ON APPROXIMATE NUMERICAL METHOD OF SOLUTION OF FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT: The present note addresses the problem of determining approximate numerical solutions of Freedholm integral equations of the second kind. Specific examples are examined in detail and the numerical results are tabulated.

Keywords: Integral equation, Approximate solution, Numerical solution.

1. INTRODUCTION

Integral equations are well known to occur, in a natural way, while trying to solve varieties of boundary value problems in different areas of applied mathematics, physics and engineering. The Fredholm integral equation of the second kind is one of the most useful ones. A number of approximate methods, such as quadrature collocation method, the Galerkin expansion method, Taylor's series expansion method, etc., have already been proposed for solving such equations. Neverthless, an efficient low-cost solution of this equation has remained a scientific inquiry. Analytical approximate methods of solution of Fredholm integral equation involves the usage of analytical approximate representations for the unknown function under consideration (see Mandal and Bhattacharya [1], Chakrabarti *et al*., [2], Chakrabarti and Mandal [3], Goldbarg and Chen [4], Kanwal [5], Mandal and Bera [6] and Polyanin and Manzhirov [7]), and this has been explained by Chakrabarti and Martha [11] by utilizing Least Squares Method. The vastly used numerical approximation to the solution of a second kind integral equation of the Fredholm type is the well-known Nystrom method (see S. Rahbar and E.Hashemizadeh [8], Richard J. Hanson [9], Jean-Paul Berrut and Manfred R. Trummer [10] etc.)

It is shown in the present note that a numerical approximate formula for obtaining the solution of a given integral equation can be derived, by utilizing a method which involves approximating the unknown function as well as the kernel directly as linear combinations of Lagrange polynomials. We find that the well-known Nystrom method, which replaces just the integral by a quadrature, is different from the presently derived method. We can use both of the solution formulas derived, for finding the numerical solutions of those integral equations for which analytical solutions are either dicult to obtain or not possible to derive.

We find that the present method produces more exact solution than the Nystrom method.

Several examples are worked out in detail to validate this fact.

2. THE DETAILS

In this note we have examined the problem of obtaining approximate numerical solution of a Fredholm integral equation of the second kind, in a style, giving to rise to a system of linear algebraic equations, which is different from the one obtained in the Nystrom method.

Integral equations are results of transformation of points in a given vector space of integrable functions to points in the same space by the use of specific integral operators.

The Fredholm integral equation of the second kind under consideration is of the form

$$\phi(x) + \lambda \int_a^b k(x,t)\phi(t)\,dt = f(x), \qquad a < x < b \tag{2.1}$$

and the goal is to determine its solution $\phi(x)$.

We assume that,any integrable function $g(x)$ of the real variable $x \epsilon (a, b)$ is approximately expressible as a linear combination of the Lagrange's polynomials $l_k(x)$, (k = 1, 2, 3, ..., n), as given by the relation;

$$g(x) \simeq \sum_{k=1}^{n} \hat{g}(x_k) l_k(x), \tag{2.2}$$

with

$$l_k(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_1)(x_k-x_2)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}, \tag{2.3}$$

where x_k's are fixed different points in (a, b).

Then we get from (2.1), the following approximate relation,

$$\sum_{j=1}^{n} \hat{\phi}(x_j) l_j(x) + \lambda \sum_{j=1}^{n} \hat{\phi}(t_j) \sum_{k=1}^{n} \alpha_{jk} \hat{k}(x, t_k) \simeq \sum_{j=1}^{n} \hat{f}(x_j) l_j(x), \tag{2.4}$$

where, $\alpha_{jk} = \int_a^b l_j(t) l_k(t)\,dt$.

