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Simple Proof of Nyquist's Criterion for Stability

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ABSTRACT

The proof of Nyquist's criterion for stability requires the assumption that frequency goes along the closed contour enclosing the right-half complex plane in a clockwise direction. This paper presents a new simple proof that does not require the assumption.

Keywords: Nyquist's criterion for stability, Cauchy's theorem.

1. INTRODUCTION

The proof of Nyquist's criterion for stability is based on Cauchy's theorem, and After that, the proof was made simple [1], [2]. The proof of the Nyquist's criterion for stability requires the assumption that frequency *s* goes along the closed contour enclosing the right-half complex plane in a clockwise direction [1]. However, the Nyquist locus does not change if *s* goes along the closed contour enclosing the left-half complex plane, because the locus becomes constant value when the magnitude of *s* is ∞. Therefore, no one can find out from the Nyquist locus that *s* goes along the closed contour enclosing the right-half complex plane. It is no problem for the proof, but it is more elegant if the assumption is removed. This paper presents a new simple proof without using the assumption.

2. PROPOSED PROOF OF NYQUIST'S CRITERION OF STABILITY

Let $L_n(s)$ and $L_d(s)$ be respectively a numerator and a denominator polynomial of a proper open loop transfer function $L(s)$. Let

$$L(\infty) \neq 1 \tag{1}$$

$1+L(s)$ and closed loop transfer function $G_{cl}(s)$ are

$$\begin{aligned} 1+L(s) &= 1+\frac{L_n(s)}{L_d(s)} \\ &= \frac{L_d(s)+L_n(s)}{L_d(s)} \end{aligned} \tag{2}$$

$$\begin{aligned} G_{cl}(s) &= \frac{L(s)}{1+L(s)} \\ &= \frac{L_n(s)}{L_d(s)+L_n(s)} \end{aligned} \tag{3}$$

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Let n_{cl} be the number of the unstable poles of $G_{cl}(s)$. Let n_{op} be the number of the unstable poles of $L(s)$. From Eqs. (2) and (3), n_{cl} equals to the number of the unstable zeros of $1 + L(s)$, and n_{op} equals to the number of the unstable poles of $1 + L(s)$. In Nyquist's criterion for stability, n_{cl} is checked using

$$n = n_{cl} - n_{op} \tag{4}$$

where n is the number of loops of the Nyquist locus of $1 + L(s)$ which encircle the origin, under the condition that the number of the stability limit poles of $L(s)$.

Let N be the order of $L_d(s)$. From Eqs. (1) and (2), the orders of the numerator and the denominator of $1 + L(s)$ are the same as N because $L(s)$ is proper. By factorizing the numerator and the denominator of $1 + L(s)$, we get

$$\begin{aligned} &1+L(s) \\ &= k\frac{(s-p_{cl1})(s-p_{cl2})\cdots(s-p_{clN})}{(s-p_{op1})(s-p_{op2})\cdots(s-p_{opN})} \end{aligned} \tag{5}$$

where k is a real constant number, p_{cl1}, p_{cl2} ··· are poles of $G_{cl}(s)$, and p_{op1}, p_{op2}, ··· are poles of $L(s)$.

(i) In Case 1 + *L*(*s*) has no Poles and Zeros on the Imaginary Axis

First, we show that the Nyquist locus of $1 + L(s)$ becomes a closed curve even when s moves only on the imaginary axis. From (5), $1 + L(s)$ becomes a constant k when the magnitude of s is ∞. Therefore, the locus on which s moves along the closed curve encloseing the right half plane corresponds with the locus on which s moves only on the imaginary axis. From (2), By differentiating $1 + L(s)$ by s, we get

$$\frac{d}{ds}(1+L(s)) = \frac{\dot{L}_n(s)}{L_d(s)} - \frac{L_n(s)\dot{L}_d(s)}{L_d^2(s)} \tag{6}$$

Because $L_d(s)$ and $L_n(s)$ are polynomials of $s, \frac{d}{ds}(1+L(s))$, becomes ∞ only when $L_d(s) = 0$, if the magnitude of s is ∞. Therefore, when $1 + L(s)$ has no poles on the imaginary axis, the Nyquist locus of $1 + L(s)$ is not discontinuous, which means the locus becomes a closed curve. When $1 + L(s)$ has no zeros on the imaginary axis, the locus does not become zero, which means the locus does not pass through the origin. Therefore, the number of loops of the closed Nyquist locus is countable.

