Communications on Stochastic Analysis
Vol. 15, No. 1-2 (2021)

Serials Publications
www.serialspublications.com

THE ITÔ FORMULA FOR A NEW STOCHASTIC INTEGRAL

HUI-HSIUNG KUO, ANUWAT SAE-TANG, AND BENEDYKT SZOZDA

Abstract. We study the new stochastic integral introduced by Ayed and Kuo in [1]. Our main results are two Itô formulas that extend the one presented in [1]. We generalize the notion of the Itô process onto the class of instantly independent stochastic processes and use it in the formulation of the two Itô formulas we derive.

1. Introduction

Let $\{B_t\colon t \geq 0\}$, be a Brownian motion and $\{\mathcal{F}_t\colon t \geq 0\}$ be a filtration such that B_t is $\mathcal{F}_t$-measurable for each $t \geq 0$ and $B_t - B_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s \leq t$. It is a well-known fact that the Itô integral is well-defined for $\{\mathcal{F}_t\}$-adapted stochastic processes $\{f(t)\colon a \leq t \leq b\}$ such that $\int_a^b |f(t)|^2\, dt < \infty$ almost surely. We will denote the class of all such processes by $\mathcal{L}_{\mathrm{ad}}(\Omega, L^2[a,b])$. The space $\mathcal{L}_{\mathrm{ad}}(\Omega, L^1[a,b])$ is defined in a similar way.

If $f(t) \in \mathcal{L}_{\mathrm{ad}}(\Omega, L^2[a,b])$ has almost surely continuous paths, the Itô integral is equal to the following limit in probability

$$\int_a^b f(t)\, dB_t = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(t_{i-1}) \Delta B_i, \tag{1.1}$$

where $\Delta_n = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ is a partition of $[a,b]$ and $\Delta B_i = B_{t_i} - B_{t_{i-1}}$ (see, for example, [5, Theorem 5.3.3].)

One of the crucial theorems in the Itô stochastic calculus is the Itô formula. It can be viewed as a stochastic counterpart of the fundamental theorem of calculus. Its most basic form is stated below.

Theorem 1.1. *For a function $f \in C^2(\mathbb{R})$, we have*

$$f(B_b) = f(B_a) + \int_a^b f'(B_t)\, dB_t + \frac{1}{2} \int_a^b f''(B_t)\, dt. \tag{1.2}$$

In this paper, we generalize the Itô formula for the new stochastic integral introduced by Ayed and Kuo [1, 2]. We also introduce a counterpart to the Itô

Received 2012-4-19; Communicated by the editors. Article is based on a lecture presented at the International Conference on Stochastic Analysis and Applications, Hammamet, Tunisia, October 10-15, 2011.

2000 *Mathematics Subject Classification.* Primary 60H05; Secondary 60H20.

Key words and phrases. Brownian motion, Itô integral, Itô formula, adapted stochastic processes, instantly independent stochastic processes, anticipating stochastic processes, stochastic integral, anticipating integral.

processes, and use it to introduce most general Itô formula for the new integral known to date.

The remainder of this paper is organized as follows. In Sections 2 and 3 we recall some basic facts about the new stochastic integral and the Itô formula derived by Ayed and Kuo [1] in their original work on the new integral. In Section 4 we introduce instantly independent stochastic processes as a counterpart of the well-known Itô processes. Sections 5 and 6 contain our main results — Theorems 5.1 and 6.1, that is Itô formulas for the new integral. We conclude with some examples in Section 7 and discussion of our results in Section 8.

2. The New Stochastic Integral

Let $\{B_t\colon t \geq 0\}$ and $\{\mathcal{F}_t\colon t \geq 0\}$ be defined as in Section 1. We say that a stochastic process $\{\varphi(t)\colon t \geq 0\}$ is *instantly independent* with respect to the filtration $\{\mathcal{F}_t\colon t \geq 0\}$ if for each $t \geq 0$, the random variable $\varphi(t)$ and the σ-field $\mathcal{F}_t$ are independent. For example $\varphi(B_1 - B_t),\ t \in [0,1]$ is instantly independent of $\mathcal{F}_t$ for any real measurable function $\varphi(x)$. However, $\varphi(B_1 - B_t)$ is adapted for $t \geq 1$.

It can be easily checked that if $\varphi(t)$ is adapted and instantly independent with respect to $\{\mathcal{F}_t\colon t \geq 0\}$, then $\varphi(t)$ is deterministic. Therefore, the family of instantly independent stochastic processes can be regarded as a counterpart to the adapted processes.

In [1], Ayed and Kuo define a new stochastic integral for adapted and instantly independent processes. Suppose that $\{f(t)\colon t \geq 0\}$ is a stochastic process adapted to $\{\mathcal{F}_t\colon t \geq 0\}$ and $\{\varphi(t)\colon t \geq 0\}$ is instantly independent with respect to $\{\mathcal{F}_t\colon t \geq 0\}$. The new stochastic integral of $f(t)\varphi(t)$ is defined as

$$\int_a^b f(t)\varphi(t)\,dB_t = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})\varphi(t_i)\Delta B_i, \tag{2.1}$$

whenever the limit exists in probability.

