = \{y \in X : d(a, y) \prec r\}.$$

Definition 3: Let (X, d) be a complex valued metric space and $0 \prec r \in \mathbb{C}$, $a \in X$. Then we define the closed ball B[a, r] as

$$B(a, r) = \{y \in X : d(a, y) \precsim r\}.$$

Now we give a topology on a complex valued metric space (X, d) induced by the complex valued metric d.

Definition 4: Let (X, d) be a complex valued metric space. We define

$$\tau = \{\varnothing\} \cup \{V \subset X : \forall a \in V, \exists\, r \in \mathbb{C}, 0 \prec r \text{ such that } B(a, r) \subset V\}.$$

Then τ is a topology on X.

Verification: Clearly $\varnothing, X \in \tau$. Let $\{V_i : i \in I\} \subset \tau$. Put $V = \cup_{i \in I} V_i$. Let $a \in V$, then $a \in V_i$ for some $i \in I$. Thus there exits $r \in \mathbb{C}$, $0 \prec r$ such that $B(a, r) \subset V_i \subset V$ and hence $V \in \tau$.

Again let $V_1, V_2 \in \tau$. Let $a \in V_1 \cap V_2$, this implies $a \in V_1$, $a \in V_2$. Thus there exist $r_1, r_2 \in \mathbb{C}$, $0 \prec r_1$, $0 \prec r_2$ such that $B(a, r_1) \subset V_1$, $B(a, r_2) \subset V_2$. Let $0 \prec r$ such that $\mathrm{Re}(r) < \min\{\mathrm{Re}(r_1), \mathrm{Re}(r_2)\}$ and $\mathrm{Im}(r) < \min\{\mathrm{Im}(r_1), \mathrm{Im}(r_2)\}$. Then clearly $r \prec r_1$ and $r \prec r_2$. Thus $B(a, r) \subset B(a, r_1) \subset V_1$ and $B(a, r) \subset B(a, r_2) \subset V_2$. Hence $B(a, r) \subset V_1 \cap V_2$. Thus $V_1 \cap V_2 \in \tau$.

Hence the complex valued metric space (X, d) can be considered as the topological space (X, τ).

2. MAIN RESULTS

In this section we present the main results of this paper.

First we show that in a complex valued metric space any open ball is an open set.

Theorem 1: Let (X, d) be a complex valued metric space, $a \in X$ and $0 \prec r \in \mathbb{C}$. Then the open ball $B(a, r)$ is an open set in X.

Proof: Let $x \in B(a, r)$. Then $0 \precsim d(a, x) \prec r$. Put $r - d(a, x) = r_1$. Then $0 \prec r_1$. Let $z \in B(x, r_1) \Rightarrow d(x, z) \prec r_1$. Thus $d(a, z) \precsim d(a, x) + d(x, z) \prec d(a, x) + r_1 = r$. Thus $z \in B(a, r)$. Hence $B(x, r_1) \subset B(a, r)$. Thus $B(a, r)$ is open in X. ■

Now it is easy to prove the following result.

Theorem 2: Let (X, d) be a complex valued metric space. Then

$$\mathcal{B} = \{B(x, r) : x \in X, 0 \prec r \in \mathbb{C}\}$$

is a basis for the topology τ of X.

Theorem 3: Every complex valued metric space is T_2.

Proof: Let (X, d) be a complex valued metric space. Let $x, y \in X$, $x \neq y$. Then $0 \precsim d(x, y)$, $d(x, y) \neq 0$. Thus $|d(x, y)| = r > 0$.

Let $0 \prec c \in \mathbb{C}$ such that $|c| = r/2$. Clearly $x \in B(x, c)$, $y \in B(y, c)$.

We claim that $B(x, c) \cap B(y, c) = \varnothing$. If not, let $z \in B(x, c) \cap B(y, c)$.

Thus $d(x, z) \prec c$, $d(y, z) \prec c$. Hence $d(x, y) \precsim d(x, z) + d(z, y) \prec 2c$. Thus $|d(x, y)| < 2|c| = r = |d(x, y)|$ and we get a contradiction.

Hence $B(x, c) \cap B(y, c) = \varnothing$. This completes the proof. ■

Theorem 4: Every complex valued metric space is first countable.

Proof: Let (X, d) be a complex valued metric space.

Let $x \in X$.

Put

$$G = \left\{ B\left(x, \frac{1}{n} + \frac{i}{n} \right) : n \in \mathbb{N} \right\}.$$

Then clearly $G \subset \mathcal{N}_x$.

