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DOUBLE-SOLITON SOLUTIONS AND PERIODIC SOLUTIONS OF BOITI-LEON-MANNA-PEMPINELLI EQUATION

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Abstract: In this paper, by using the bilinear method of Hirota and the Cole-Hopf transformation method, Boiti-Leon-Manna-Pempinelli equation is studied. Double-soliton solutions and periodic solutions of Boiti-Leon-Manna-Pempinelli equation are obtained. The dynamic properties of double-soliton solutions and periodic solutions are discussed. By using software Maple, the graphs of some double-soliton solutions and periodic solutions are drawn.

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1. INTRODUCTION

In recent years, there has been an increasing interest in the study of the exact solutions of nonlinear equations, which can be used to simulate many phenomena in physics, chemistry, and biology. The bilinear method developed by Hirota is a powerful tool and direct approach to construct exact solution of nonlinear equations. By applying the bilinear method, people obtained a series of multi-soliton solutions of many nonlinear equations, see references [1-8]. In this paper, by using the bilinear method of Hirota and the Cole-Hopf transformation method, we will study a (2 + 1)-dimensional model equation for shallow water waves, that is, the following (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation [9-11]

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} = 0. \tag{1}$$

This equation can be reduced to the potential KdV equation for $y = x$. In 1994, Clarkson and Mansfield [9] studied this (2 + 1)-dimensional shallow water wave equation. Later, Wazwaz studied Multiple soliton solutions and multiple-singular soliton solutions for Eq. (1) in [10]. Recently, Luo investigated Quasi-Periodic Wave solutions of Eq. (1) in [11]. Different from Refs. [9-11], we will use a new auxiliary function to investigate the two-soliton solutions of Eq. (1) based on the bilinear method of Hirota and the Cole-Hopf transformation method.

2. BILINEAR FORM OF BOITI-LEON-MANNA-PEMPINELLI EQUATION

In order to obtain more new two-soliton solutions of Eq. (1), we need to introduce bilinear form and new auxiliary function. Suppose that Eq. (1) has a bilinear transformation

$$u = M(\ln f)_x, \tag{2}$$

this is Hope-Cole transformation, where $f = f(x, y, t)$ is a auxiliary function and we will know $M = -2$ in the below. When $M = -2$, substituting the Hope-Cole transformation (2) into the Eq. (1), the Eq. (1) can be rewritten as bilinear form as follows:

$$(D_y D_t + D_y D_x^3) f \cdot f = 0, \tag{3}$$

where the bilinear arithmetic operators can be defined as

$$D_x^m D_y^n D_t^k f(x, y, t) \cdot g(x, y, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial yx} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k f(x, y, t) \cdot g(x', y', t')\Big|_{x = x',\, y = y',\, t = t'}. \tag{4}$$

Suppose that the auxiliary function can be written as

$$f(x, y, t) = a_{-2} e^{-\xi_2} + a_{-1} e^{-\xi_1} + a_1 e^{\xi_1} + a_2 e^{\xi_2} + \mathrm{A} e^{\xi_1 + \xi_2}, \tag{5}$$

where $\xi_i = k_i x + v_i y - w_i t$ and $k_i, v_i, w_i, a_i, A, (i = 1, 2)$ are constants to be determined further. Especially, when $A = a_{-1} = a_2 = 0$, $a_{-2} = a_1 = 1$, $k_2 = v_2 = w_2 = 0$ (i.e., $f(x, y, t) = 1 + e^{\xi_1}$), from Eqs. (2) and (1), it is easy to know that

$$M = -2, \quad w_1 = k_1^3. \tag{6}$$

For convenience, without loss generally we suppose that $M = -2$, $w_i = k_i^3$ in Eq. (2). Thus, the Cole-Hopf transformation of Eq. (1) can be written as

$$u = -2(\ln f)_x, \tag{7}$$

where $w_i = k_i^3$, that is, $\xi_i = k_i x + v_i y - k_i^3 t$.

3. TWO-SOLITON SOLUTIONS OF (2 + 1)-DIMENSIONAL MODEL EQUATION FOR SHALLOW WATER WAVES

Substituting (5) into (3), we let all the coefficients of the polynomials e^{ξ_i} as zero yields

$$-21Aa_{-1}a_{-2}k_1^3k_2v_1 - 15Aa_{-1}a_{-2}k_1^3k_2v_2 - 36Aa_{-1}a_{-2}k_1^2k_2^2v_1 - 36Aa_{-1}a_{-2}k_1^2k_2^2v_2 - 15Aa_{-1}a_{-2}k_1k_2^3v_1 - 21Aa_{-1}a_{-2}k_1k_2^3v_2 = 0,$$

$$-8a_{-1}^2 a_1 k_1^4 v_1 + 2a_{-1} a_{-2} a_2 k_1^3 k_2 v_2 - 4a_{-1} a_{-2} a_2 k_1^2 k_2^2 v_1 - 14a_{-1} a_{-2} a_2 k_1 k_2^3 v_2 = 0,$$

$$8a_{-1} a_1^2 k_1^4 v_1 - 2a_{-2} a_1 a_2 k_1^3 k_2 v_2 + 4a_{-2} a_1 a_2 k_1^2 k_2^2 v_1 + 14a_{-2} a_1 a_2 k_1 k_2^3 v_2 = 0,$$

$$3Aa_{-2} a_1 k_1^3 k_2 v_1 + 5Aa_{-2} a_1 k_1^3 k_2 v_2 + 8Aa_{-2} a_1 k_1^2 k_2^2 v_1 + 12Aa_{-2} a_1 k_1^2 k_2^2 v_2 + 5Aa_{-2} a_1 k_1 k_2^3 v_1 + 7Aa_{-2} a_1 k_1 k_2^3 v_2 = 0,$$