The crucial distinction between the classical Itô definition and the one proposed by Ayed and Kuo is the fact that the evaluation point of the instantly independent process is the right-endpoint of the sub-interval, while the evaluation point of the adapted process is the left-endpoint, as in the definition of the Itô integral (see Equation (1.1)). This choice of evaluation points ensures that the Itô integral is a special case of the new stochastic integral because if $\varphi(t) \equiv 1$, then Equation (2.1) reduces to Equation (1.1).

For some evaluation formulas, the discussion of the near-martingale property, and an isometry formula for the new integral, see [6, 7] by Kuo, Sae-Tang and Szozda. Application of the formulas derived in this paper can be found in [4] by Khalifa et al.

3. The First Itô Formula for the New Stochastic Integral

In this section we recall, very briefly, the first Itô formula for the new integral that was derived by Ayed and Kuo in [1].

Theorem 3.1. *Let $f(x)$ and $\varphi(x)$ be C^2-functions and $\theta(x,y) = f(x)\varphi(y-x)$. Then the following equality holds for $a \le t \le b$,*

$$\begin{aligned}\theta(B_t, B_b) = \theta(B_a, B_b) &+ \int_a^t \frac{\partial\theta}{\partial x}(B_s, B_b)\, dB_s \\ &+ \int_a^t \left[\frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(B_s, B_b) + \frac{\partial^2\theta}{\partial x \partial y}(B_s, B_b)\right] ds.\end{aligned} \tag{3.1}$$

Theorem 3.1 facilitates computation of many integrals of the new type, however one of its main drawbacks is the fact that θ can only be a function of B_t and $B_b - B_t$. There are two extensions that we wish to present in the forthcoming sections. First, we will allow for the evaluation processes of θ to be an Itô processes and their instantly independent counterparts. we will also present an Itô formula applicable to integrals like

$$\int_0^1 B_1\, dB_t.$$

Note that Theorem 3.1 can be applied to the above integral, only after decomposing the integrand as $B_1 = (B_1 - B_t) + B_t$.

4. Itô Processes and Their Counterpart

An *Itô process* is a stochastic process of the form

$$X_t = X_a + \int_a^t g(s)\, dB_s + \int_a^t \gamma(s)\, ds, \qquad a \le t \le b, \tag{4.1}$$

where X_a is an $\mathcal{F}_a$-measurable random variable, $g \in \mathcal{L}_{\mathrm{ad}}(\Omega, L^2[a,b])$, and $\gamma \in \mathcal{L}_{\mathrm{ad}}(\Omega, L^1[a,b])$.

Observe that if $X_a = 0$, $g(t) \equiv 1$, and $\gamma(t) \equiv 0$, then $X_t = B_t$. This gives us an idea on how to find the instantly independent counterpart to the Itô processes. Consider

$$Y^{(t)} = Y^{(b)} + \int_t^b h(s)\, dB_s + \int_t^b \chi(s)\, ds, \qquad a \le t \le b, \tag{4.2}$$

where $Y^{(b)}$ is independent of $\mathcal{F}_b$, functions $h \in L^2[a,b]$ and $\chi \in L^1[a,b]$ are deterministic. Notice that if $Y^{(b)} = 0$, $h(t) \equiv 1$ and $\chi(t) \equiv 0$, then $Y^{(t)} = B_b - B_t$. Thus $Y^{(t)}$ is to $B_b - B_t$ what X_t is to B_t.

For convenience, we will often use the differential notation instead of the integral one, and so Equation (4.1) is equivalent to

$$dX_t = g(t)\, dB_t + \gamma(s)\, dt,$$

while Equation (4.2) is equivalent to

$$dY^{(t)} = -h(t)\, dB_t - \chi(t)\, dt.$$

Before we recall the Itô formula for the Itô processes (see [5, Theorem 7.4.3]), we wish to note that using the differential notation introduced above is very easy when combined with the fact that

$$(dB_t)(dt) = 0, \quad (dt)^2 = 0, \quad \text{and} \quad (dB_t)^2 = dt.$$

For example, $(dX_s)^2 = g(s)^2\, ds$ and $\left(dY^{(t)}\right)^2 = h(t)^2\, dt$.

Finally, we can recall the well-known Itô formula for the Itô processes.