Again the relation (2.4) can be expressed as,

$$\sum_{j=1}^{n} \left(\hat{\phi}(x_j) + \lambda \sum_{p=1}^{n} \hat{\phi}(t_p) \sum_{k=1}^{n} \alpha_{pk} \hat{\hat{k}}(x_j, t_k) \right) l_j(x) \simeq \sum_{j=1}^{n} \hat{f}(x_j) l_j(x).$$

This implies that

$$\hat{\phi}(x_j) + \lambda \sum_{p=1}^{n} \hat{\phi}(t_p) \sum_{k=1}^{n} \alpha_{pk} \hat{\hat{k}}(x_j, t_k) \simeq \hat{f}(x_j) \tag{2.5}$$

$(j = 1, 2, 3, ..., n)$

Now, with $x_j = t_p$, $j = p = 1, 2, ..., n$ the relation (2.5) is written in matrix notation as,

$$(I + \lambda \hat{\alpha} K)\Phi \simeq \bar{f}\ , \tag{2.6}$$

where, $K = [\hat{\hat{k}}(x_j, t_k)]^T$, $\hat{\alpha} = [\alpha_{pk}]^T$, $\hat{\Phi} = [\hat{\phi}(x_j)]^T$ and $\bar{f} = [\hat{f}(x_j)]^T$ with $[a_{ij}]^T$ representing the transpose of the matrix A_{ij} and I representing the $n \times n$ identity matrix.

We can thus obtain the numerical values of $\hat{\phi}(x_j)$ after solving the linear algebraic system of equations (2.6).

The solution formula for the unknown function $\phi(x)$ in question is then given by

$$\phi(x) \simeq \sum_{k=1}^{n} \hat{\phi}(x_k) l_k(x)\ . \tag{2.7}$$

Now, our new system of n algebraic equations (2.5), for the n unknowns is different from the well-known Nystrom system as describe below.

The Nystrom method for solving the integral equation (2.1), after using a quadrature involving Lagrange polynomials $l_k(x)$,) $(k = 1, 2, 3, ..., n)$ gives the approximate solution as

$$\phi(x) \simeq f(x) - \lambda \sum_{i=1}^{n} w_i k(x, t_i) \phi(t_i)\ , \tag{2.8}$$

where $w_i = \int_a^b l_i(t)\, dt$ are the weights of the quadrature, with $l_i(t)$ representing the Lagrange polynomial as given by the relation (2.3).

We obtain from the relation (2.8), by taking $x = x_i$ $(i = 1, 2, 3, ..., n)$, the following Nystrom system of equations, to determine the unknowns $\phi(x_i)$ $[x_i = t_i; i = 1, 2, 3, ..., n]$

$$\phi(x_i) \simeq f(x_i) - \lambda \sum_{j=1}^{n} w_j K(x_i, t_j) \phi(t_j)\ , \tag{2.9}$$

$(i = 1, 2, 3, ..., n)$

i.e., in matrix notation,

$$(I + \lambda \bar{W} \bar{K})\Phi \simeq \bar{f}\ , \tag{2.10}$$

where, $W = [w_j]^T$, $K = [K(x_i, t_j)]^T$, $\Phi = [\phi(x_i)]^T$ and $f = [f(x_i)]^T$.

The two different approximate formulas (2.7) and (2.8) produce different errors, and the mathematical expressions for these errors can be written down easily, by utilizing the well-known error formulas of Lagrange interpolation and the numerical quadrature associated with it, as discussed in the next section.

3. THE ERRORS INVOLVED

Here we have discussed the errors related with the Nystrom method and the new method. In Nystrom method, by using $e_j (j = 1, 2, 3, ..., n)$ as the various errors involved, we have

$$\phi(x) + \lambda \int_a^b K(x, t)\phi(t)\, dx = f(x), \qquad a < x < b$$

giving

$$\phi(x) + \lambda \sum_{i=1}^{n} w_i K(x, t_i)\phi(t_i) + e_1 = f(x)$$

i.e.,

$$\sum_{i=1}^{n} \phi(x) l_i(x) + e_2 + \lambda \sum_{i=1}^{n} w_i \left(\sum_{j=1}^{n} k(x_j, t_i) l_j(x) + e_3 \right) \phi(t_i) + e_1 = \sum_{i=1}^{n} f(x_i) l_i(x) + e_4$$