Next, Eq. (4) is proved. For complex number z_1 and z_2, it follows

$$\angle(z_1 z_2) = \angle z_1 + \angle z_2 \tag{7}$$

and

$$\angle(z_1^{-1}) = -\angle z_1. \tag{8}$$

So, substituting

$$s = j\omega$$

in Eq. (5), it follows

$$\begin{aligned} &\angle(1 + L(j\omega)) \\ &= \angle k + \angle(j\omega - p_{cl1}) + \cdots + \angle(j\omega - p_{clN}) \\ &\quad - \angle(jw - p_{op1}) - \cdots - \angle(j\omega - p_{opN}) \end{aligned} \tag{9}$$

when

$$\angle k = \tan^{-1}\frac{\mathrm{lm}[k]}{\mathrm{Re}[k]} = const.$$

and

$$\angle(j\omega - p) = \tan^{-1}\frac{\mathrm{lm}[j\omega - p]}{\mathrm{Re}[j\omega - p]} \\ = \tan^{-1}\frac{\omega - \mathrm{lm}[p]}{-\mathrm{Re}[p]} \tag{10}$$

In the case that p ($= p_{cl1}$) is a stable pole (Re $[p_{cl1}] < 0$), we consider the phase of Nyquist locus of $(j\omega - p_{cl1})$ shown in Fig. 1 when ω goes from $-\infty$ to ∞. The denominator in Eq. (10) becomes a plus constant number because Re $[p_{cl1}] < 0$.

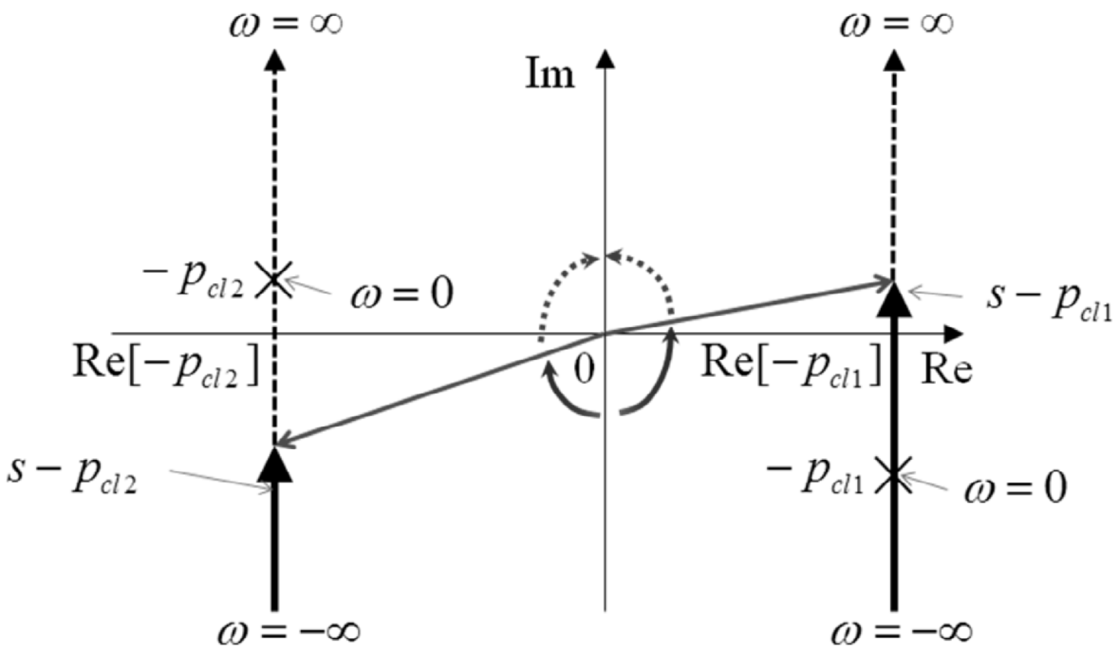

Figure 1: Vector Plot of

When $\omega = -\infty$, the phase is –90° from Eq. (10). Then, ω becomes larger, and when ω = lm $[p_{cl1}]$, the phase becomes 0°. Then, when $\omega = +\infty$, the phase becomes + 90°. Therefore, the phase increases in turn as –90, 0, and + 90°. In other words, the phase changes half rotation in a counterclockwise direction. In the case that p ($= p_{cl2}$) is unstable (Re $[p_{cl2}] > 0$), in the same manner, the phase decreases in turn as –90, –180, and –270°. In other words, the phase changes half rotation in a clockwise direction. These properties are summarized as:

- 1) If the pole is stable (Re $[p_{cl1}] < 0$), $\angle(j\omega - p_{cl1})$ that is the phase of $(s - p_{cl1})$ changes half rotation in a counterclockwise direction.
- 2) If the pole is not stable (Re $[p_{cl2}] > 0$, Re $[p_{cl3}] = 0$), the phase of $(s - p_{cl2})$ or $(s - p_{cl3})$ changes half rotation in a clockwise direction.

The number of poles of $G_{cl}(s)$ $(p_{cl1}, p_{cl2} \cdots)$ is N. Among these poles, n_{cl} poles are unstable, so $N - n_{cl}$ poles are stable. Therefore, the following relation holds:

- R1) The phase of $(s-p_{cl1})\,(s-p_{cl2})\cdots(s-p_{clN})$ changes 0.5 $(N-n_{cl})$ rotations in a counterclockwise direction, and $0.5n_{cl}$ rotations in a clockwise direction. Therefore, The phase changes $-0.5(N-n_{cl}) + 0.5n_{cl} = -0.5N + n_{cl}$ rotations in a clockwise direction.

Next, we consider the phase of $(s-p_{op1})\,(s-p_{op2})\cdots(s-p_{opN})$. From Eq. (9), the phases have the minus symbol. So, the direction in which the locus goes around the origin becomes inverse direction. Therefore, the following relation is derived setting minus symbol to the numbers of rotation:

- R2) The phase of $(s-p_{op1})\,(s-p_{op2})\cdots(s-p_{opN})$ changes $-0.5(N-n_{op})$ rotations in a counterclockwise direction, and $-0.5n_{op}$ rotations in a clockwise direction. Therefore, the phase changes $0.5(N-n_{op}) - 0.5n_{op} = 0.5N - n_{op}$ rotations in a clockwise direction.

From Eq. (9), R1, and R2, when ω changes from $-\infty$ to $+\infty$, it follows

$$\begin{aligned} n &= (-0.5N + n_{cl}) + (0.5N - n_{op}) \\ &= n_{cl} - n_{op} \end{aligned} \tag{11}$$

and, then, Eq. (4) is proved.

(ii) In Case has Poles and Zeros on the Imaginary Axis

The conventional proof is based on Cauchy's argument principle [1]. The principle assumes that $1 + L(s)$ has no poles and zeros on the pathway of s. Therefore, in the conventional proof, the pathway of s is modified in order to regard that the stable limit poles of $1 + L(s)$ exist in the left half plane. If $1 + L(s)$ has stable limit zeros, the proof only shows $G_{cl}(s)$ has stability limit poles [1], [2].

From here, similar modification is done, and it is shown that similar results are derived. Fig. 2(a) shows the modification of the pathway of s. The path is modified in order to regard the stable limit poles and zeros of $1 + L(s)$ exist inside the stable left half plane. The pathway is shifted δ toward right from the imaginary axis. The limit of δ = 0 is considered.