Theorem 4.1. *Suppose that X_t is as in Equation (4.1) and $f(x)$ is a C^2-function. Then for any $a \leq t \leq b$,*

$$f(X_t) = f(X_a) + \int_a^t f'(X_s)\, dX_s + \frac{1}{2}\int_a^t f''(X_s)\,(dX_s)^2\,. \tag{4.3}$$

5. The Itô Formula for Instantly Independent Itô Processes

As we have seen, the elementary Itô formula (Equation (1.2)) for the Brownian motion can be generalized to be applicable to Itô processes (Equation (4.3)). Our goal is to generalize Theorem 3.1 in a similar way to how Theorem 4.1 generalizes Theorem 1.1 in classical Itô calculus. That is, we will show that it is possible to change $f(B_t)$ and $\varphi(B_b - B_t)$ into $f(X_t)$ and $\varphi\left(Y^{(t)}\right)$ where X_t is as in Equation (4.1) and $Y^{(t)}$ is as in Equation (4.2). Note that due to the way the Itô integral can be computed (see Equation (1.1)), it is clear that the process $Y^{(t)}$ is instantly independent of $\{\mathcal{F}_t\colon t \geq 0\}$. Therefore we can integrate functions of the form $f(X_t)\varphi\left(Y^{(t)}\right)$ using the new integral. This observation leads us to the following theorem.

Theorem 5.1. *Suppose that $\theta(x,y) = f(x)\varphi(y)$, where $f, \varphi \in C^2(\mathbb{R})$. Let X_t be as in Equation (4.1) and $Y^{(t)}$ be as in Equation (4.2). Then for $a \leq t \leq b$,*

$$\begin{aligned}\theta(X_t, Y^{(t)}) &= \theta(X_a, Y^{(a)}) \\ &\quad + \int_a^t \frac{\partial\theta}{\partial x}(X_s, Y^{(s)})\, dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}(X_s, Y^{(s)})\,(dX_s)^2 \\ &\quad + \int_a^t \frac{\partial\theta}{\partial y}(X_s, Y^{(s)})\, dY^{(s)} - \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial y^2}(X_s, Y^{(s)})\left(dY^{(s)}\right)^2.\end{aligned} \tag{5.1}$$

Proof. Throughout this proof, we will use the standard notation introduced earlier, namely $\Delta_n = \{a = t_0 < t_1, \cdots < t_{n-1} < t_n = t\}$ and $\Delta X_i = X_{t_i} - X_{t_{i-1}}$. To establish the formula in Equation (5.1), we begin by writing the difference $\theta\left(X_t, Y^{(t)}\right) - \theta\left(X_a, Y^{(a)}\right)$ in the form of a telescoping sum.

$$\begin{aligned}&\theta\left(X_t, Y^{(t)}\right) - \theta\left(X_a, Y^{(a)}\right) \\ &= \sum_{i=1}^n \left[\theta\left(X_{t_i}, Y^{(t_i)}\right) - \theta\left(X_{t_{i-1}}, Y^{(t_{i-1})}\right)\right] \\ &= \sum_{i=1}^n \left[f\left(X_{t_i}\right)\varphi\left(Y^{(t_i)}\right) - f\left(X_{t_{i-1}}\right)\varphi\left(Y^{(t_{i-1})}\right)\right].\end{aligned} \tag{5.2}$$

Since for $k > 2$, $(\Delta X_i)^k = (\Delta Y_i)^k = o(\Delta t_i)$, we can use the second order Taylor expansion of f and φ to obtain

$$\begin{aligned}f\left(X_{t_i}\right) &\approx f\left(X_{t_{i-1}}\right) + f'\left(X_{t_{i-1}}\right)(\Delta X_i) + \frac{1}{2}f''\left(X_{t_{i-1}}\right)(\Delta X_i)^2\,, \\ \varphi\left(Y^{(t_{i-1})}\right) &\approx \varphi\left(Y^{(t_i)}\right) + \varphi'\left(Y^{(t_i)}\right)(-\Delta Y_i) + \frac{1}{2}\varphi''\left(Y^{(t_i)}\right)(-\Delta Y_i)^2\,.\end{aligned} \tag{5.3}$$

Putting Equations (5.2) and (5.3) together, we have

$$\begin{aligned}
&\theta\left(X_t, Y^{(t)}\right) - \theta\left(X_a, Y^{(a)}\right) \\
&\approx \sum_{i=1}^{n}\left[\left(f\left(X_{t_{i-1}}\right) + f'\left(X_{t_{i-1}}\right)\left(\Delta X_i\right) + \frac{1}{2}f''\left(X_{t_{i-1}}\right)\left(\Delta X_i\right)^2\right)\varphi\left(Y^{(t_i)}\right)\right. \\
&\qquad \left. - f\left(X_{t_{i-1}}\right)\left(\varphi\left(Y^{(t_i)}\right) + \varphi'\left(Y^{(t_i)}\right)\left(-\Delta Y_i\right) + \frac{1}{2}\varphi''\left(Y^{(t_i)}\right)\left(-\Delta Y_i\right)^2\right)\right] \\
&= \sum_{i=1}^{n}\left[\left(f'\left(X_{t_{i-1}}\right)\varphi\left(Y^{(t_i)}\right)\left(\Delta X_i\right) + \frac{1}{2}f''\left(X_{t_{i-1}}\right)\varphi\left(Y^{(t_i)}\right)\left(\Delta X_i\right)^2\right)\right. \\
&\qquad \left. - \left(f\left(X_{t_{i-1}}\right)\varphi'\left(Y^{(t_i)}\right)\left(-\Delta Y_i\right) + \frac{1}{2}f\left(X_{t_{i-1}}\right)\varphi''\left(Y^{(t_i)}\right)\left(-\Delta Y_i\right)^2\right)\right] \\
&= \sum_{i=1}^{n}\left[\frac{\partial\theta}{\partial x}\left(X_{t_{i-1}}, Y^{(t_i)}\right)\Delta X_i + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\left(X_{t_{i-1}}, Y^{(t_i)}\right)\left(\Delta X_i\right)^2\right. \\
&\qquad \left. + \frac{\partial\theta}{\partial y}\left(X_{t_{i-1}}, Y^{(t_i)}\right)\Delta Y_i - \frac{1}{2}\frac{\partial^2\theta}{\partial y^2}\left(X_{t_{i-1}}, Y^{(t_i)}\right)\left(\Delta Y_i\right)^2\right].
\end{aligned} \tag{5.4}$$

