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AGGREGATION OPERATORS ON DISCRETE FUZZY NUMBER

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Abstract: In this paper some special general aggregation operators are given with respect to different ordering on the set of all discrete fuzzy number. Further point wise extension of aggregation operators and general aggregation operators with respect to the ordering is established.

Keywords: general aggregation operator, discrete fuzzy number, extension principle, ordering on fuzzy numbers

1. INTRODUCTION

Chang and Zadeh[2] introduced the concept of fuzzy number with the consideration of the properties of probability functions. Since then a lot of mathematicians have been studying on fuzzy numbers. In 2001 Voxman [10] introduced the concept of discrete fuzzy numbers, it can be used to represent the pixel value in the centre point of a window [11].

In order to expand the classical fuzzy arithmetic, general aggregation operators are introduced as special functions on discrete fuzzy numbers of some universe. Classical fuzzy arithmetic is based on the extension principle ([3],[4]) and operations on the values of membership functions of the arguments [5].

In this paper general aggregation operators are defined and also types of possible general aggregation operators are introduced. Further the ordering on the discrete fuzzy number of the set are given. Also general aggregation operators of type1 and type2 are viewed with respect to the orderings are developed.

2. PRELIMINARIES

Definition 2.1 [7] (Special fuzzy set)

Let $u \in \mathcal{F}(\mathcal{R})$, the α-cut of u is defined by ${}^{\alpha}u = \{x \in \mathcal{R} / u(x) \geq \alpha\}$ and the special fuzzy set α^{u} is defined by ${}_{\alpha}u = \alpha.{}^{\alpha}u$

2.2. First Decomposition Theorem [7]

Let $u \in \mathcal{F}(\mathcal{R})$ and ${}_{\alpha}u = \alpha.{}^{\alpha}u$ then $u = \bigcup_{\alpha \in (0,1]} {}_{\alpha}u$.

Definition 2.3

A continuous mapping $A:\bigcup_{n\in N}[0,1]^n \to [0,1]$ is an aggregation operator on the unit interval if the following conditions hold.

(A1) A (0, ..., 0) = 0

(A2) A (1, ..., 1) = 1

(A 3) if for every i = 1, 2 ...., n, $x_i \le y_i$ then $A(x_1, x_2, ..., x_n) \le A(y_1, y_2, ..., y_n)$.

Conditions (A1) and (A2) are called boundary conditions, and (A3) resembles the monotonicity property of the operator A.

Definition 2.4 (Ordering on the set F(X))(Fuzzy subset ordering $\subseteq_F$)

For any two fuzzy subsets P, Q of the universe. P is a fuzzy subset of Q denoted by $P \subseteq_F Q$ if the following holds

For every $x \in X$; $\mu_P(x) \le \mu_Q(x)$ where ≤ is an ordering on [0,1].

3. AGGREGATION OPERATORS ON DISCRETE FUZZY NUMBER

Definition 3.1

Let $\tilde{A}$ be a continuous mapping $\tilde{A}:\bigcup_{n\in N}\mathcal{F}(\mathcal{R}) \to \mathcal{F}(\mathcal{R})$ where $\mathcal{F}(\mathcal{R})$ is the set of all discrete fuzzy numbers and $\preccurlyeq$ be the ordering on $\mathcal{F}(\mathcal{R})$. $\tilde{A}$ is called a general aggregation operator on $\mathcal{F}(\mathcal{R})$ if the following properties hold.

($\tilde{A}1$) $\tilde{A}(\tilde{0},..............,\tilde{0}) = \tilde{0}$

($\tilde{A}2$) $\tilde{A}(\tilde{1},..............,\tilde{1}) = \tilde{1}$

($\tilde{A}3$) For any $u_i, v_i \in \mathcal{F}(\mathcal{R}),\ i = 1,2,........,n$ if $u_i \le v_i$

then $\tilde{A}(\mathrm{u}_1, \mathrm{u}_2, ...,\mathrm{u}_n) \le \tilde{A}(\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n)$ where $u_{1,} u_2, ...u_{n,} v_{1,} v_2, ...v_n, \tilde{0}, \tilde{1} \in \mathcal{F}(\mathcal{R})$.