$$\sum_{i=1}^{n} \phi(x_i) l_i(x) + e_2 + \lambda \sum_{j=1}^{n} \sum_{i=1}^{n} w_i k(x_j, t_i) l_j(x) + e_3 \phi(t_i) + e_1 + \sum_{i=1}^{n} w_i \phi(t_i) e_3 + e_1 = \sum_{i=1}^{n} f(x_i) l_i(x) + e_4 \quad (3.11)$$

where

$$\left.\begin{aligned} e_1 &= \frac{[k(x, \xi)\phi(\xi)]^{n+1}}{(n+1)!} \int_a^b \prod_{i=1}^{n} (t - t_i)\, dt, \quad \xi \epsilon [a, b] \\ e_2 &= \frac{\phi^{n+1}(\beta)}{(n+1)!} \prod_{i=1}^{n} (x - x_i), \quad \beta \epsilon [a, b] \\ e_3 &= \frac{[k(x, \xi)]^{n+1}}{(n+1)!} \int_a^b \prod_{i=1}^{n} (t - t_i)\, dt, \quad \eta \epsilon [a, b] \\ e_4 &= \frac{f^{n+1}(\theta)}{(n+1)!} \prod_{i=1}^{n} (x - x_i), \quad \theta \epsilon [a, b] \end{aligned}\right\} \quad (3.12)$$

Let, the total error in the left hand side in the Nystrom method be E_1 i.e.,

$$e_1 + e_2 + \sum_{i=1}^{n} w_i \phi(t_i) e_3 = E_1$$

and the error in the right hand side be $e_4 = E_2$.

We note that ξ, β, η and θ∊ [*a*, *b*] are certain special points in the open interval (*a*, *b*), and $w_i = \int_a^b l_i(t)\,dt$.

So, if $E_1 = E_2$ for some numbers ξ, β, η and θ∊ [*a*, *b*] then the Nystrom method gives the exact solutions. The error related with the new method can be studied in a manner as described below.

We have

$$\phi(x) + \lambda \int_a^b K(x,t)\,\phi(t)\,dx = f(x), \qquad a < x < b,$$

giving

$$\hat{\phi}(x)l_i(x)+\hat{e}_1+\lambda\int_a^b\left(\sum_{k=1}^{n}\hat{K}(x,t_k)l_k(t)+\hat{e}_2\right)\left(\sum_{p=1}^{n}\hat{\phi}(t_p)l_p(t)+\hat{e}_3\right)dt=\sum_{i=1}^{n}\hat{f}(x_i)l_i(x)+\hat{e}_4$$

i.e.,

$$\hat{\phi}(x)l_i(x)+\hat{e}_1+\lambda\int_a^b\left[\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\hat{K}(x,t_k)l_i(x)+\hat{e}_5\right)l_k(t)+\hat{e}_2\right]\left(\sum_{p=1}^{n}\hat{\phi}(t_p)l_p(t)+\hat{e}_3\right)dt=\sum_{i=1}^{n}\hat{f}(x_i)l_i(x)+\hat{e}_4 \quad (3.13)$$

where,

$$\alpha_{kp} = \int_a^b l_k(t)\,l_p(t)\,dt$$

$$w_k = \int_a^b l_k(t)\,dt$$

$$\left.\begin{aligned}
\hat{e}_1 &= \frac{\phi^{n+1}(\xi)}{(n+1)!}\prod_{i=1}^{n}(x-x_i), \quad \xi\epsilon[a,b]\\
\hat{e}_2 &= \frac{[\hat{k}(x,\eta)]^{n+1}}{(n+1)!}\prod_{i=1}^{n}(t-t_i), \quad \eta\epsilon[a,b]\\
\hat{e}_3 &= \frac{\hat{\phi}^{n+1}(\beta)}{(n+1)!}\prod_{i=1}^{n}(t-t_i), \quad \beta\epsilon[a,b]\\
\hat{e}_4 &= \frac{f^{n+1}(\beta)}{(n+1)!}\prod_{i=1}^{n}(x-x_i), \quad \theta\epsilon[a,b]\\
\hat{e}_5 &= \frac{[\hat{k}(\rho,\eta)]^{n+1}}{(n+1)!}\prod_{i=1}^{n}(x-x_i), \quad \rho\epsilon[a,b]
\end{aligned}\right\} \quad (3.14)$$