$$-14a_{-1} a_{-2} a_1 k_1^3 k_2 v_1 - 4a_{-1} a_{-2} a_1 k_1^2 k_2^2 v_2 + 2a_{-1} a_{-2} a_1 k_1 k_2^3 v_1 - 8a_{-2}^2 a_2 k_2^4 v_2 = 0,$$

$$14a_{-1} a_1 a_2 k_1^3 k_2 v_1 + 4a_{-1} a_1 a_2 k_1^2 k_2^2 v_2 - 2a_{-1} a_1 a_2 k_1 k_2^3 v_1 + 8a_{-2} a_2^2 k_2^4 v_2 = 0,$$

$$7Aa_{-1} a_2 k_1^3 k_2 v_1 + 5Aa_{-1} a_2 k_1^3 k_2 v_2 + 12Aa_{-1} a_2 k_1^2 k_2^2 v_1 + 8Aa_{-1} a_2 k_1^2 k_2^2 v_2 + 5Aa_{-1} a_2 k_1 k_2^3 v_1 + 3Aa_{-1} a_2 k_1 k_2^3 v_2 = 0,$$

$$a_{-1}^2 a_{-2} k_1^3 k_2 v_1 - a_{-1}^2 a_{-2} k_1^3 k_2 v_2 - 2a_{-1}^2 a_{-2} k_1^2 k_2^2 v_1 + 2a_{-1}^2 a_{-2} k_1^2 k_2^2 v_2 + a_{-1}^2 a_{-2} k_1 k_2^3 v_1 - a_{-1}^2 a_{-2} k_1 k_2^3 v_2 = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$a_{-1} a_2^2 k_1^3 k_2 v_1 + a_{-1} a_2^2 k_1^3 k_2 v_2 + 2a_{-1} a_2^2 k_1^2 k_2^2 v_1 + 2a_{-1} a_2^2 k_1^2 k_2^2 v_2 + a_{-1} a_2^2 k_1 k_2^3 v_1 + a_{-1} a_2^2 k_1 k_2^3 v_2 = 0.$$

By using software Maple, we solve the above group of equations, thus the twenty kinds of parametric conditions are obtained. Corresponding to every group of parametric conditions, we can obtain different kinds of exact solutions, such as double-soliton solutions, smooth and non-smooth periodic solutions which are shown the below:

Case 1: Under the parametric conditions $A = A$, $a_{-1} = 0$, $a_{-2} = 0$, $a_1 = 0$, $a_2 = a_2$, $k_1 = k_1$, $v_1 = v_1$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_1(x, y, t) = a_2 e^{k_2 x + v_2 y - k_2^3 t} + Ae^{(k_1 + k_2)x + (v_1 + v_2)y - (k_1^3 + k_2^3)t}. \tag{8}$$

Substituting Eq. (8) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_1 = \frac{-2a_2 k_2 e^{k_2 x + v_2 y - k_2^3 t} + A(k_1 + k_2) e^{(k_1 + k_2)x + (v_1 + v_2)y - (k_1^3 + k_2^3)t}}{a_2 e^{k_2 x + v_2 y - k_2^3 t} + Ae^{(k_1 + k_2)x + (v_1 + v_2)y - (k_1^3 + k_2^3)t}}, \tag{9}$$

where A, a_2, k_1, k_2, v_1, v_2 are arbitrary constants.

Case 2: Under the parametric conditions $A = A$, $a_{-1} = a_{-1}$, $a_{-2} = a_{-2}$, $a_1 = 0$, $a_2 = 0$, $k_1 = -k_2$, $k_2 = k_2$, $v_1 = v_2$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_2(x, y, t) = a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_{-1} e^{k_2 x - v_2 y - k_2^3 t} + Ae^{2v_2 y}. \tag{10}$$

Substituting Eq. (10) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_2 = \frac{2k_2(a_{-2}e^{-k_2x - v_2y + k_2^3t} - a_{-1}e^{k_2x - v_2y - k_2^3t})}{a_{-2}e^{-k_2x - v_2y + k_2^3t} + a_{-1}e^{k_2x - v_2y - k_2^3t} + Ae^{2v_2y}}, \tag{11}$$

where A, a_{-1}, a_{-2}, k_2, v_2 are arbitrary constants. If $a_{-2} = -a_{-1}$, then $f_2(x, y, t)$ becomes

$$\tilde{f}_2(x, y, t) = -2a_{-1}\sinh(k_2(-x + k_2^2t)) + Ae^{2v_2y}. \tag{12}$$

Substituting Eq. (12) into the Cole-Hopf transformation (7), we obtain a soliton-like solution as follows:

$$\tilde{u}_2 = \frac{4a_{-1}k_2\cosh(k_2(-x + k_2^2t))}{2a_{-1}\sinh(k_2(-x + k_2^2t)) - Ae^{2v_2y}}. \tag{13}$$

Especially, when $k_2 = K_2 i$, $a_{-1} = A_{-1} i$, $i = \sqrt{-1}$, the $\tilde{u}_2$ becomes a periodic solution

$$\tilde{\tilde{u}}_2 = \frac{4A_{-1}K_2\cos(K_2(x + K_2^2t))}{-2a_{-1}\sin(K_2(x + K_2^2t)) - Ae^{2v_2y}}, \tag{14}$$

where A, A_{-1}, K_2, v_2 are arbitrary constants.