Finally, as $\|\Delta_n\| \to 0$, the expression in Equation (5.4) converges to the right-hand side of Equation (5.1), hence the theorem holds. □

Arguments similar to the ones in the proof of Theorem 5.1 can be used to prove the following corollary. It introduces a purely deterministic part that depends only on t.

Corollary 5.2. *Suppose that $\theta(t,x,y) = \tau(t)f(x)\varphi(y)$, where $f, \varphi \in C^2(\mathbb{R})$ and $\tau \in C^1([a,b])$. Let X_t be as in Equation (4.1) and $Y^{(t)}$ be as in Equation (4.2). Then*

$$\begin{aligned}
\theta(t, X_t, Y^{(t)}) = \theta(a, X_a, Y^{(a)}) &+ \int_a^t \frac{\partial\theta}{\partial s}(s, X_s, Y^{(s)})\, ds \\
&+ \int_a^t \frac{\partial\theta}{\partial x}(s, X_s, Y^{(s)})\, dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}(s, X_s, Y^{(s)})\,(dX_s)^2 \\
&+ \int_a^t \frac{\partial\theta}{\partial y}(s, X_s, Y^{(s)})\, dY^{(s)} - \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial y^2}(s, X_s, Y^{(s)})\left(dY^{(s)}\right)^2.
\end{aligned}$$

6. The Itô Formula for More General Processes

As we have already mentioned in Section 3, upon appropriate decomposition of the integrand, it is possible to use the new definition of the stochastic integral to compute the integral of processes that are not instantly independent, for example $\int_0^1 B_1\, dB_t$. Our next goal is to establish an Itô formula for such processes. Note that using the notation of Equation (4.2), with $h(s) \equiv 1$ and $\chi(s) \equiv 0$, on the interval $[0,1]$, we have

$$Y^{(a)} = \int_0^1 1\, dB_t = B_1 - B_0 = B_1.$$

Thus $Y^{(a)}$ to $Y^{(t)}$ is what B_1 to $B_1 - B_t$. Hence we wish to establish an Itô formula for $\theta\left(X_t, Y^{(a)}\right)$, with X_t and $Y^{(t)}$ as defined in Equations (4.1) and (4.2).

Keeping in mind that the definition of the new integral does not allow processes that are anticipating and not instantly independent, we have to impose an additional structure on function θ in order to move freely between $Y^{(a)}$ and $Y^{(t)}$. Following the ideas of [6, 7], we use functions whose Maclaurin series expansion has infinite radius of convergence. Such approach gives us the additional structure we need in order to apply the new theory of stochastic integration.

Theorem 6.1. *Suppose that $\theta(x,y) = f(x)\varphi(y)$, where $f \in C^2(\mathbb{R})$, and $\varphi \in C^\infty(\mathbb{R})$ has Maclaurin expansion with infinite radius of convergence. Let X_t be as in Equation* (4.1) *and $Y^{(t)}$ be as in Equation* (4.2)*. Then for $a \le t \le b$,*

$$\begin{aligned}\theta(X_t, Y^{(a)}) &= \theta(X_a, Y^{(a)}) \\ &\quad + \int_a^t \frac{\partial\theta}{\partial x}\left(X_s, Y^{(a)}\right) dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}\left(X_s, Y^{(a)}\right)(dX_s)^2 \\ &\quad - \int_a^t \frac{\partial^2\theta}{\partial x \partial y}\left(X_s, Y^{(a)}\right)(dX_s)\left(dY^{(s)}\right).\end{aligned} \tag{6.1}$$

Proof. We will derive the formula in Equation (6.1) symbolically using the differential notation introduced earlier. That is, we need to establish that

$$\begin{aligned}d\theta(X_t, Y^{(a)}) &= \frac{\partial\theta}{\partial x}\left(X_t, Y^{(a)}\right) dX_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\left(X_t, Y^{(a)}\right)(dX_t)^2 \\ &\quad - \frac{\partial^2\theta}{\partial x \partial y}\left(X_t, Y^{(a)}\right)(dX_t)\left(dY^{(t)}\right).\end{aligned} \tag{6.2}$$