Definition 3.2 General aggregation operators of type1

General aggregation operators $\tilde{A}$ of type 1 are derived from the classical aggregation operators A in the following way as

For every $t \in R$; $\mu_{\tilde{A}(u_1,.........u_n)}(t) = \mathrm{A}(\mu_{u_1}(\mathrm{t}), \mu_{u_2}(\mathrm{t}),........., \mu_{u_n}(\mathrm{t}))$

Definition 3.3

Let $u_{1,}u_2,........u_n \in \mathcal{F}(\mathcal{R})$, $\tilde{A}: \bigcup_{n\in N} \mathcal{F}(\mathcal{R})^n \rightarrow \mathcal{F}(\mathcal{R})$ and A be an ordinary aggregation operator on the unit interval

1. $\tilde{A}$ is a point wise extension of A if the following holds.

   For every $t \in R$, $\mu_{\tilde{A}(u_1,.........u_n)}(t) = A(\mu_{u_1}(t), \mu_{u_2}(t),........., \mu_{u_n}(t))$

   $\mu_{\tilde{A}(u_1,.........u_n)}$ is the membership function of the resulting fuzzy set obtained by applying the operator $\tilde{A}$ to the discrete fuzzy numbers $u_{1,}u_2,........u_n$

2. Let T be a t-norm $\tilde{A}$ is defined as a T extension of an aggregation operator A if $\mu_{\tilde{A}(u_1,.........u_n)}(t) = \sup_{t=A(x_1,.......x_n)} \{\mathrm{T}(\mu_{u_1}(x_1), \mu_{u_2}(x_2),........., \mu_{u_n}(x_n))\}$

3. Let $X_{1,}X_2,.......\mathrm{X}_n, \mathrm{Y}$ be universe and let $u_{1,}u_2,........u_n$ be a discrete fuzzy number of those universes $X_{1,}X_2,.......\mathrm{X}_n$ respectively $\tilde{A}$ is defined as an extension of some increasing operator $\phi: X_1 \times X_2 \times .... \times X_n \rightarrow Y$ with $\mu_{\tilde{A}(u_1,.........u_n)}(t) = \sup_{t=\phi(x_1,.......x_n), x_i \in u_i} \{\mathrm{A}(\mu_{u_1}(x_1), \mu_{u_2}(x_2),........., \mu_{u_n}(x_n))\}$

Definition 3.4

Let $\preccurlyeq$ be an ordering on $\mathcal{R}$ and $u \in \mathcal{F}(\mathcal{R})$

Define ${}_{\alpha}LH(x) = \max\{{}_{\alpha}u(y) / y \le x\}$ and ${}_{\alpha}RH(x) = \max\{{}_{\alpha}u(y) / \mathrm{x} \le y\}$

Define $L(x) = \max_{\alpha\in(0,1]} {}_{\alpha}LH(x)$ and $\mathrm{R}(x) = \max_{\alpha\in(0,1]} {}_{\alpha}RH(x)$

Definition 3.5

Let *u, v* be a discrete fuzzy number such that its corresponding supports ${}^{\alpha}u, {}^{\alpha}v$ are finite,

The ordering ≤ on *u, v* are defined by $u \le v$ if and only if $L(u) \le L(\mathrm{v})$ and $R(v) \le R(u)$

(ie) $u \leq_\alpha v$ if and only if $_\alpha LH(u) \leq {}_\alpha LH(\mathrm{v})$ and $_\alpha RH(v) \leq {}_\alpha RH(u)$.

