International Review of Pure and Applied Mathematics
Volume 14 • Number 2 • July-December 2018

Some Generalized Fractional Integral Formulae Involving the Family of Extended Generalized Gauss Hypergeometric Functions

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Abstract: We aim at establishing certain new image formulas of family of some extended generalized Gauss hypergeometric functions by applying generalized form of Riemann-Liouville fractional integrals operators. Furthermore, by employing some integral transforms on the resulting formulas, we presented some more image formulas the results obtained here are quite general in nature and capable of yielding a very large number of known and (presumably) new results.

Keywords: Pochhemmer symbol, Fractional Calculus, Family of extended Gauss hypergeometric function, Laplace Transform, Beta Transform, Fractional Integration.

2010 MSC: 26A33, 33C45, 33C60, 33C70

1. INTRODUCTION

For our present study we recall some required special functions.

The generalized Beta function $B_p^{(\delta,\zeta;\kappa,\mu)}(x,y)$ is defined by (see [1])

$$B_p^{(\delta,\zeta;\kappa,\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1(\delta;\zeta;-\frac{p}{t^{\kappa}(1-t)^{\mu}})dt, \tag{1.1}$$

$$\left(\Re(p)\geq 0;\min\left[\Re(x),\Re(y),\Re(\delta),\Re(\zeta)\right]>0;\min\left[\Re(\kappa),\Re(\mu)\right]>0\right)$$

When $\kappa=\mu$ (1.1) reduces to the generalized extended beta function $B_p^{(\delta,\zeta;\mu)}(x,y)$ defined by (see [2, p.37]).

$$B_p^{(\delta,\zeta;\mu)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\delta;\zeta;-\frac{p}{t^{\mu}(1-t)^{\mu}}\right)dt, \tag{1.2}$$

$$\left(\Re(p)\geq 0;\min\left[\Re(x),\Re(y),\Re(\delta),\Re(\zeta)\right]>0;\Re(\mu)>0\right).$$

The special case of (1.2), when $\mu=1$ reduces to the generalized Beta type function as follows (see [3, p.4602])

$$B_p^{(\delta,\zeta)}(x,y) = B_p^{(\delta,\zeta;1)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\delta;\zeta;-\frac{p}{t(1-t)}\right)dt, \tag{1.3}$$

$$\left(\Re(p)\geq 0;\min\left[\Re(x),\Re(y),\Re(\delta),\Re(\zeta)\right]>0\right).$$

The further special case of (1.3) when $\delta=\zeta$ is given due to Choudhary et al. [4] by

$$B_p(x,y) = B_p^{(\delta,\delta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t(1-t)}\right)dt,\ \left(\Re(p)\geq 0\right) \tag{1.4}$$

The classical Beta function $\mathrm{B}(x,y)$ is defined by

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ \left(\Re(x)>0,\Re(y)>0\right) \tag{1.5}$$

It is clear that there is following relationship between the classical Beta function $\mathrm{B}(x,y)$ and its extensions:

$$\mathrm{B}(x,y) = B_0(x,y) = B_0^{(\delta,\zeta)}(x,y) = B_0^{(\delta,\zeta;1)}(x,y) = B_0^{(\delta,\zeta;1,1)}(x,y)$$

The generalized hypergeometric series ${}_pF_q\left(p,q\in\mathbb{N}\right)$ is defined as (see [5,p.73]) and (6,pp. 71-75)]:

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots\ldots\ldots,\alpha_p;\\ \beta_1,\ldots\ldots\ldots,\beta_q;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots\ldots(\alpha_p)_n}{(\beta_1)_n\ldots\ldots(\beta_q)_n}\frac{z^n}{n!} \tag{1.6}$$

$$= {}_pF_q\left(\alpha_1,.........,\alpha_p;\beta_1,.........,\beta_q;z\right),$$

Where $(\lambda)_n$ is the Pochhammer symbol defined for $(\lambda \in \mathbb{C})$ by (see [6. P.2 and p.5]):

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)...........(\lambda+n-1) & (n\in\mathbb{N}) \end{cases} \tag{1.7}$$

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{1.8}$$

and $\mathbb{Z}_0^-$ denotes the set of Non-Positive integers, where $\Gamma(\lambda)$ is familiar Gamma function.