To simplify the notation, we will write $\mathcal{D} = d\left(\theta\left(X_t, Y^{(a)}\right)\right)$. Let us consider

$$\begin{aligned}\mathcal{D} &= d\left(f(X_t)\varphi\left(Y^{(a)}\right)\right) \\ &= d\left(f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\left(Y^{(a)}\right)^n\right) \\ &= d\left(f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\left(Y^{(a)} - Y_t + Y_t\right)^n\right).\end{aligned}$$

Applying the binomial theorem and the fact that $Y^{(a)} - Y_t = Y^{(t)}$ allows us to rewrite $\mathcal{D}$ as

$$\begin{aligned}\mathcal{D} &= d\left(f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(Y^{(t)}\right)^k (Y_t)^{n-k}\right) \\ &= \sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k} d\left(f(X_t)\,(Y_t)^{n-k}\left(Y^{(t)}\right)^k\right).\end{aligned} \tag{6.3}$$

Note that $Z_t = f(X_t)Y_t^{n-k}$ as a product of Itô processes is an Itô process itself, hence we can use the Itô formula from Theorem 5.1 to evaluate the differential

under the sum in Equation (6.3). We take $\eta(z,y) = zy^k$ to obtain $\eta_z(z,y) = y^k, \eta_{zz}(z,y) = 0, \eta_y(z,y) = kzy^{k-1}$ and $\eta(z,y)_{yy} = k(k-1)zy^{k-2}$ which yields

$$\begin{aligned} d\left(\eta\left(Z_t, Y^{(t)}\right)\right) = &\left(Y^{(t)}\right)^k d(Z_t) + kZ_t\left(Y^{(t)}\right)^{k-1} dY^{(t)} \\ &- \frac{1}{2}k(k-1)Z_t\left(Y^{(t)}\right)^{k-2}\left(dY^{(t)}\right)^2. \end{aligned} \tag{6.4}$$

Using the Itô product rule for Itô processes, we easily see that dZ_t can be expressed as

$$\begin{aligned} dZ_t &= f(X_t)\, d\left(Y_t^{n-k}\right) + Y_t^{n-k}\, df(X_t) + \left(df(X_t)\right)\left(dY_t^{n-k}\right) \\ &= f(X_t)\left[(n-k)Y_t^{n-k-1}\, dY_t + \frac{1}{2}(n-k)(n-k-1)Y_t^{n-k-2}\left(dY_t\right)^2\right] \\ &\quad + Y_t^{n-k}\left[f'(X_t)\, dX_t + \frac{1}{2}f''(X_t)\left(dX_t\right)^2\right] \\ &\quad + (n-k)f'(X_t)Y_t^{n-k-1}\left(dX_t\right)\left(dY_t\right) \\ &= f(X_t)(n-k)Y_t^{n-k-1}\, dY_t \\ &\quad + \frac{1}{2}(n-k)(n-k-1)f(X_t)Y_t^{n-k-2}\left(dY_t\right)^2 + f'(X_t)Y_t^{n-k}\, dX_t \\ &\quad + \frac{1}{2}f''(X_t)Y_t^{n-k}\left(dX_t\right)^2 + (n-k)f'(X_t)Y_t^{n-k-1}\left(dX_t\right)\left(dY_t\right). \end{aligned} \tag{6.5}$$

Putting together Equations (6.3), (6.4) and (6.5), we see that in order to complete this proof, we have to evaluate

$$\begin{aligned} \mathcal{D} = &f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}(n-k)Y_t^{n-k-1}\left(Y^{(t)}\right)^k dY_t \\ &+ \frac{1}{2}f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}(n-k)(n-k-1)Y_t^{n-k-2}\left(Y^{(t)}\right)^k\left(dY_t\right)^2 \\ &+ f'(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}Y_t^{n-k}\left(Y^{(t)}\right)^k dX_t \\ &+ \frac{1}{2}f''(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}Y_t^{n-k}\left(Y^{(t)}\right)^k\left(dX_t\right)^2 \\ &+ f'(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}(n-k)Y_t^{n-k-1}\left(Y^{(t)}\right)^k\left(dX_t\right)\left(dY_t\right) \\ &+ f(X_t)\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}kY_t^{n-k}\left(Y^{(t)}\right)^{k-1} dY^{(t)} \\ &- f(X_t)\frac{1}{2}\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{n!}\sum_{k=0}^{n}\binom{n}{k}k(k-1)Y_t^{n-k}\left(Y^{(t)}\right)^{k-2}\left(dY^{(t)}\right)^2 \end{aligned}$$

$$\begin{aligned}&= f(X_t)\Sigma_1\, dY_t + \frac{1}{2}f(X_t)\Sigma_2\,(dY_t)^2 + f'(X_t)\Sigma_3\, dX_t + \frac{1}{2}f''(X_t)\Sigma_3\,(dX_t)^2 \\ &\quad + f'(X_t)\Sigma_1\,(dX_t)\,(dY_t) + f(X_t)\Sigma_4\, dY^{(t)} - f(X_t)\frac{1}{2}\Sigma_5\left(dY^{(t)}\right)^2.\end{aligned} \tag{6.6}$$

In order to simplify $\mathcal{D}$ in Equation (6.6), we need to evaluate the 5 sums denoted above by Σ_i, with $i \in \{1, 2, \ldots, 5\}$.