Definition 3.6

The special fuzzy supersets $_\alpha L(u)$ and $_\alpha R(u)$ are defined by

$$_\alpha L(\mathrm{u})(\mathrm{x}) = \sup\left\{\bigcup_{\alpha\in(0,1]} {}_\alpha u(y) / y \leq x\right\},\ {}_\alpha \mathrm{R}(\mathrm{u})(\mathrm{x}) = \sup\left\{\bigcup_{\alpha\in(0,1]} {}_\alpha u(y) / \mathrm{x} \leq y\right\}$$

Define the ordering $\leq_\alpha$ by

$$u \leq_\alpha v \Leftrightarrow [{}_\alpha L(u)(\mathrm{x}) \supseteq {}_\alpha L(\mathrm{v})(\mathrm{x})] \wedge [{}_\alpha \mathrm{R}(u)(\mathrm{x}) \subseteq {}_\alpha \mathrm{R}(\mathrm{v})(\mathrm{x})]$$

Theorem 3.7

Let $\tilde{A}$ be a general aggregation operator of type 1 (i.e) a point wise extension of an aggregation operator of type 1 (i.e) a point wise extension of an aggregation operator A and the membership functions of fuzzy sets $\tilde{0}, i, \tilde{1}$ respectively are defined by

(1) $\mu_{\tilde{0}}(x) = 0 \quad x \in X$

(2) $\mu_{\tilde{1}}(x) = 1 \quad x \in X$

Then for the operator $\tilde{A}$ and fuzzy sets $\tilde{0}, i, \tilde{1}$ the following conditions hold.

$(\tilde{A}1)$ $\tilde{A}(\tilde{0}, \ldots\ldots\ldots\ldots, \tilde{0}) = \tilde{0}$

$(\tilde{A}2)$ $\tilde{A}(\tilde{1}, \ldots\ldots\ldots\ldots, \tilde{1}) = \tilde{1}$

$(\tilde{A}3)$ if for every $i = 1, 2, \ldots\ldots\ldots, n$ $u_i \subseteq_F v_i$ then $\tilde{A}(\mathrm{u}_1, \mathrm{u}_2, \ldots\ldots, \mathrm{u}_n)$ $\subseteq_F \tilde{A}(\mathrm{v}_1, \mathrm{v}_2, \ldots\ldots, \mathrm{v}_n)$ where $u_{1,} u_2, \ldots\ldots u_{n,} v_{1,} v_2, \ldots\ldots\ldots v_n$ are discrete fuzzy numbers.

Proof

since $(\tilde{A}1)$ $\mu_{\tilde{A}(\tilde{0}, \ldots\ldots\ldots \tilde{0})}(t) = \mathrm{A}(\mu_{\tilde{0}}(\mathrm{t}), \mu_{\tilde{0}}(\mathrm{t}), \ldots\ldots\ldots, \mu_{\tilde{0}}(\mathrm{t}))$

$$= \mathrm{A}(0, \ldots\ldots\ldots\ldots, 0)$$
$$= 0$$

Since the upper presumption holds for every $\mathcal{R}$ it concludes that $\tilde{A}(\tilde{0},.............,\tilde{0}) = \tilde{0}$

Similarly, $(\tilde{A}2)$ $\mu_{\tilde{A}(\tilde{1},...........\tilde{1})}(t) = \mathrm{A}(\mu_{\tilde{1}}(\mathrm{t}), \mu_{\tilde{1}}(\mathrm{t}), ..., \mu_{\tilde{1}}(\mathrm{t}))$

$$= \mathrm{A}(1, ..., 1)$$
$$= 1$$

Now to prove $(\tilde{A}3)$ assume discrete fuzzy numbers

$u_{1,} u_2, ...u_{n,} v_{1,} v_2, ...v_n \in \mathcal{F}(\mathcal{R})$ for every $i = 1, 2, ..., n$ with $u_i \subseteq_F v_i$

If $u \leq v$ then ${}_{\alpha}L(u) \leq {}_{\alpha}L(\mathrm{v}) \Rightarrow {}_{\alpha}L(u)(\mathrm{t}) \leq {}_{\alpha}L(\mathrm{v})(\mathrm{t})$

$$\Rightarrow \sup\{\bigcup_{\alpha} u(s) / s \leq t\} \leq \sup\{\bigcup_{\alpha} v(s) / s \leq t\}$$

$$\Rightarrow \bigcup_{\alpha} u(s_0) \text{ (for some } s_0 \leq t\text{)} \subseteq \bigcup_{\alpha} \mathrm{v}(s_0^{/}) \text{ (for some } s_0^{/} \leq t\text{)}$$