Case 3: Under the parametric conditions $A = A$, $a_{-1} = 0$, $a_{-2} = a_{-2}$, $a_1 = 0$, $a_2 = a_2$, $k_1 = -k_2$, $k_2 = k_2$, $v_1 = v_1$, $v_2 = 0$, the auxiliary function (5) becomes

$$f_3(x, y, t) = a_{-2}e^{-k_2x + k_2^3t} + a_2e^{k_2x - k_2^3t} + Ae^{v_1y}. \tag{15}$$

Substituting Eq. (15) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_3 = \frac{2k_2(a_{-2}e^{-k_2x + k_2^3t} - a_2e^{k_2x - k_2^3t})}{a_{-2}e^{-k_2x + k_2^3t} + a_2e^{k_2x - k_2^3t} + Ae^{v_1y}}, \tag{16}$$

where A, a_{-2}, a_2, k_2, v_1 are arbitrary constants. If $a_{-2} = a_2$, then $f_3(x, y, t)$ becomes

$$\tilde{f}_3(x, y, t) = 2a_2\cosh(k_2(-x + k_2^2t)) + Ae^{v_1y}. \tag{17}$$

Substituting Eq. (17) into the Cole-Hopf transformation (7), we obtain a soliton-like solution as follows:

$$\tilde{u}_3 = \frac{4a_2 k_2 \sinh(k_2(-x + k_2^2 t))}{2a_2 \cosh(k_2(-x + k_2^2 t)) + Ae^{v_1 y}}. \tag{18}$$

Especially,When $k_2 = K_2 i,\ i = \sqrt{-1}$, $\tilde{u}_3$ becomes a periodic solution

$$\tilde{\tilde{u}}_3 = \frac{4a_2 K_2 \sin(K_2(x + K_2^2 t))}{2a_2 \cos(K_2(x + K_2^2 t)) - Ae^{v_1 y}}, \tag{19}$$

where A, a_2, K_2, v_1 are arbitrary constants.

Case 4: Under the parametric conditions $A = A$, $a_{-1} = a_{-1}$, $a_{-2} = 0$, $a_1 = 0$, $a_2 = 0$, $k_1 = k_1$, $k_2 = k_2$, $v_1 = \frac{-v_2}{2}$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_4(x, y, t) = a_{-1} e^{-k_1 x + \frac{v_2 y}{2} + k_1^3 t} + Ae^{(k_1 + k_2)x + \frac{v_2 y}{2} - (k_1^3 + k_2^3)t}. \tag{20}$$

Substituting Eq. (20) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_4 = \frac{-2(-a_{-1} k_1 e^{-k_1 x + \frac{v_2 y}{2} + k_1^3 t} + A(k_1 + k_2) e^{(k_1 + k_2)x + \frac{v_2 y}{2} - (k_1^3 + k_2^3)t}}{a_{-1} e^{-k_1 x + \frac{v_2 y}{2} + k_1^3 t} + Ae^{(k_1 + k_2)x + \frac{v_2 y}{2} - (k_1^3 + k_2^3)t}}, \tag{21}$$

where A, a_{-1}, k_1, k_2, v_2 are arbitrary constants.

Case 5: Under the parametric conditions $A = A$, $a_{-1} = a_{-1}$, $a_{-2} = 0$, $a_1 = 0$, $a_2 = a_2$, $k_1 = -k_2$, $k_2 = k_2$, $v_1 = v_1$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_5(x, y, t) = a_{-1} e^{k_2 x - v_1 y - k_2^3 t} + a_2 e^{k_2 x + v_2 y - k_2^3)t} + Ae^{(v_1 + v_2)y}. \tag{22}$$

Substituting Eq. (22) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_5 = \frac{-2k_2(a_{-1} e^{k_2 x - v_1 y - k_2^3 t} + a_2 e^{k_2 x + v_2 y - k_2^3 t})}{a_{-1} e^{k_2 x - v_1 y - k_2^3 t} + a_2 e^{k_2 x + v_2 y - k_2^3 t} + Ar^{(v_1 + v_2)y}}, \tag{23}$$

where A, a_{-1}, a_2, k_2, v_1, v_2 are arbitrary constants.

Case 6: Under the parametric conditions $A = A$, $a_{-1} = 0$, $a_{-2} = a_{-2}$, $a_1 = 0$, $a_2 = 0$, $k_1 = k_1$, $k_2 = k_2$, $v_1 = -2v_2$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_6(x, y, t) = a_{-2} e^{-k_2 x + v_2 y + k_2^3 t} + Ae^{(k_1 + k_2)x + v_2 y - (k_1^3 + k_2^3)t}. \tag{24}$$

Substituting Eq. (24) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_6 = \frac{-2(-a_{-2}k_2 e^{-k_2 x - v_2 y + k_2^3 t} + A(k_1 + k_2)e^{(k_1+k_2)x - v_2 y - (k_1^3 + k_2^3)t}}{a_{-2}e^{-k_2 x - v_2 y + k_2^3 t} + Ae^{(k_1+k_2)x - v_2 y - (k_1^3+k_2^3)t}}, \tag{25}$$

where A, a_{-2}, k_1, k_2, v_2 are arbitrary constants.

Case 7: Under the parametric conditions $A = A$, $a_{-1} = a_{-1}$, $a_{-2} = a_{-2}$, $a_1 = 0$, $a_2 = a_2$, $k_1 = k_1$, $k_2 = 0$, $v_1 = \frac{-v_2}{2}$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_7(x, y, t) = a_{-2}e^{-v_2 y} + a_{-1}e^{-k_1 x + \frac{v_2 y}{2} + k_1^3 t} + Ae^{k_1 x + \frac{v_2 y}{2} - k_1^3 t}. \tag{26}$$

Substituting Eq. (26) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_7 = \frac{2k_1(a_{-1}e^{-k_1 x + \frac{v_2 y}{2} + k_1^3 t} - Ae^{k_1 x + \frac{v_2 y}{2} - k_1^3 t}}{a_{-2}e^{-v_2 y} + a_{-1}e^{-k_1 x + \frac{v_2 y}{2} + k_1^3 t} + Ae^{k_1 x + \frac{v_2 y}{2} - k_1^3 t}}, \tag{27}$$

where A, a_{-1}, a_{-2}, a_2, k_1, v_2 are arbitrary constants.