Σ_1. The first sum is given by

$$\Sigma_1 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} (n-k) Y_t^{n-k-1} \left(Y^{(t)}\right)^k. \tag{6.7}$$

Note that for $n = k$ the expression under the sum is equal to zero, so we have

$$\Sigma_1 = \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n-1} \binom{n}{k} (n-k) Y_t^{n-k-1} \left(Y^{(t)}\right)^k.$$

Now, since $\frac{1}{n!}\binom{n}{k}(n-k) = \frac{1}{(n-1)!}\binom{n-1}{k}$, we get

$$\Sigma_1 = \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} Y_t^{(n-1)-k} \left(Y^{(t)}\right)^k,$$

and application of the binomial theorem yields

$$\Sigma_1 = \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{(n-1)!} \left(Y_t + Y^{(t)}\right)^{n-1}.$$

Since, by definition, $Y_t + Y^{(t)} = Y^{(a)}$ and $\sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{(n-1)!} x^{n-1} = \varphi'(x)$, we obtain

$$\Sigma_1 = \varphi'\left(Y^{(a)}\right). \tag{6.8}$$

Σ_2. The second sum we have to evaluate is

$$\Sigma_2 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} (n-k)(n-k-1) Y_t^{n-k-2} \left(Y^{(t)}\right)^k. \tag{6.9}$$

Due to the $n-k$ and $n-k-1$ factors, the terms with $k = n$ and $k = n-1$ do not contribute to the sum, hence

$$\Sigma_2 = \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n-2} \binom{n}{k} (n-k)(n-k-1) Y_t^{n-k-2} \left(Y^{(t)}\right)^k.$$

Since $\frac{1}{n!}\binom{n}{k}(n-k)(n-k-1) = \frac{1}{(n-2)!}\binom{n-2}{k}$, we have

$$\Sigma_2 = \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{(n-2)!} \sum_{k=0}^{n-2} \binom{n-2}{k} Y_t^{n-k-2} \left(Y^{(t)}\right)^k.$$

Using the binomial theorem, we obtain

$$\Sigma_2 = \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{(n-2)!} \left(Y_t + Y^{(t)}\right)^{n-2}.$$

Using the facts that $\varphi''(x) = \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{(n-2)} x^{n-2}$ and $Y_t + Y^{(t)} = Y^{(a)}$ we get

$$\Sigma_2 = \varphi'' \left(Y^{(a)} \right). \tag{6.10}$$

Σ_3. Using the same reasoning as previously, we can write the next sum appearing in Equation (6.6) as

$$\begin{aligned} \Sigma_3 &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{\infty} \binom{n}{k} Y_t^{n-k} \left(Y^{(t)} \right)^k \\ &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \left(Y_t + Y^{(t)} \right)^n \\ &= \varphi \left(Y^{(a)} \right). \end{aligned} \tag{6.11}$$

Σ_4. Now, we evaluate the following sum

$$\Sigma_4 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} k Y_t^{n-k} \left(Y^{(t)} \right)^{k-1}.$$

Notice that substitution $j = n - k$ together with the fact that $\binom{n}{n-j} = \binom{n}{j}$ yields

$$\begin{aligned} \Sigma_4 &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{j=0}^{n} \binom{n}{n-j} (n-j) Y_t^j \left(Y^{(t)} \right)^{n-j-1} \\ &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{j=0}^{n} \binom{n}{j} (n-j) Y_t^j \left(Y^{(t)} \right)^{n-j-1}, \end{aligned}$$

and this is the same sum we have evaluated in Equation (6.7), hence

$$\Sigma_4 = \varphi \left(Y^{(a)} \right). \tag{6.12}$$

Σ_5. The last sum needed is

$$\Sigma_5 = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{k} k(k-1) Y_t^{n-k} \left(Y^{(t)} \right)^{k-2}.$$

Using the substitution $j = n - k$ and the fact that $\binom{n}{n-j} = \binom{n}{j}$ again, we obtain

$$\begin{aligned} \Sigma_5 &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{n-j} (n-j)(n-j-1) Y_t^j \left(Y^{(t)} \right)^{n-j-2} \\ &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n}{j} (n-j)(n-j-1) Y_t^j \left(Y^{(t)} \right)^{n-j-2}. \end{aligned}$$

