International Journal of Mathematical Science
Vol. 2, No. 3-4, July-December 2003, pp. 283-287

ISSN: 0972-754X

APPROXIMATION METHODS FOR NONLINEAR M-MONOTONIC OPERATOR EQUATIONS

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We suppose H is a real refelexive Hilbert space and $T : D(T) \subset H \to H$ is a locally Lipschitzian m-monotonic operator, where the domain of T, D(T), is a proper subset of H. For any f ε H, approximation methods are constructed which converge strongly to the solution of the equation $x + Tx = f$.

Let H be a real Hilbert space.

An operator T with domain D(T) and range R(T) in H is called monotonic if the inequality

$$\| x - y \| \le \| x - y + \lambda(Tx - Ty) \| \qquad (1)$$

holds for all $x, y \, \varepsilon \, D(T)$ and all $\lambda > 0$. If T is monotonic and (I + rt) (D(T)) = H for all r > 0 then T is called m - monotonic, where I denotes the identify operator.

If T is m - monotonic, then for any given $f \, \varepsilon \, H$ the equation

$$x + Tx = f \qquad (2)$$

has a unique solution.

Here we study methods of approximating the solution of Eq. (2) (when T is m - monotonic) in Hilbert spaces where the domain of T, D(T), is a proper subset of H and T maps D(T) into H. We have taken real reflexive Hilbert spaces instead of uniformly smooth Hilbert spaces.

We denotes L the Local Lipschitz constant of T.

Theorem 1 : Suppose H is a real reflexive Banach space, and $T : D(T) \subset H \to H$ is m - monotonic and Locally Lipschitzian. Suppose D(T) is open and that $x^* \, \varepsilon \, D(T)$ is the unique solution of the equation $x + Tx = f$, $f \, \varepsilon \, H$. Suppose $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence satisfying the conditions.

(i) $0 \le a_n < 1/2\,[L2 + 2L + 2]$, $n \ge 0$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then there exists a closed convex neighbourhood B of x* contained in D(T) and for any given $x_0 \in B$, a sequence $\{x_n\}_{n=0}$ of elements of B such that on setting.

$$p_n = (1-\alpha_n)x_n + \alpha_n (f - Tx_n), \quad n \geq 0$$

the sequence {pn} satisfying the condition

$$\| p_{n-1} - x_n \| = \inf\{\| p_{n-1} - x \| : x \in B\}'' \; n \geq 1 \tag{3}$$

and converges strongly to x*. Moreover, if $\alpha_n = 1/2[L^2 + 2L + 2]$ for all n $\geq$ 0 the

$$\| p_{n-1} - x^* \| \leq p^n \; \| p_n - x^* \|$$

where $p = (1 - 1/4[L^2 + 2L + 2]) \in (0,1)$.

Proof : Define $S : D(T) \to H$ by Sx = f – Tx. Observe that x* is a fixed point of S and that S is locally Lipschitz with constant L. Furthermore (–S) is monotonic so that for all r > 0, and x, y ε D(T).

$$\| x - y \| \leq \| x - y - r\,(Sx - Sy) \| \tag{4}$$

Without loss of generality we may assume L $\geq$ 1. Let B(y, r) = $\{x \in H : \| x - y \| \leq r\}$. Then there exists $r_1 > 0$ such that $B(x^*, r) \subseteq D(T)$. Since S is locally Lipschitzian, there exists $r_2 > 0$ such that S is Lipschitzian on $B(x^*, r_2)$. Let r = min $\{r_1, r_2\}$. Then $B(x^*, r) \subseteq D(T)$ and S is Lipschitzia on $B(x^*, r_2)$ Let B = B(x*, r/ 2L). Given any $x_0 \in B$, the $\| Sx_0 - x^* \| \leq r/2 < r$ so that $Sx_0 \in B\,(x^*, r)$. Hence $p_0 = (1 - \alpha_0)\, x_0 + \alpha_0 Sx_0 \in (x^*, r)$. Hence $p_0 = (1 - \alpha_0)\, x_0 + \alpha_0 Sx_0 \in B\,(x^*, r)$. Since H is reflexive, there exists x1 ε B such that

$$\| p_0 - x_1 \| = \inf(\| p_0 - x \| : x \in B\}.$$

Thus

$$p_1 = (1-\alpha_1)x_1 + \alpha_1 Sx_1 \in B(x^*, r).$$

By continuing this process we obtain $\{p_n\}$ in B(x*, r) and $\{x_n\}$ in B satisfying the conditions.

$$p_n = (1 - \alpha_n)x_n + \alpha_n Sx_n \,,\; n \geq 0$$

$$\| p_{n-1} - x_n \| = \inf\{\| p_{n-1} - x \| : x \in B\},\; n \geq 1 \tag{5}$$