Case 8: Under the parametric conditions $A = A$, $a_{-1} = 0$, $a_{-2} = 0$, $a_1 = a_1$, $a_2 = a_2$, $k_1 = k_1$, $k_2 = k_2$, $v_1 = v_2$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_8(x, y, t) = a_1 e^{k_1 x + v_2 y - k_1^3 t} + a_2 e^{k_2 x + v_2 y - k_2^3 t} + Ae^{(k_1+k_2)x + 2v_2 y - (k_1^3 + k_2^3)t}. \tag{28}$$

Substituting Eq. (28) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_8 = \frac{-2(a_1 k_1 e^{k_1 x + v_2 y - k_1^3 t} + a_2 k_2 e^{k_2 x + v_2 y - k_2^3 t} + A(k_1 + k_2)e^{(k_1+k_2)x + 2v_2 y - (k_1^3+k_2^3)t}}{a_1 e^{k_1 x + v_2 y - k_1^3 t} + a_2 e^{k_2 x + v_2 y - k_2^3 t} + Ae^{(k_1+k_2)x + 2v_2 y - (k_1^3+k_2^3)t}}, \tag{29}$$

where A, a_1, a_2, k_1, k_2, v_2 are arbitrary constants. If $k_1 = -k_2$, $v_2 = 0$, $a_1 = a_2$, then $f_8(x, y, t)$ becomes

$$\tilde{f}_8(x, y, t) = 2a_2 \cosh(k_2(-x + k_2^2 t)) + A. \tag{30}$$

Substituting Eq. (30) into the Cole-Hopf transformation (7), we obtain a kink wave solution as follows:

$$\tilde{u}_8 = \frac{4a_2 k_2 \sinh(k_2(-x + k_2^2 t))}{2a_2 \cosh(k_2(-x + k_2^2 t)) + A}. \tag{31}$$

Especially,when $k_2 = K_2 i,\ i=\sqrt{-1}$, $\tilde{u}_8$ becomes a periodic solution

$$\tilde{\tilde{u}}_8 = \frac{4a_2 k_2 \sinh\left(K_2(x+K_2^2 t)\right)}{2a_2 \cosh\left(K_2(x+K_2^2 t)\right)+A}, \tag{32}$$

where A, a_2, K_2 are arbitrary constants.

Case 9: Under the parametric conditions $A = A$, $a-1 = 0$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = 0$, $k_1 = -k_2$, $k_2 = k_2$, $v_1 = v_1$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_9(x,y,t) = a_{-2}e^{-k_2 x - v_2 y + k_2^3 t} + a_1 e^{-k_2 x + v_1 y + k_2^3 t} + Ae^{(v_1+v_2)y}. \tag{33}$$

Substituting Eq. (33) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_9 = \frac{2k_2\left(a_{-2}e^{-k_2 x - v_2 y + k_2^3 t} + a_1 e^{-k_2 x + v_1 y + k_2^3 t}\right)}{a_{-2}e^{-k_2 x - v_2 y + k_2^3 t} + a_1 e^{-k_2 x + v_1 y + k_2^3 t} + Ae^{(v_1+v_2)y}}, \tag{34}$$

where A, a_{-2}, a_1, k_2, v_1, v_2 are arbitrary constants.

Case 10: Under the parametric conditions $A = A$, $a_{-1} = 0$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = a_2$, $k_1 = k_1$, $k_2 = 0$, $v_1 = v_1$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_{10}(x,y,t) = a_{-2}e^{-v_2 y} + a_1 e^{k_1 x + v_1 y - k_1^3 t} + a_2 e^{v_2 y} + Ae^{k_1 x + (v_1+v_2)y - k_1^3 t}. \tag{35}$$

Substituting Eq. (35) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_{10} = \frac{-2k_1\left(a_1 e^{k_1 x + v_1 y - k_1^3 t} + Ae^{k_1 x + (v_1+v_2)y - k_1^3 t}\right)}{a_{-2}e^{-v_2 y} + a_1 e^{k_1 x + v_1 y - k_1^3 t} + a_2 e^{v_2 y} + Ae^{k_1 x + (v_1+v_2)y - k_1^3 t}}, \tag{36}$$

where A, a_{-2}, a_1, a_2, k_1, v_1, v_2 are arbitrary constants.

Case 11: Under the parametric conditions $A = A$, $a_{-1} = a_{-1}$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = 0$, $k_1 = 0$, $k_2 = k_2$, $v_1 = -2v_2$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_{11}(x,y,t) = a_{-2}e^{-k_2 x - v_2 y + k_2^3 t} + a_{-1}e^{2v_2 y} + a_1 e^{-2v_2 y} + Ae^{k_2 x - v_2 y - k_2^3 t}. \tag{37}$$

Substituting Eq. (37) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_{11} = \frac{-2k_2(-a_{-2}e^{-k_2x - v_2y + k_2^3t} + Ae^{k_2x - v_2y - k_2^3t})}{a_{-2}e^{-k_2x - v_2y + k_2^3t} + a_{-1}e^{2v_2y} + a_1e^{-2v_2y} + Ae^{k_2x - v_2y - k_2^3t}}, \tag{38}$$

where A, a_{-1}, a_{-2}, a_1, k_2, v_2 are arbitrary constants. If $a_{-1} = a_1$, then $f_{11}(x, y, t)$ becomes

$$\tilde{f}_{11}(x, y, t) = 2a_1 \cosh(2v_2y) + a_{-2}e^{-k_2x - v_2y + k_2^3t} + Ae^{k_2x - v_2y - k_2^3t}. \tag{39}$$

Substituting Eq. (39) into the Cole-Hopf transformation (7), we obtain a soliton-like solution as follows:

$$\tilde{u}_{11} = \frac{2k_2(a_{-2}e^{-k_2x - v_2y + k_2^3t} - Ae^{k_2x - v_2y - k_2^3}}{2a_1 \cosh(2v_2y) + a_{-2}e^{-k_2x - v_2y + k_2^3t} + Ae^{k_2x - v_2y - k_2^3}}, \tag{40}$$

where A, a_1, a_{-2}, k_2, v_2 are arbitrary constants.