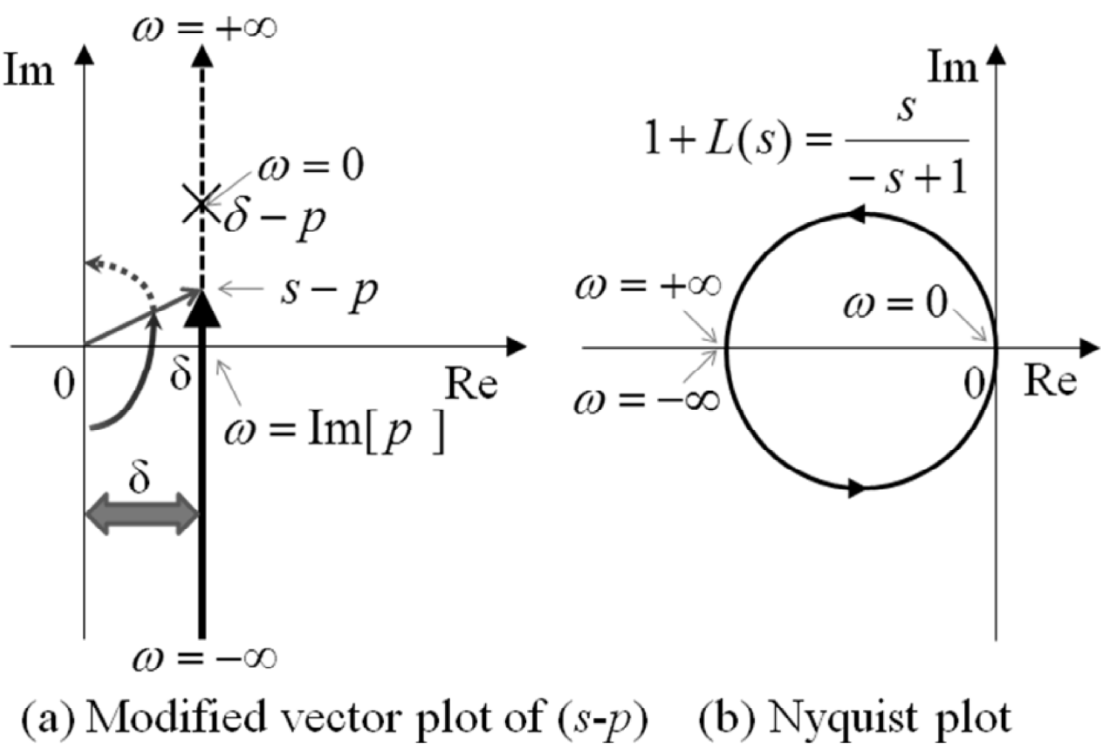

Figure 2: Modified Vector Plot of (s-p) with $s = j\omega + \delta$ when Re [p] = 0, and Nyquist Plot of $\frac{s}{-s+1}$

When p is a stable limit zeros on the imaginary axis, that is Re $[p] = 0$, Fig. 2(a) shows a locus of $s - p$ in which $s = j\omega + \delta$ and ω changes from $-\infty$ to ∞. The phase of the locus $s - p$ changes half rotation in a counterclockwise direction. When $\omega = $ lm $[p]$, it follows

$$\begin{aligned} s - p &= j\omega + \delta - p \\ &= \delta \end{aligned} \tag{12}$$

Then, from Eq. (5), the magnitude of $1 + L(s)$ becomes zero at the limit of $\delta = 0$, and the locus passes through the origin. In other words, it is discriminated that $G_{cl}(s)$ has stable limit poles when the locus pass through the origin. However, n is not able to be defined because the multiplicity of the pole on the imaginary axis is not known. Therefore, it is not able to discriminate $G_{cl}(s)$ has unstable poles or not.

For example, we consider the Nyquist locus of $\frac{s}{-s+1}$ shown in Fig. 2(b). When the locus encircle the origin 0.5 rotation in a counterclockwise direction, we get

$$n = -1 \tag{13}$$

In case of 3 multiple root, then the locus encircle the origin 1.5 rotation in a counterclockwise direction, and we get

$$n = 2 \tag{14}$$

As a result, n is not able to be defined from the locus.

If p in Fig. 2(a) is a stable limit pole on the imaginary axis, that is Re $[p] = 0$, from Eq. (6), the locus $\frac{1}{s-p}$ encircles the origin 0.5 rotation in a clockwise direction because the sign of the phase is minus. When $\omega = $ lm $[p]$, it follows

$$\begin{aligned} s - p &= j\omega + \delta - p \\ &= \delta \end{aligned} \tag{15}$$

From Eq. (5), the magnitude of $1 + L(s)$ becomes ∞ at the limit of $\delta = 0$, and the locus encircles the origin 0.5 rotation in a clockwise direction with infinite distance. When p has n_1 multiplicity, the locus encircles $0.5n_1$ rotation. Therefore, when the multiplicity number n_1 of the stable limit poles of $L(s)$ at $\omega = $ lm $[p]$ is known, the locus encircles $0.5n_1$ rotation, which decides and enables to check the stability of $G_{cl}(s)$ using Eq. (4).

2. CONCLUSION

This paper presented a new simple proof without using the assumption that frequency s went along the closed contour enclosing the right-half complex plane.

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