So, the new method gives exact solution if

$$\hat{e}_1+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{p=1}^{n}\left[\hat{\hat{k}}(x_i,t_k)l_i(x)w_k\,\hat{e}_3+\hat{e}_5\,\hat{\phi}(t_p)\alpha_{pk}+\hat{e}_5\hat{e}_3w_k+\hat{e}_2p\hat{h}i(t_p)w_p\right]+\hat{e}_2\hat{e}_3(b-a)=\hat{e}_4$$

where,

$$\alpha_{kp}=\int_a^b l_k(t)\,l_p(t)\,dt$$

$$w_k=\int_a^b l_k(t)\,dt\,.$$

We have taken up four illustrative examples and tabulated various numerical results.

4. ILLUSTRATIVE EXAMPLES

Example 1:

$$\phi(x)+\int_0^1(x,t)\phi(t)\,dt=2x,\qquad 0<x<1\,.$$

Solution: The exact solution of the integral equation is

$$\phi(x)=\frac{36x}{23}-\frac{8}{23}\,.$$

Now we consider five points, such as $x_1=0.1$, $x_2=0.3$, $x_3=0.5$, $x_4=0.7$, $x_5=0.9$.

Then by applying the Algorithm (3.12) to (3.13) with $n=5$, we get the the exact solution for any chosen $\xi, \beta, \eta, \theta \epsilon [0.1]$.

Again by applying Algorithm (3.14) to (3.15),we get the exact solution for any chosen $\xi, \beta, \eta, \theta, \rho \epsilon [0.1]$.

The numerical results for $\phi(x_1)$, $\phi(x_2)$, $\phi(x_3)$, $\phi(x_4)$, $\phi(x_5)$ are as given below;

Result by new method	*Result by Nystrom method*	*Exact solutions*
−0.191304347826087	−0.191304347826087	−0.191304347826087
0.121739130434782	0.121739130434782	0.121739130434782
0.434782608695652	0.434782608695652	0.434782608695652
0.747826086956521	0.747826086956521	0.747826086956521
1.060869565217391	1.060869565217391	1.060869565217391

Example 2:

$$\phi(x)-\int_0^1(x^2+t^2)\phi(t)\,dt=x^2,\qquad 0<x<1\,.$$

Solution: The exact solution of the integral equation is

$$\phi(x) = \frac{30}{11}x^2 + \frac{9}{11}.$$

Now we consider five points, such as $x_1 = 0.1$, $x_2 = 0.3$, $x_3 = 0.5$, $x_4 = 0.7$, $x_5 = 0.9$.

Then by applying the Algorithm (3.12) to (3.13) with $n = 5$, we get the the exact solution for any chosen $\xi, \beta, \eta, \theta \epsilon [0.1]$.

Again by applying Algorithm (3.14) to (3.15),we get the exact solution for any chosen $\xi, \beta, \eta, \theta, \rho \epsilon [0.1]$.

The numerical results for $\phi(x_1)$, $\phi(x_2)$, $\phi(x_3)$, $\phi(x_4)$, $\phi(x_5)$ are as given below;

Result by new method	*Result by Nystrom method*	*Exact solutions*
0.845454545454546	0.845454545454546	0.845454545454545
1.063636363636364	1.063636363636364	1.063636363636364
1.500000000000001	1.500000000000001	1.500000000000000
2.154545454545456	2.154545454545456	2.154545454545454
3.027272727272730	3.027272727272730	3.027272727272727

Example 3:

$$\phi(x) - \int_0^1 (1 - 3xt)\phi(t)\,dt = 1, \qquad 0 < x < 1.$$

Solution: The exact solution of the integral equation is

$$\phi(x) = \frac{8}{3} - 2x.$$

For this example errors do not exist in both the methods described above if we consider five nodal points as given by $x_1 = 0.1$, $x_2 = 0.3$, $x_3 = 0.5$, $x_4 = 0.7$, $x_5 = 0.9$.

The following table shows the above situation.