Case 12: Under the parametric conditions $A = A$, $a_{-1} = 0$, $a_{-2} = 0$, $a_1 = a_1$, $a_2 = a_2$, $k_1 = k_2$, $k_2 = k_2$, $v_1 = v_1$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_{12}(x, y, t) = a_1e^{k_2x + v_1y - k_2^3t} + a_2e^{k_2x + v_2y - k_2^3t} + Ae^{2k_2x + (v_1 + v_2)y - 2k_2^3t}. \tag{41}$$

Substituting Eq. (41) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_{12} = \frac{-2k_2(a_1e^{k_2x + v_1y - k_2^3t} + a_2e^{k_2x + v_2y - k_2^3t} + Ae^{2k_2x + (v_1 + v_2)y - 2k_2^3t})}{a_1e^{k_2x + v_1y - k_2^3t} + a_2e^{k_2x + v_2y - k_2^3t} + Ae^{2k_2x + (v_1 + v_2)y - 2k_2^3t}}, \tag{42}$$

where A, a_1, a_2, k_2, v_1, v_2 are arbitrary constants.

Case 13: Under the parametric conditions $A = A$, $a- 1 = a_2 a_{-2}$ a$_1$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = a_2$, $k_1 = -k_2$, $k_2 = k_2$, $v_1 = v_2$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_{13}(x, y, t) = a_{-2}e^{-k_2x - v_2y + k_2^3t} + \frac{a_2a_{-2}}{a_1} e^{k_2x - v_2y - k_2^3t} + a_1e^{-k_2x + v_2y + k_2^3t} + a_2e^{k_2x + v_2y - k_2^3t} + Ae^{2v_2y}. \tag{43}$$

Substituting Eq. (43) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_{13}=\frac{2k_2(a_1a_{-2}e^{-k_2x-v_2y+k_2^3t}-a_2a_{-2}e^{k_2x-v_2y-k_2^3t}+a_1^2e^{-k_2x+v_2y+k_2^3t}-a_1a_2e^{k_2x+v_2y-k_2^3t})}{a_1a_{-2}e^{-k_2x-v_2y+k_2^3t}+a_2a_{-2}e^{k_2x-v_2y-k_2^3t}+a_1^2e^{-k_2x+v_2y+k_2^3t}+a_1a_2e^{k_2x+v_2y-k_2^3t}+Aa_1e^{2v_2y}}, \quad (44)$$

where A, a_{-2}, a_1, a_2, k_2, v_2 are arbitrary constants. If $a_{-2} = a_2$, $a_1 = -a_2$, $a_2 = a_2$, then $f_{13}(x, y, t)$ becomes

$$\tilde{f}_{13}(x, y, t)=2a_2(\cosh(k_2x+v_2y-k_2^3t)-\cosh(k_2x-v_2y-k_2^3t))+Ae^{2v_2y}. \quad (45)$$

Substituting Eq. (45) into the Cole-Hopf transformation (7), we obtain a soliton-like solution as follows:

$$\tilde{u}_{13}=\frac{2a_2k_2(\sinh(-k_2x-v_2y+k_2^3t)-\sinh(-k_2x+v_2y+k_2^3t))}{2a_2\cosh(-k_2x-v_2y+k_2^3t)-2a_2\cosh(-k_2x+v_2y+k_2^3t))+Ae^{2v_2y}}, \quad (46)$$

where A, a_2, k_2, v_2 are arbitrary constants.

Case 14: Under the parametric conditions $A = A$, $a_{-1} = a_{-1}$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = a_2$, $k_1 = k_1$, $k_2 = k_2$, $v_1 = 0$, $v_2 = 0$, the auxiliary function (5) becomes

$$f_{14}(x, y, t)=a_{-2}e^{-k_2x+k_2^3t}+a_{-1}e^{-k_1x+k_1^3t}+a_1e^{k_1x-k_1^3t}+a_2e^{k_2x-k_2^3t}-Ae^{(k_1+k_2)x-(k_1^3+k_2^3)t}). \quad (47)$$

Substituting Eq. (47) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_{14}=\frac{2(a_{-2}k_2e^{-k_2x+k_2^3t}+a_{-1}k_1e^{-k_1x+k_1^3t}-a_1k_1e^{k_1x-k_1^3t}-a_2k_2e^{k_2x-k_2^3t}-A(k_1+k_2)e^{(k_1+k_2)x-(k_1^3+k_2^3)t})}{a_{-2}e^{-k_2x+k_2^3t}+a_{-1}e^{-k_1x+k_1^3t}+a_1e^{k_1x-k_2^3t}+a_2e^{k_2x-k_2^3t}+Ae^{(k_1+k_2)x-(k_1^3+k_2^3)t})}, \quad (48)$$

where A, a_{-1}, a_{-2}, a_1, a_2, k_1, k_2 are arbitrary constants. If $a_{-1} = a_1$, $a_{-2} = a_2$, then $f_{14}(x, y, t)$ becomes

$$\tilde{f}_{14}(x,y,t)=2a_1\cosh(k_1(-x+k_1^2t))+2a_2\cosh(k_2(-x+k_2^2t))+Ae^{(k_1+k_2)(x-(k_1^2-k_1k_2+k_2^2)t)}. \quad (49)$$

Substituting Eq. (49) into the Cole-Hopf transformation (7), we obtain a soliton-like solution as follows:

$$\tilde{u}_{14}=\frac{2(2a_1k_1\sinh(k_1(-x+k_1^2t))+2a_2k_2\sinh(k_2(-x+k_2^2t))-A(k_1+k_2)e^{(k_1+k_2)(x-(k_1^2-k_1k_2+k_2^2)t)}}{2a_1\cosh(k_1(-x+k_1^2t))+2a_2\cosh(k_2(-x+k_2^2t))+Ae^{(k_1+k_2)(x-(k_1^2-k_1k_2+k_2^2)t)}}. \quad (50)$$

Especially,when $k_1 = K_1 i$, $k_2 = K_2 i$, $A = 0$, $i = \sqrt{-1}$, $\tilde{u}_{14}$ becomes a periodic solution

$$\tilde{\tilde{u}}_{14} = \frac{2(a_1 K_1 \sin(K_1(x + K_1^2 t)) + a_2 K_2 \sin(K_2(x + k_2^2 t))}{a_1 \cos(K_1(x + k_1^2 t)) + a_2 \cos(K_2(x + k_2^2 t))}, \tag{51}$$

where a_1, a_2, K_1, K_2 are arbitrary constants.