And this sum appears in Equation (6.9), thus

$$\Sigma_5 = \varphi'' \left(Y^{(a)} \right). \tag{6.13}$$

Now, putting together Equations (6.6), (6.8), (6.10), (6.11), (6.12) and (6.13) we obtain

$$\begin{aligned}\mathcal{D} &= f(X_t)\Sigma_1\,dY_t + \frac{1}{2}f(X_t)\Sigma_2\,(dY_t)^2 + f'(X_t)\Sigma_3\,dX_t + \frac{1}{2}f''(X_t)\Sigma_3\,(dX_t)^2 \\ &\quad + f'(X_t)\Sigma_1\,(dX_t)\,(dY_t) + f(X_t)\Sigma_4\,dY^{(t)} - \frac{1}{2}f(X_t)\Sigma_5\left(dY^{(t)}\right)^2 \\ &= f(X_t)\varphi'\left(Y^{(a)}\right)\,dY_t + \frac{1}{2}f(X_t)\varphi''\left(Y^{(a)}\right)(dY_t)^2 + f'(X_t)\varphi\left(Y^{(a)}\right)\,dX_t \\ &\quad + \frac{1}{2}f''(X_t)\varphi\left(Y^{(a)}\right)(dX_t)^2 + f'(X_t)\varphi'\left(Y^{(a)}\right)(dX_t)\,(dY_t) \\ &\quad + f(X_t)\varphi'\left(Y^{(a)}\right)\,dY^{(t)} - \frac{1}{2}f(X_t)\varphi''\left(Y^{(a)}\right)\left(dY^{(t)}\right)^2.\end{aligned}$$

Since $dY_t = -dY^{(t)}$, we have

$$\begin{aligned}\mathcal{D} = &-f(X_t)\varphi'\left(Y^{(a)}\right)\,dY^{(t)} + \frac{1}{2}f(X_t)\varphi''\left(Y^{(a)}\right)\left(dY^{(t)}\right)^2 \\ &+ f'(X_t)\varphi\left(Y^{(a)}\right)\,dX_t + \frac{1}{2}f''(X_t)\varphi\left(Y^{(a)}\right)(dX_t)^2 \\ &- f'(X_t)\varphi'\left(Y^{(a)}\right)(dX_t)\left(dY^{(t)}\right) + f(X_t)\varphi'\left(Y^{(a)}\right)\,dY^{(t)} \\ &- \frac{1}{2}f(X_t)\varphi''\left(Y^{(a)}\right)\left(dY^{(t)}\right)^2 \\ = &\frac{\partial\theta}{\partial x}\left(X_t, Y^{(a)}\right)\,dX_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\left(X_t, Y^{(a)}\right)(dX_t)^2 \\ &- \frac{\partial^2\theta}{\partial x\partial y}\left(X_t, Y^{(a)}\right)(dX_t)\left(dY^{(t)}\right),\end{aligned}$$

which completes the proof. □

As with Corollary 5.2, we can easily deduce, that if we had a component of θ that is deterministic and depends only on t, the following Corollary will hold.

Corollary 6.2. *Suppose that $\theta(t,x,y) = \tau(t)f(x)\varphi(y)$, where $\tau \in C^1(\mathbb{R})$, $f \in C^2(\mathbb{R})$, and $\varphi \in C^\infty(\mathbb{R})$ has Maclaurin expansion with infinite radius of convergence. Let X_t be as in Equation (4.1) and $Y^{(t)}$ be as in Equation (4.2). Then*

$$\begin{aligned}\theta(t, X_t, Y^{(a)}) = \theta(a, X_a, Y^{(a)}) &+ \int_a^t \frac{\partial\theta}{\partial t}\left(s, X_s, Y^{(a)}\right)\,ds \\ &+ \int_a^t \frac{\partial\theta}{\partial x}\left(s, X_s, Y^{(a)}\right)\,dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}\left(s, X_s, Y^{(a)}\right)(dX_s)^2 \\ &- \int_a^t \frac{\partial^2\theta}{\partial x\partial y}\left(s, X_s, Y^{(a)}\right)(dX_s)\left(dY^{(s)}\right). \qquad (6.14)\end{aligned}$$

7. Examples

To illustrate the usage of the new Itô formulas introduced in previous sections, we will establish simpler versions of Theorems 5.1 and 6.1 for functions of Brownian motion. Let $X_a = B_a$, $g(t) \equiv 1$, and $\gamma(t) \equiv 0$, so that the process X_t becomes B_t.

Moreover, let $Y^{(b)} = 0,\ h(t) \equiv 1$, and $\chi(t) \equiv 0$, so that the process $Y^{(t)}$ becomes $B_b - B_t$. Using the above in Theorem 5.1 gives us the following corollary.