$$\Rightarrow {}_{\alpha}u(s_0) \leq_{\alpha} \mathrm{v}(s_0^{/}) \text{ with } s_0, s_0^{/} \leq t$$

$$\Rightarrow \tilde{u}(s) \leq \tilde{v}(s) \; for \;\; s \leq t$$

$$\Rightarrow \tilde{u}(\mathrm{t}) \leq \tilde{v}(\mathrm{t})$$

Now for any $t \in R$, $\tilde{A}(\mathrm{u}_1, \mathrm{u}_2, ..., \mathrm{u}_n)(\mathrm{t}) = \mathrm{A}(\mu_{\tilde{u}_1}(\mathrm{t}), \mu_{\tilde{u}_2}(\mathrm{t}), ..., \mu_{\tilde{u}_n}(\mathrm{t}))$

$$\leq \mathrm{A}(\mu_{\tilde{v}_1}(\mathrm{t}), \mu_{\tilde{v}_2}(\mathrm{t}), ..., \mu_{\tilde{v}_n}(\mathrm{t}))$$

$$= \tilde{A}(\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_n)(\mathrm{t}).$$

3.8. General aggregation operators of type 1 and the ordering $\leq_\alpha$

We will now prove that the general aggregation operators $\tilde{A}$ of type 1 with respect to the ordering $\leq_\alpha$, which do not fulfill the condition $(\tilde{A}3)$ in the general case, the proof is given by a counter example.

Theorem 3.9

Let $\tilde{A}$ be a general aggregation operator of type 1.The next implication does not hold for discrete fuzzy number u_i, v_i and arbitrary operator $\tilde{A}$:

$$\text{If } u \leq_\alpha v \text{ then} [{}_\alpha L(u)(\mathrm{x}) \supseteq {}_\alpha L(\mathrm{v})(\mathrm{x})] \wedge [{}_\alpha \mathrm{R}(u)(\mathrm{x}) \subseteq {}_\alpha \mathrm{R}(\mathrm{v})(\mathrm{x})]$$

Proof

Let $u = \left\{ \frac{0.1}{x_1}, \frac{0.2}{x_2}, \frac{0.1}{x_3}, \frac{0.4}{x_4}, \frac{0.3}{x_5}, \frac{0.6}{x_6}, \frac{0.4}{x_7}, \frac{0.3}{x_8}, \frac{0.2}{x_9} \right\}$ be a discrete fuzzy number

By definition (3.6) ${}_\alpha L(\mathrm{u})(\mathrm{x}) = \sup \left\{ \bigcup_{\alpha \in (0,1]} {}_\alpha u(y) \,/\, y \leq x \right\}$,

${}_\alpha \mathrm{R}(\mathrm{u})(\mathrm{x}) = \sup \left\{ \bigcup_{\alpha \in (0,1]} {}_\alpha u(y) \,/\, \mathrm{x} \leq y \right\}$ the values are tabulated in Table (3.1) and pictorially represented in Fig. (3.1)

Table 3.1

X	${}_\alpha L(u)\,(x)$	${}_\alpha R(u)\,(x)$
x_1	0.1	0.6
x_2	0.2	0.6
x_3	0.2	0.6
x_4	0.4	0.6
x_5	0.4	0.6
x_6	0.6	0.6
x_7	0.6	0.4
x_8	0.6	0.4
x_9	0.6	0.4