Case 15: Under the parametric conditions $A = 0$, $a_{-1} = 0$, $a_{-2} = a_{-2}$, $a_1 = 0$, $a_2 = a_2$, $k_1 = k_1$, $k_2 = k_2$, $v_1 = v_1$, $v_2 = 0$, the auxiliary function (5) becomes

$$f_{15}(x, y, t) = a_{-2}\, e^{-k_2 x + k_2^3 t} + a_2\, e^{k_2 x - k_2^3 t}. \tag{52}$$

Substituting Eq. (52) into the Cole-Hopf transformation (7), we obtain a soliton solution as follows:

$$u_{15} = \frac{2k_2(a_{-2}\, e^{-k_2 x + k_2^3 t} - a_2\, e^{k_2 x - k_2^3 t})}{a_{-2}\, e^{-k_2 x + k_2^3 t} + a_2\, e^{k_2 x - k_2^3 t}}, \tag{53}$$

where a_{-2}, a_2, k_1, k_2, v_1 are arbitrary constants. If $a_{-2} = -a_2$, then $f_{15}(x, y, t)$ becomes

$$\tilde{f}_{15}(x, y, t) = -2\sinh(k_2(-x + k_2^2 t)). \tag{54}$$

Substituting Eq. (54) into the Cole-Hopf transformation (7), we obtain a anti-kink wave solution as follows:

$$\tilde{u}_{15} = \frac{2K_2 \cosh(k_2(-x + k_2^2 t))}{\sinh(k_2(-x + k_2^2 t))} = -2k_2 \coth(k_2(-x + k_2^2 t)). \tag{55}$$

Especially,when $k_2 = K_2 i$, $i = \sqrt{-1}$, $\tilde{u}_{15}$ becomes

$$\tilde{\tilde{u}}_{15} = \frac{-2K_2 \cos(K_2(x + K_2^2 t))}{\sin(K_2(x + K_2^2 t))} = -2K_2 \cot(K_2(x + K_2^2 t)), \tag{56}$$

where K_2 are arbitrary constants.

Case 16: Under the parametric conditions $A = 0$, $a_{-1} = 0$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = 0$, $k_1 = k_1$, $k_2 = k_2$, $v_1 = -v_2$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_{16}(x, y, t) = a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_1 e^{k_1 x - v_2 y - k_2^3 t}. \tag{57}$$

Substituting Eq. (57) into the Cole-Hopf transformation (7), we obtain a soliton solution as follows:

$$u_{16} = \frac{2(a_{-2}k_2 e^{-k_2 x - v_2 y + k_2^3 t} - a_1 k_1 e^{k_1 x - v_2 y - k_1^3 t})}{a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_1 e^{k_1 x - v_2 y - k_1^3 t}}, \tag{58}$$

where a_{-2}, a_1, k_1, k_2, v_2 are arbitrary constants.

Case 17: Under the parametric conditions $A = 0$, $a_{-1} = a_{-1}$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = 0$, $k_1 = 0$, $k_2 = k_2$, $v_1 = v_1$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_{17}(x, y, t) = a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_{-1} e^{-v_1 y} + a_1 e^{v_1 y}. \tag{59}$$

Substituting Eq. (59) into the Cole-Hopf transformation (7), we obtain a double-soliton solution as follows:

$$u_{17} = \frac{2a_{-2}k_2 e^{-k_2 x - v_2 y + k_2^3 t}}{a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_{-1} e^{-v_1 y} + a_1 e^{v_1 y}}, \tag{60}$$

where a_{-1}, a_{-2}, a_1, k_2, v_1;, v_2 are arbitrary constants.

Case 18: Under the parametric conditions $A = 0$, $a_{-1} = a_{-1}$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = a_2$, $k_1 = 0$, $k_2 = k_2$, $v_1 = v_1$, $v_2 = 0$;the auxiliary function (5) becomes

$$f_{18}(x, y, t) = a_{-2} e^{-k_2 x + k_2^3 t} + a_{-1} e^{-v_1 y} + a_1 e^{v_1 y} + a_2 e^{k_2 x - k_2^3 t}. \tag{61}$$

Substituting Eq. (61) into the Cole-Hopf transformation (7), we obtain a soliton solution as follows:

$$u_{18} = \frac{2k_2(a_{-2} e^{-k_2 x + k_2^3 t} - a_2 e^{k_2 x - k_2^3 t})}{a_{-2} e^{-k_2 x + k_2^3 t} + a_{-1} e^{-v_1 y} + a_1 e^{v_1 y} + a_2 e^{k_2 x - k_2^3 t}}, \tag{62}$$

where a_{-1}, a_{-2}, a_1, a_2, k_2, v_1 are arbitrary constants. If $a_2 = -a_{-2}$, $a_{-1} = a_1$, then $f_{18}(x, y, t)$ becomes

$$\tilde{f}_{18}(x, y, t) = 2a_{-2} \sinh(k_2(-x + k_2^2 t)) + 2a_1 \cosh(v_1 y). \tag{63}$$