Corollary 7.1. *Suppose that $\theta(x, y) = f(x)\varphi(y)$ with $f, \varphi \in C^2(\mathbb{R})$. Then*

$$\begin{aligned}\theta(B_t, B_b - B_t) = \theta(B_a, B_b - B_a) \\ + \int_a^t \frac{\partial\theta}{\partial x}(B_s, B_b - B_s)\, dB_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}(B_s, B_b - B_s)\, ds \\ - \int_a^t \frac{\partial\theta}{\partial y}(B_s, B_b - B_s)\, dB_s - \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial y^2}(B_s, B_b - B_s)\, ds.\end{aligned}$$

Similarly, Theorem 6.1 becomes

Corollary 7.2. *Suppose that $\theta(x, y) = f(x)\varphi(y)$, where $f \in C^2(\mathbb{R})$, and $\varphi \in C^\infty(\mathbb{R})$ has Maclaurin expansion with infinite radius of convergence. Then*

$$\begin{aligned}\theta(B_t, B_b - B_a) = \theta(B_a, B_b - B_a) + \int_a^t \frac{\partial\theta}{\partial x}(B_t, B_b - B_a)\, dB_s \\ + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}(B_t, B_b - B_a)\, ds + \int_a^t \frac{\partial^2\theta}{\partial x\partial y}(B_t, B_b - B_a)\, ds.\end{aligned}$$

Example 7.3. Applying Corollary 7.2 on the interval $[0, 1]$ to a function $\theta(x, y) = \frac{x^{n+1}}{n+1}y^m$, with $m, n \in \mathbb{N}$, we obtain

$$\int_0^1 B_t^n B_1^m\, dB_t = \frac{B_1^{m+n+1}}{n+1} - B_1^{m-1}\int_0^1 B_t^{n-1}\left(\frac{n}{2}B_1 + mB_t\right)\, dt.$$

This shows how we can express the integral of an anticipating process in terms of a random variable and a Riemann integral of a stochastic process.

8. Conclusions

We have derived two Itô formulas for Itô processes and their instantly independent counterparts. Our results are applicable in a variety of situations and extend the result of Ayed and Kuo [1]. Below, we compare the two formulas derived in this paper as Theorems 5.1 and 6.1.

Notice that there are 4 main components to the Itô formula in the setting of the new stochastic integral, namely, $x, y, f(x)$ and $\varphi(y)$. In both formulas derived above, x is an Itô process and $f \in C^2(\mathbb{R})$. This assumptions are natural because the new stochastic integral is an extension of the stochastic integral of Itô, and putting $\varphi(y) \equiv 1$ shows that our formulas are in fact extensions of the classical result cited earlier as Theorems 1.1 and 3.1.

The main differences between Theorem 5.1 and Theorem 6.1 are the properties of y and $\varphi(y)$. In Theorem 5.1, the process substituted for y is an instantly independent process that arises in a similar way as the Itô processes do in the theory of adapted processes. This allows us to work with functions $\varphi \in C^2(\mathbb{R})$. In Theorem 6.1, we have derived a formula that allows us to work with random variables that arise from the instantly independent Itô processes in the place of y. However, this extension comes at a price of an additional smoothness conditions on the function φ. That is φ has to have infinite radius of convergence of its

Maclaurin series expansion. In many applications, such a requirement should not be too restrictive.

One of the applications of the formulas established in this paper is a solution of a class of linear stochastic differential equations with anticipating initial conditions. This will be presented in a forthcoming paper by Khalifa et al. [4].

References

1. Ayed, W. and Kuo, H.-H.: *An extension of the Itô integral*, Communications on Stochastic Analysis **2** (2008), no. 3, 323–333.
2. Ayed, W. and Kuo, H.-H.: *An extension of the Itô integral: toward a general theory of stochastic integration*, Theory of Stochastic Processes **16(32)** (2010), no. 1, 1–11.
3. Itô, I.: *Stochastic integral*, Proceedings of the Imperial Academy **20** (1944), no. 8, 519–524.
4. Khalifa, N., Kuo, H.-H., Ouerdiane, H. and Szozda, B.: *Linear stochastic differential equations with anticipating initial conditions*, pre-print, (2012).
5. Kuo, H.-H.: *Introduction to stochastic integration*, Universitext, Springer, 2006.
6. Kuo, H.-H., Sae-Tang, A. and Szozda, B.: *A stochastic integral for adapted and instantly independent stochastic processes*, in "Advances in Statistics, Probability and Actuarial Science Vol. I", Stochastic Processes, Finance and Control: A Festschrift in Honour of Robert J. Elliott (eds.: Cohen, S., Madan, D., Siu, T. and Yang, H.), 53–71, World Scientific, 2012.
7. Kuo, H.-H., Sae-Tang, A. and Szozda, B.: *An isometry formula for a new stochastic integral*, Accepted for the Proceedings of International Conference on Quantum Probability and Related Topics, May 29–June 4, 2011, Levico, Italy, (to appear in QP – PQ, Vol. 32).

Hui-Hsiung Kuo: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

E-mail address: kuo@math.lsu.edu

URL: http://www.math.lsu.edu/∼kuo

Anuwat Sae-Tang: Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok, Thailand

E-mail address: anuwat.sae@kmutt.ac.th

Benedykt Szozda: The T.N. Thiele Centre for Mathematics in Natural Science, Department of Mathematical Sciences, & CREATES, School of Economics and Management, Aarhus University, DK-8000 Aarhus C, Denmark

E-mail address: szozda@imf.au.dk

URL: http://home.imf.au.dk/szozda