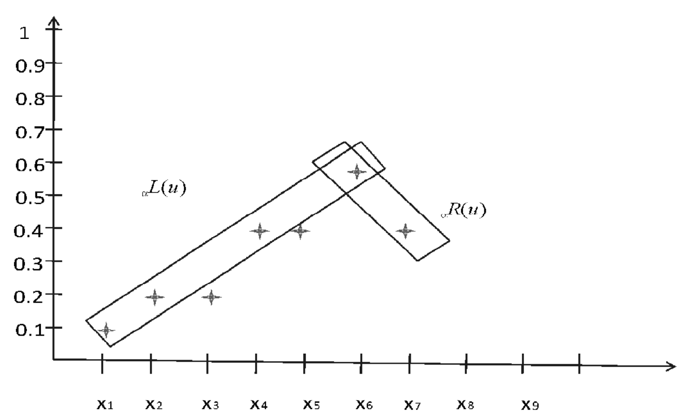

Figure 3.1

Let $v = \left\{ \frac{0.2}{x_1}, \frac{0.4}{x_2}, \frac{0.6}{x_3}, \frac{0.7}{x_4}, \frac{0.3}{x_5}, \frac{0.2}{x_6}, \frac{0.4}{x_7} \right\}$ be a discrete fuzzy number

By definition (3.6) $_{\alpha}L(\text{v})(\text{x}) = \sup \left\{ \bigcup_{\alpha \in (0,1]} {}_{\alpha}\text{v}(y) / y \leq x \right\}$,

$_{\alpha}\text{R}(\text{v})(\text{x}) = \sup \left\{ \bigcup_{\alpha \in (0,1]} {}_{\alpha}\text{v}(y) / \text{x} \leq y \right\}$ the values are tabulated in Table(3.2) and pictorially represented in Fig. (3.2)

Table 3.2

X	$_{\alpha}L(v)\ (x)$	$_{\alpha}R(v)\ (x)$
x_1	0.2	0.7
x_2	0.4	0.7
x_3	0.6	0.7
x_4	0.7	0.7
x_5	0.7	0.4
x_6	0.7	0.4
x_7	0.7	0.4

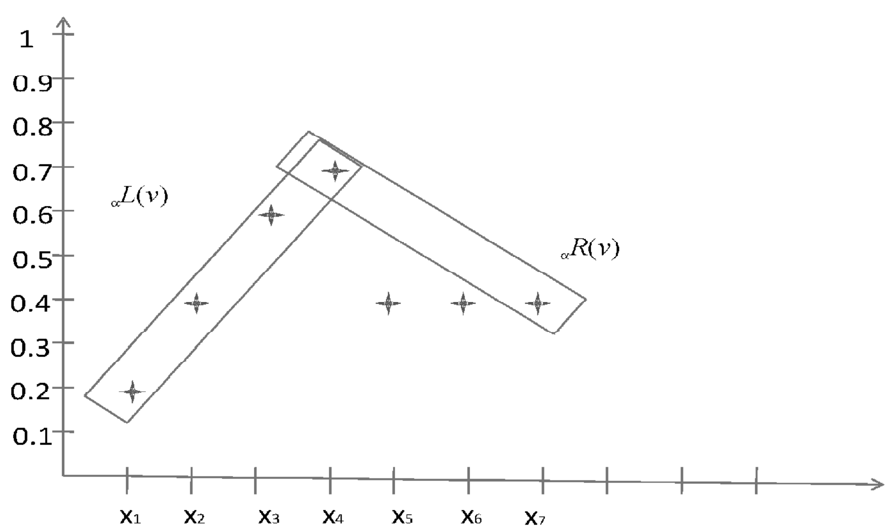

Figure 3.2

On comparing the discrete fuzzy numbers u and v we observe that $u \leq_{\alpha} v$ since $\tilde{u}(\mathrm{t}) \leq \tilde{v}(\mathrm{t})$ but $[{}_{\alpha}L(u)(\mathrm{x}) \supseteq {}_{\alpha}L(\mathrm{v})(\mathrm{x})] \wedge [{}_{\alpha}\mathrm{R}(u)(\mathrm{x}) \subseteq {}_{\alpha}\mathrm{R}(\mathrm{v})(\mathrm{x})]$ does not hold from the above figure (3.1) and figure(3.2) .Hence the theorem is proved.