Substituting Eq. (63) into the Cole-Hopf transformation (7), we obtain a soliton-like solution as follows:

$$\tilde{u}_{18} = \frac{2a_{-2}k_2 \cosh(k_2(-x + k_2^2 t))}{a_{-2} \sinh(k_2(-x + K_2^2 t)) + a_1 \cosh(v_1 y)}. \tag{64}$$

Especially,when $k_2 = K_2 i$, $v_1 = V_1 i$, $a_{-2} = A_{-2} i$, $i = \sqrt{-1}$, $\tilde{u}_{18}$ becomes

$$\tilde{\tilde{u}}_{18} = \frac{-2A_{-2}K_2 \cos(K_2(x + K_2^2 t))}{A_{-2} \sin(K_2(x + K_2^2 t)) + a_1 \cos(V_1 y)}, \tag{65}$$

where A_{-2}, K_2, v_1 are arbitrary constants.

Case 19: Under the parametric conditions $A = 0$, $a_{-1} = a_{-1}$, $a_{-2} = a_{-2}$, $a_1 = 0$, $a_2 = 0$, $k_1 = k_1$, $k_2 = k_2$, $v_1 = v_2$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_{19}(x, y, t) = a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_{-1} e^{-k_1 x - v_2 y + k_1^3 t}. \tag{66}$$

Substituting Eq. (66) into the Cole-Hopf transformation (7), we obtain a soliton solution as follows:

$$u_{19} = \frac{2(a_{-2} k_2 e^{-k_2 x - v_2 y + k_2^3 t} + a_{-1} k_1 e^{-k_1 x - v_2 y + k_1^3 t})}{a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_{-1} e^{-k_1 x - v_2 y + k_1^3 t}}, \tag{67}$$

where a_{-1}, a_{-2}, k_1, k_2, v_2 are arbitrary constants.

Case 20: Under the parametric conditions $A = 0$, $a_{-1} = \frac{a_2 a_{-2}}{a_1}$, $a_{-2} = a_{-2}$, $a_1 = a_1$, $a_2 = a_2$, $k_1 = k_2$, $k_2 = k_2$, $v_1 = -v_2$, $v_2 = v_2$, the auxiliary function (5) becomes

$$f_{20}(x,y,t) = a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + \frac{a_2 a_{-2}}{a_1} e^{-k_2 x + v_2 y + k_2^3 t} + a_1 e^{k_2 x - v_2 y + k_2^3 t} + a_2 e^{k_2 x + v_2 y - k_2^3 t}. \tag{68}$$

Substituting Eq. (68) into the Cole-Hopf transformation (7), we obtain a soliton solution as follows:

$$u_{20} = \frac{2k_2(a_1 a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_2 a_{-1} e^{-k_2 x + v_2 y + k_2^3 t} - a_1^2 e^{k_2 x - v_2 y + k_2^3 t} - a_1 a_2 e^{k_2 x + v_2 y - k_2^3 t})}{a_1 a_{-2} e^{-k_2 x - v_2 y + k_2^3 t} + a_2 a_{-2} e^{-k_2 x + v_2 y + k_2^3 t} + a_1^2 e^{k_2 x - v_2 y - k_2^3 t} + a_1 a_2 e^{k_2 x + v_2 y - k_2^3 t}}, \tag{69}$$

where a_{-2}, a_1, a_2, k_2, v_2 are arbitrary constants.If $a_{-2} = a_2$, $a_1 = -a_2$, $a_2 = a_2$, then $f_{20}(x, y, t)$ becomes

$$\tilde{f}_{20}(x, y, t) = 2a_2(\cosh(k_2 x + v_2 y - k_2^3 t) - \cosh(k_2 x - v_2 y - k_2^3 t)). \tag{70}$$

Substituting Eq. (71) into the Cole-Hopf transformation (7), we obtain a soliton-like solution as follows:

$$\tilde{u}_{20} = \frac{2k_2(\sinh(-k_2 x - v_2 y + k_2^3 t) - \sinh(-k_2 x + v_2 y + k_2^3 y))}{\cosh(-k_2 x - v_2 y + k_2^3 t) - \cosh(-k_2 x + v_2 y + k_2^3 y)}. \tag{71}$$

where k_2, v_2 are arbitrary constants.

In order to show the dynamic properties of above double-soliton solutions, kink wave solutions and peiodic solutions intuitively, as examples, we plot the 3D-graphs of double-soliton solutions and kink wave solutions (11), (13), (31), (46), (64) and (71) (see Fig. 1), the the 3D-graphs of periodic solutions (14), (32), (51) and (65) (see Fig. 2.)