Chaudhry et al. [7,p.591,Eqs. (2.1 and (2.2))] made use of the extended Beta functions $B_p(x,y)$ in (1.4) to extend the Gauss hypergeometric function ${}_2F_1$ as follow: The extended Gauss hypergeometric function $F_p(a,b;c;z)$ is defined as

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.9}$$

$$\left(|z|<1;\Re(c)>\Re(b)>0;\Re(p)\geq 0\right)$$

Similarly, by appealing to the generalized Beta function $B_p^{(\delta,\delta)}(x,y)$ in (1.3) Ozergin [8] and Ozergin et al. [3] introduced and investigated a further potentially useful extensions of the generalized hypergeometric functions as follows: The extended generalized Gauss hypergeometric function $F_p^{(\delta,\zeta)}(.)$ is defined by

$$F_p^{(\delta,\zeta)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\delta,\zeta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.10}$$

$$\left(|z|<1;\min\left\{\Re(\delta),\Re(\zeta)\right\}>0;\Re(c)>\Re(b)>0;\Re(p)\geq 0\right)$$

Based upom the generalized Beta function in (1.2), Parmar [2] introduced and studied a family of the generalized Gauss hypergeometric function $F_p^{(\delta,\zeta,\mu)}(.)$ defined by

$$F_p^{(\delta,\zeta;\mu)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.11}$$

$$\left(|z|<1;\min\{\Re(\delta),\Re(\zeta),\Re(\mu)\}>0;\Re(c)>\Re(b)>0;\Re(p)\geq 0\right)$$

Recently, Srivastava et al. [9] used the generalized Beta function in (1.1) to introduce a family of some extended generalized Gauss hypergeometric function defined by

$$F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.12}$$

$$\left(|z|<1;\min\{\Re(\delta),\Re(\zeta),\Re(\kappa),\Re(\mu)\}>0;\Re(c)>\Re(b)>0;\Re(p)\geq 0\right)$$

It is easy to see the following relationship:

$$F_p^{(\delta,\zeta;1,1)}(a,b;c;z)=F_p^{(\delta,\zeta)}(a,b;c;z)$$

$$F_p^{(\delta,\zeta;1)}(a,b;c;z)=F_p^{(\delta,\zeta)}(a,b;c;z)$$

$$F_p^{(\delta,\delta;1)}(a,b;c;z)=F_p(a,b;c;z)$$

$$F_0^{(\delta,\delta;1)}(a,b;c;z)={}_2F_1(a,b;c;z).$$

The Fox-Wright function ${}_p\Psi_q$ defined as (see, for details, Srivastava and Karlsson 1985, [10]).

$${}_p\Psi_q[z]={}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots\ldots\ldots\ldots(a_p,\alpha_p);\\(b_1,\beta_1),\ldots\ldots\ldots\ldots(b_p,\beta_p);\end{matrix}\;z\right]$$

$$= {}_p\Psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p};\\(b_j,\beta_j)_{1,q};\end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{\Pi_{i=0}^{p}\Gamma(a_i+\alpha_i n)}{\Pi_{i=0}^{p}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}, \tag{1.13}$$

Where the coefficients $\alpha_1,.............,\alpha_p,\beta_1,..........,\beta_q \in \mathbb{R}^+$ such that

$$1+\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i \geq 0 \tag{1.14}$$

To establish the image formulas, we require the following concept of the Hadamard products (see[11]).

DEFINITIONS - 1.

Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ and $g(z) := \sum_{n=0}^{\infty} b_n z^n$ be two power series whose radii of convergence are given by R_f and R_g , respectively, then their Hadamard product is power series defined by

$$\left(f * g\right)(z) := \sum_{n=0}^{\infty} a_n b_n z^n, \tag{1.15}$$

whose radius of convergence R satisfies $R_f.R_g \leq R$.

2. FRACTIONAL INTEGRAL FORMULAE INVOLVING THE GENERALIZED GAUSS HYPERGEOMETRIC FUNCTIONS

In this section, we will establish some fractional integral formulas for the generalized Gauss hypergeometric type functions $F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;z)$ by using certain general pair of fractional integral operators. To do this, we need to recall the following pair of generalized fractional integral operators (which are generalized form of Riemann-Liouville fractional integrals) introduced by Katugampola [12].

LEMMA - 1.

Let $\Omega = [a,b](-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The generalized fractional integral ${}^{\rho}I_{a+}^{\sigma} f$ of order $\sigma \in \mathbb{C}$ for $x > a$ and $\Re(\sigma) > 0$ is defined as

$$\left({}^{\rho}I_{a+}^{\sigma} f\right)(x) = \frac{(\rho)^{1-\sigma}}{\Gamma(\sigma)} \int_a^x \frac{t^{\rho} f(t)}{(x^{\rho} - t^{\rho})^{1-\sigma}} dt, \tag{2.1}$$

Similarly the generalized fractional integral ${}^{\rho}I_{b-}^{\sigma} f$ of order $\sigma \in \mathbb{C}$ for $x < b$ and $\mathbb{R}(\sigma) > 0$ is defined as

$$\left({}^{\rho}I_{b-}^{\sigma} f\right)(x) = \frac{(\rho)^{1-\sigma}}{\Gamma(\sigma)} \int_x^b \frac{t^{\rho} f(t)}{(t^{\rho} - x^{\rho})^{1-\sigma}} dt, \tag{2.2}$$

In our investigation, we choose a = b = 0 the above Lemma 1 reduces to the following form:

LEMMA - 2.