Thus

$$\| x_n - x^* \| \leq \| p_{n-1} - x^* \| \; n \geq 1,$$

We now prove that lim pn = x*. From (5) we obtain

$$x_n = p_n + \alpha_n x_n - \alpha_n Sx_n$$

$$n = (1 + \alpha_n)\, p_n - \alpha_n Sp_n + \alpha^2 (x_n - Sx_n) + \alpha_n (Sp_n - Sx_n) \tag{6}$$

Observe that

$$x^* = (1 + \alpha_n)x^* - \alpha_n Sx^* \tag{7}$$

Thus from (6) and (7) we obtain

$$\begin{aligned}
\| x_n - x^* \| &= \| (1+\alpha_n)(p_n - x^*) - \alpha_n (Sp_n - Sx^*) \\
&\quad + \alpha_n^2 (x_n - Sx_n) + \alpha_n (Sp_n - Sx_n) \| \\
&\geq (1+\alpha_n) \| p_n - x^* - \frac{\alpha_n}{1+\alpha_n}(Sp_n - Sx^*) \| \\
&\quad - \alpha_n^2 \| x_n - Sx_n \| - \alpha_n \| Sp_n - Sx_n \| \\
&\geq (1+\alpha_n) \| p_n - x^* \| - \alpha_n^2 \| x_n - Sx_n \| \\
&\quad - \alpha_n \| Sp_n - Sx_n \| \quad \text{(using (4))}
\end{aligned}$$

Hence

$$\begin{aligned}
\| p_n - x^* \| &\leq \frac{1}{1+\alpha_n} \| x_n - x^* \| + \alpha_n^2 (x_n - Sx_n) + \alpha_n (Sp_n - Sx_n) \| \\
&\geq [1 - \alpha_n + \alpha_n^2] \| x_n - x^* \| \\
&\quad + (1+L)\alpha_n^2 \| x_n - x^* \| + L(1+L)\alpha_n^2 \| x_n - x^* \|, \ldots (8) \\
&= [1 - \alpha_n + (L^2 + 2L + 2)\alpha_n^2] \| x_n - x^* \| \\
&\leq \left[1 - \frac{1}{2}\alpha_n\right] \| p_{n-1} - x^* \| \\
&\leq \exp\left(-\frac{1}{2}\sum_{j=0}^{n} \alpha_j\right) \| p_n - x^* \| \to 0 \text{ as } n \to \infty
\end{aligned}$$

If we set $\alpha_n = 1/2[L^2 + 2L + 2]'' \, n \geq 0$ in (8) we obtain

$$\begin{aligned}
\| p_n - x^* \| &\leq \left[1 - \frac{1}{4(L^2 + 2L + 2}\right] \| p_{n-1} - x^* \| \\
&\leq p^2 \| p_n - x^* \|
\end{aligned}$$

This completes the proof of Theorem 1.

Remark 1: In Proving Theorem 1 we first observed that the monocity of T implies that (–S) is monotonic and hence inequality (4) holds. In order to apply inequality (4) we used (5) to express x_n in the form (6). Furthermore, we expressed x* in the form (7). Using (6) and (7) and an application of inequality (4) and the Lipschitz condition of S we were able to prove Theorem 1 without the application of an inequality which hold in uniformly smooth Hilbert Spaces.

Remark 2 : Using Straightforward modifications of the proof of Theorem 1 one can easily prove the following theorem for a more general iteration scheme.

Theorem 2 : Suppose H, T, D(T), S, and x* are as in Theorem 1. Suppose $\{\alpha_n)_{n=0}^{\infty}$ and $\{\beta_n)_{n=0}^{\infty}$ are real sequence satisfying the conditions:

(*i*) $0 \leq \alpha_n \leq 1/2\ [L^2 + 2L + 2]$, $n \geq 0$

(*ii*) $0 \leq \beta_n \leq 1/4\ [L^2 + 2L + 2]$, $n \geq 0$

(*iii*) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

Then there exists a closed convex neighbourhood B of x* contained in D(T) and for any given $x_0\ \varepsilon\ B$, a sequence $\{x_n\}_{n=0}$ of elements of B such that on setting.

$$y_n = (1-\beta_n)_{x_n} + \beta_n Sx_n,\quad n \geq 0$$

$$p_n = (1-\alpha_n)_{x_n} + \alpha_n Sx_n),\quad n \geq 0$$

the sequence $\{p_n\}$ satisfying the condition

$$\| p_{n-1} - x_n \| = \inf\{\| p_{n-1} - x \| : x\,\varepsilon\, B\}\ \forall\, n \geq 1 \tag{3}$$

and converges strongly to x*. Moreover, if $\alpha_n = 1/\ 2[L^2 + 2L + 2]\ \forall\ n \geq 0$.then

$$\| p_{n-1} - x^* \| \leq p^n \| p_n - x^* \|$$

where $p = (1 - 1/\ 4[L^2 + 2L + 2])\ \varepsilon\ (0,1)$.

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