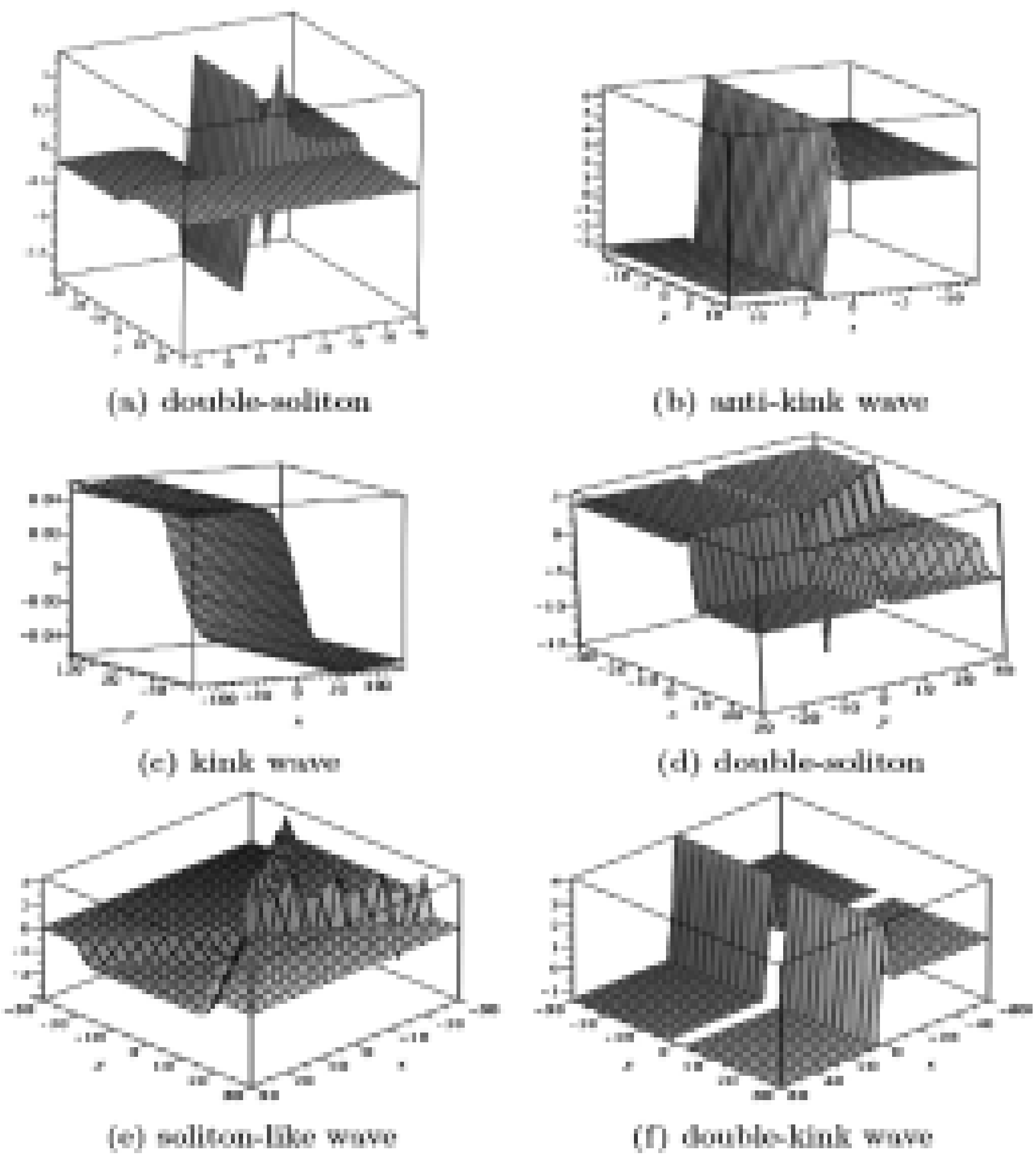

Figure 1: The Profiles of Solutions (11), (13), (31), (46), (64) and (71)

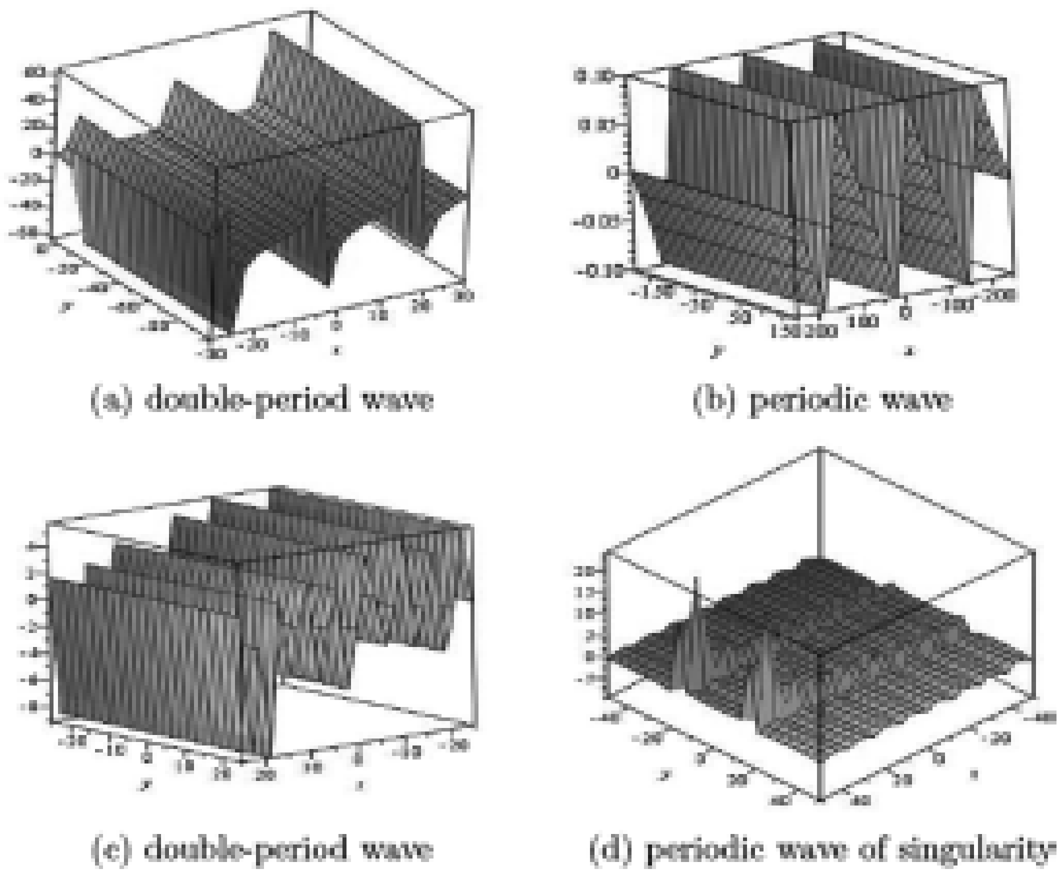

Figure 2: The Profiles of Periodic Solutions (14), (32), (51) and (65)

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