The numerical results for $\phi(x_1)$, $\phi(x_2)$, $\phi(x_3)$, $\phi(x_4)$, $\phi(x_5)$ are as given below;

Result by new method	*Result by Nystrom method*	*Exact solutions*
2.466666666666669	2.466666666666667	2.466666666666666
2.066666666666669	2.066666666666666	2.066666666666666
1.666666666666668	1.666666666666667	1.666666666666667
1.266666666666668	1.266666666666667	1.266666666666667
0.866666666666667	0.866666666666666	0.866666666666666

Example 4:

$$\phi(x) = 1 + \int_0^1 (1 + e^{x+t})\phi(t)\,dt, \qquad 0 < x < 1.$$

Solution: The exact solution of the integral equation is

$$\phi(x) = \frac{e^2 - 3}{2(e-1)^2} - \frac{e^x}{e-1}.$$

Taking, $x_1 = 0.1$, $x_2 = 0.3$, $x_3 = 0.5$, $x_4 = 0.7$, $x_5 = 0.9$.

Now applying the Algorithm (3.12) to (3.13) with $n = 5$, there exist some error, for any chosen $\xi, \beta, \eta, \theta \in [0.1]$.

Again by applying Algorithm (3.14) to (3.15), there exist some error for any chosen $\xi, \beta, \eta, \theta, \rho \in [0.1]$.

The numerical results for $\phi(x_1)$, $\phi(x_2)$, $\phi(x_3)$, $\phi(x_4)$, $\phi(x_5)$ are as given below;

Result by new method	*Result by Nystrom method*	*Exact solutions*
0.100091321128117	0.100073947282368	0.100096088101445
−0.042311381805276	−0.042328755651023	−0.042306564040777
−0.216242435937583	−0.216259809783331	−0.216237556136611
−0.428682305184490	−0.428699679030238	−0.428677349611997
−0.688156947425849	−0.688174321271596	−0.688151899305811

Error in New method	*Error in Nystrom method*
0.00000476697	0.00002214081
0.00000481776	0.00002219161
0.0000048798	0.00002225364
0.00000495557	0.00002232941
0.00000504812	0.00002242196

Example 5:

$$\phi(x) + \int_0^1 (x^4 + t^4)\phi(t)\,dt = x^x, \qquad 0 < x < 1.$$

Solution: The exact solution of the integral equation is

$$\phi(x) = x^4\left(\frac{270}{299}\right) - \frac{25}{299}.$$

For this example the approximated result by using Nystrom method, is so far from the exact solution whereas the new method gives the exact solution if we consider five points as $x_1 = 0.1$, $x_2 = 0.3$, $x_3 = 0.5$, $x_4 = 0.7$, $x_5 = 0.9$.

The following table proves the above situation.

The numerical results for $\phi(x_1)$, $\phi(x_2)$, $\phi(x_3)$, $\phi(x_4)$, $\phi(x_5)$ are respectively

Result by new method	*Result by Nystrom method*	*Exact solutions*
−0.083521739130435	−0.081910520171204	−0.083521739130435
−0.076297658862876	−0.074697182258570	−0.076297658862876
−0.027173913043478	−0.025646484452658	−0.027173913043478
0.133200668896321	0.134489617207818	0.133200668896321
0.508852842809365	0.509583188664791	0.508852842809365

Now we will take a special type of integral equation called Love's equation for which analytical solution does not exist. The Love's equation is of the form

$$\phi(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{f(t)\,dt}{1+(x-t)^2} = 1, \qquad -1 \le x \le 1.$$

Let us take five points as, $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = 0$, $x_4 = -\frac{1}{2}$, $x_5 = -1$.

Then the numerical results for $\phi(x_1)$, $\phi(x_2)$, $\phi(x_3)$, $\phi(x_4)$, $\phi(x_5)$ are respectively

Result by new method	*Result by Nystrom method*
1.650874296555539	1.641840908299119
1.847426086605967	1.847059667925486
1.914686914520872	1.912582653473487
1.847097306264008	1.847059667925485
1.641390079147116	1.641840908299119

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