Let $N_x \in \mathcal{N}_x$.

Then $\exists\, c \in \mathbb{C}$, $0 \prec c$ such that

$$B(x, c) \subset N_x.$$

Let $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \min\{\text{Re}(c), \text{Im}(c)\}.$$

Thus

$$B\left(x, \frac{1}{N} + \frac{i}{N}\right) \subset B(x, c) \subset N_x.$$

Hence X is first countable. ■

Since any complex valued metric space is first countable we have the following two important results.

Theorem 5: Let (X, d) be a complex valued metric space and $A \subset X$. Then $a \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ in A such that $x_n \to a$ as $n \to \infty$.

Theorem 6: Let (X, d), (X, σ) be two complex valued metric spaces and $f : X \to Y$ be a function. Then f is continuous at $a \in X$ if and only if for every sequence $\{x_n\}$ in X, $f(x_n) \to f(a)$ as $n \to \infty$ whenever $x_n \to a$ as $n \to \infty$.

The proofs of the above two theorems are omitted because for first countable topological spaces they are available in any standard book of general topology, see [6] or [7].

We know every second countable topological space is separable so every second countable complex valued metric space is also separable. In general, the converse is not true for topological spaces, but the converse is true for metric spaces. Is the converse true for complex valued metric spaces? Here we show this in affirmative.

Theorem 7: Every separable complex valued metric space is second countable.

Proof: Let (X, d) be a complex valued metric space.

Then X has a countable dense subset A (say).

Let

$$G = \{B(a, c) : a \in A, c = r + ir \in \mathbb{C}, r \in \mathbb{Q}^+\}.$$

Clearly G is a countable collection of open sets.

Let V be an open set in X and $x \in V$.

Thus there exists $c \in C$, $0 \prec c$ such that

$$B(x, c) \subset V.$$

Choose $r \in \mathbb{Q}^+$ such that

$$r < \min\{\text{Re}(c), \text{Im}(c)\}.$$

Put $c^* = r + ir$.

Then $0 \prec c^* \in \mathbb{C}$ and $c^* \prec c$.

Thus

$$B(x, c^*) \subset B(x, c) \subset V.$$

Since $\overline{A} = X$,

$$B\left(x, \frac{c^*}{2}\right) \cap A \neq \varnothing.$$

Let

$$a \in B\left(x, \frac{c^*}{2}\right) \cap A$$

$\Rightarrow$ $$d(x, a) \prec \frac{c^*}{2}.$$

Thus

$$x \in B\left(a, \frac{c^*}{2}\right) \subset B\left(x, c^*\right) \subset B\left(x, c\right) \subset V$$

Also $B\left(a, \frac{c^*}{2}\right) \in G$

Thus G is a countable base for X.

Hence X is second countable. ■

Now we show that every complex valued metric space is regular, to prove this we need the following theorem.

Theorem 8: Let (X, d) be a complex valued metric space, $a \in X$ and $0 \prec r \in \mathbb{C}$. Then the closed ball B[a, r] is a closed set in X.

Proof: Put $V = X - B[a, r]$.

Let $x \in V$.

Then

$$x \notin B[a, r]$$

$\Rightarrow \qquad \mathrm{Re}(d(x, a)) > \mathrm{Re}(r) \text{ or } \mathrm{Im}(d(x, a)) > \mathrm{Im}(r).$

Case 1: $\mathrm{Re}(d(x, a)) > \mathrm{Re}(r)$

Put

$$\mathrm{R} = d(x, a) - r + i\,|\,d(x, a) - r\,|.$$

Thus $\mathrm{Re}(\mathrm{R}) > 0$ and

$$\mathrm{Im}(\mathrm{R}) = |\,d(x, a) - r\,| + \mathrm{Im}(d(x, a) - r) > 0.$$

Thus $0 \prec \mathrm{R}$.

Let

$y \in \mathrm{B}(x, \mathrm{R})$

$\Rightarrow \; d(x, y) \prec \mathrm{R} = d(x, a) - r + i\,|\,d(x, a) - r\,|$

$\Rightarrow \qquad r \prec d(x, a) - d(x, y) + i\,|\,d(x, a) - r\,| \precsim d(a, y) + i\,|\,d(x, a) - r\,|$

$\Rightarrow \qquad r \prec d(a, y) + i\,|\,d(x, a) - r\,|.$

Thus

$$\mathrm{Re}(d(a, y)) > \mathrm{Re}(r)$$

$\Rightarrow \qquad y \notin \mathrm{B}[a, r].$

Hence

$$\mathrm{B}(x, \mathrm{R}) \subset \mathrm{X} - \mathrm{B}[a, r].$$

Thus V is open and hence $\mathrm{B}[a, r]$ is closed.