4. GENERAL AGGREGATION OPERATORS OF TYPE 2 AND THE ORDERING $\leq_{\alpha}$

General aggregation operators of type 2 are a min-extension of an arbitrary aggregation operator A on any interval [a, b];

$$\mu_{\tilde{A}(\mathrm{X}_1,.........\mathrm{X}_n)}(t) = \sup_{t=A(x_1,......x_n),x_i\in X_i} \min(\mu_{u_1}(\mathrm{x}_1),,.........,\mu_{u_n}(\mathrm{x}_n))$$

Theorem4.1

Let $\tilde{A}$ be a general aggregation of type 2 which is a min extension of a classical aggregation operator $A:[a,b]^n \rightarrow [a,b]$ then the operator $\tilde{A}$ is a general aggregation operator of type 2 with respect to $\leq_{\alpha}$ and the discrete fuzzy numbers $\tilde{0}$ and $\tilde{1}$ are defined by $\tilde{0} = \{a\}$ and $\tilde{1} = \{b\}$ respectively.

Proof

To prove that $\tilde{A}$ satisfies condition $(\tilde{\mathrm{A}}1)-(\tilde{\mathrm{A}}3)$

$(\tilde{\mathrm{A}}1)$ $\tilde{A}(\{\mathrm{b}\},...............\{\mathrm{b}\}) = \{\mathrm{b}\}$. Since $\tilde{A}$ is a min extension of A $\tilde{A}(\{b\},........\{b\})(t) = \sup_{t=A(x_1,......x_n)} \min\left\{\mu_{\{b\}}(\mathrm{x}_1),,.........,\mu_{\{b\}}(\mathrm{x}_n)\right\}$, If atleast one of the values x_i is different from b then $\mu_{\{b\}}(x_i) = 0$ and $\min\left\{\mu_{\{b\}}(\mathrm{x}_1),,.........,\mu_{\{b\}}(\mathrm{x}_n)\right\} = 0$

Again if $x_1 = ... = x_n = b$ then $\min\left\{\mu_{\{b\}}(\mathrm{b}),,.........,\mu_{\{b\}}(\mathrm{b})\right\} = 1$

Hence only at the point b the value is 1 and in every other point the value is 0 which proves condition $(\tilde{\mathrm{A}}1)$

Analogously to prove $(\tilde{\mathrm{A}}2)$

Now to prove $(\tilde{A}3)$ that is to prove $u \leq_{\alpha} v \Leftrightarrow$ $[{}_{\alpha}L(u)(\mathrm{x}) \supseteq {}_{\alpha}L(\mathrm{v})(\mathrm{x})] \wedge [{}_{\alpha}\mathrm{R}(u)(\mathrm{x}) \subseteq {}_{\alpha}\mathrm{R}(\mathrm{v})(\mathrm{x})]$

By definition (3.6) ${}_{\alpha}L(\mathrm{u})(\mathrm{x}) = \sup\left\{\bigcup_{\alpha} u(y) / y \leq x\right\}$

That is $v \subseteq {}_{\alpha}L(u)(x)$ is implies there exists $y \in u$ such that $y \leq x$

Moreover, deduce from the minimality that $v \subseteq {}_{\alpha}L(\mathrm{v})(x)$ hence it follows that

$$v \subseteq {}_{\alpha}L(u)(x) \text{ if and only if } {}_{\alpha}L(u)(\mathrm{x}) \supseteq {}_{\alpha}L(\mathrm{v})(\mathrm{x})$$

So the equivalence condition for every $x \in v$ or there exists $y \in u$ such that $y \leq x$ if and only if ${}_{\alpha}L(u)(\mathrm{x}) \supseteq {}_{\alpha}L(\mathrm{v})(\mathrm{x})$ holds

On applying analogous arguments it follows that for every $x \in u$ there exists $y \in v$ such that $\mathrm{x} \leq y$ if and only if ${}_{\alpha}\mathrm{R}(u)(\mathrm{x}) \subseteq {}_{\alpha}L(\mathrm{v})(\mathrm{x})$

From the above two equivalences we conclude $u \leq_{\alpha} v$ if and only if

$$[{}_{\alpha}L(u)(\mathrm{x}) \supseteq {}_{\alpha}L(\mathrm{v})(\mathrm{x})] \wedge [{}_{\alpha}\mathrm{R}(u)(\mathrm{x}) \subseteq {}_{\alpha}\mathrm{R}(\mathrm{v})(\mathrm{x})]$$

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