Case 2: $\mathrm{Im}(d(x, a)) > \mathrm{Im}(r)$

Put

$$\mathrm{R} = d(x, a) - r + |\,d(x, a) - r\,|.$$

Then $\mathrm{Im}(\mathrm{R}) > 0$ and

$$\mathrm{Re}(\mathrm{R}) = |\,d(x, a) - r\,| + \mathrm{Re}(d(x, a) - r) > 0.$$

Thus $0 \prec \mathrm{R}$.

Let

$y \in \mathrm{B}(x, \mathrm{R})$

$\Rightarrow \; d(x, y) \prec \mathrm{R} = d(x, a) - r + |\,d(x, a) - r\,|$

$\Rightarrow \qquad r \prec d(x, a) - d(x, y) + |\, d(x, a) - r \,| \precsim d(a, y) + |\, d(x, a) - r \,|$

$\Rightarrow \qquad r \prec d(a, y) + |\, d(x, a) - r \,|.$

Thus

$$\mathrm{Im}(d(a, y)) > \mathrm{Im}(r)$$

$$\Rightarrow \qquad y \notin \mathrm{B}[a, r].$$

Hence

$$\mathrm{B}(x, \mathrm{R}) \subset \mathrm{X} - \mathrm{B}[a, r].$$

Thus V is open and hence B[a, r] is closed. ■

Theorem 9: Every complex valued metric space is regular.

Proof: Let (X, d) be a complex valued metric space.

Let $x \in \mathrm{X}$.

Let V be any open set such that $x \in \mathrm{V}$.

Then there exists $r \in \mathbb{C}$, $0 \prec r$ such that

$$\mathrm{B}(x, r) \subset \mathrm{V}.$$

Thus

$$x \in \mathrm{B}\left(x, \frac{r}{2}\right) \subset \mathrm{B}(x, r) \subset \mathrm{V}.$$

Since $\mathrm{B}\left[x, \frac{r}{2}\right]$ is a closed set we have

$$x \in \overline{\mathrm{B}\left(x, \frac{r}{2}\right)} \subset \mathrm{B}\left[x, \frac{r}{2}\right] \subset \mathrm{B}(x, r) \subset \mathrm{V}.$$

Thus

$$x \in \mathrm{B}\left(x, \frac{r}{2}\right) \subset \overline{\mathrm{B}\left(x, \frac{r}{2}\right)} \subset \mathrm{V}.$$

Hence X is regular. ■

Since every complex valued metric space is T_2 and hence T_1 we have the following corollary.

Corollary 1: Every complex valued metric space is T_3.

References

[1] S. Ali: Some Common Fixed Point Theorems For Two Weakly Compatible Mappings in Complex Valued Metric Spaces, Thai Journal of Mathematics (Accepted, Article in Press).

[2] A. Azam, F. Brain and M. Khan: Common fixed point theorems in complex value metric spaces, Numer. Funct. Anal. Optim., Vol. 32, No. 3 (2011), pp. 243-253.

[3] S. Bhatt, S. Chaukiyal and R. C. Dimri: A common fixed point theorem for weakly compatible maps in complex valued metric space, Int. J. Math. Sci. Appl., Vol. 1, No. 3(September 2011), pp. 1385-1389.

[4] B. S. Choudhury, N. Metiya and P. Maity: Best Proximity Point Results in Complex Valued Metric Spaces, International Journal of Analysis, Vol. 2014, Article ID 827862.

[5] B. S. Choudhury, N. Metiya and P. Konar: Fixed Point Results in Partially ordered Complex Valued Metric Spaces For Rational Type Expressions, Bangmod Int. J. Math. & Comp. Sci. Vol. 1, No. 1, 2015: Pages 54 - 62.

[6] John L. Kelley: General Topology, Springer, 2012.

[7] James R. Munkres: Topology, Pearson, 2013.

[8] W. Sintunavarat and P. Kumam: Generalized common fixed point theorems in complex valued metric spaces and applications, Journal of Inequalities and Applications, Vol. 84(2012), doi:10.1186/1029-242X-2012-84.

[9] R. K. Verma and H. K. Pathak: Common fixed point theorems using property (E.A) in complex valued metric spaces, Thai Journal of Mathematics, Vol. 11, No 2 (2013), pp. 275-283.