The generalized fractional integral ${}^{\rho}I_{0+}^{\sigma} f$ of order $\sigma \in \mathbb{C}$ for $x > 0$ and $\mathbb{R}(\sigma) > 0$ is defined as

$$\left({}^{\rho}I_{0+}^{\sigma} f\right)(x) = \frac{(\rho)^{1-\sigma}}{\Gamma(\sigma)} \int_0^x \frac{t^{\rho} f(t)}{(x^{\rho} - t^{\rho})^{1-\sigma}} dt, \tag{2.3}$$

Similarly the generalized fractional integral ${}^{\rho}I_{0-}^{\sigma} f$ of order $\sigma \in \mathbb{C}$ for $x < 0$ and $\mathbb{R}(\sigma) > 0$ is defined as

$$\left({}^{\rho}I_{0-}^{\sigma} f\right)(x) = \frac{(\rho)^{1-\sigma}}{\Gamma(\sigma)} \int_x^0 \frac{t^{\rho} f(t)}{(t^{\rho} - x^{\rho})^{1-\sigma}} dt, \tag{2.4}$$

The main results are given in the following theorem.

THEOREM - 1.

Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$, $\min\{\Re(\delta),\Re(\zeta),\Re(\kappa),\Re(\mu)\}>0$; $\Re(c)>\Re(b)>0;\Re(p)\geq 0$, $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$.for $x>0$, we have the following formulae:

$$\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;t)\right)(x)$$

$$=\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right] \tag{2.5}$$

Proof. For convenience, we denote the left-hand side of the result (2.5) by $\mathcal{I}$ and using the definition given in equation (1.12). Further changing the order of integration and summation, which is valid under the conditions of Theorem 1, we have

$$\mathcal{I}=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{1}{n!}\left({}^{\rho}I_{0+}^{\sigma}t^{n+\lambda-1}\right)(x), \tag{2.6}$$

employing the definition of fractional integral given in equation (2.1), the above equation (2.6) reduces to

$$\mathcal{I}=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{1}{n!}\frac{(\rho)^{1-\sigma}}{\Gamma(\sigma)}\int_0^x\frac{t^{\rho+n+\lambda-1}}{\left(x^{\rho}-t^{\rho}\right)^{1-\sigma}}dt, \tag{2.7}$$

Choose $t^{\rho}=x^{\rho}z$ in equation (2.7), after simplification we have

$$\mathcal{I}=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{1}{n!}x^{n+\lambda+\sigma\rho}\frac{(\rho)^{-\sigma}}{\Gamma(\sigma)}\int_0^1 z^{\frac{n+\lambda}{\rho}}(1-z)^{\sigma-1}dz, \tag{2.8}$$

further using the definition of the Beta integral in (2.8), after simplification we get

$$\mathcal{J}=\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{x^n}{n!}\frac{\Gamma\left(\frac{\lambda}{\rho}+1+\frac{1}{\rho}n\right)}{\Gamma\left(\frac{\lambda}{\rho}+\sigma+1+\frac{1}{\rho}n\right)}, \tag{2.9}$$

Interpreting the above the above (2.9) in the view of the Hypergeometric function given in equation (1.12) and Wright function following the definition of Hadmard product given in equation (1.15) for two Series of function, we have the required result.

THEOREM - 2.

Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$, $\min\{\Re(\delta),\Re(\zeta),\Re(\kappa),\Re(\mu)\}>0;\ \Re(c)>\Re(b)>0;\Re(p)\geq 0$, $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, for $x<0$, we have the following formulae:

$$\left({}^{\rho}I_{0-}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;t)\right)(x)$$

$$=(-1)^{\sigma}\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\\left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right] \tag{2.10}$$

Proof.

The proof of the Theorem 2 would run parallel to that of Theorem 1. Therefore, the same is skipped here.

In the following sections, we establish certain theorems involving the results obtained in this section associated with the integral transforms like Beta transform and Laplace Transform.

3. IMAGE FORMULAE ASSOCIATED WITH BETA TRANSFORM

The Beta transform of f(z) is defined as [13]

$$B\{f(z):a,b\}=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz \tag{3.1}$$

THEOREM 3.

Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$, $\min\{\Re(\delta),\Re(\zeta),\Re(\kappa),\Re(\mu)\}>0$; $\Re(c)>\Re(b)>0;\Re(p)\geq 0$, $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, for $x>0$, we have the following formulae:

$$B\left\{\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}F_{p}^{(\delta,\zeta;\kappa,\mu)}(a,b;c;tz)\right)(x):l,m\right\}$$

$$=\Gamma(m)\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_{p}^{(\delta,\zeta;\kappa,\mu)}(a,b;c;x)*{}_2\Psi_2\left[\begin{matrix}(l,1),\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ (l+m,1),\left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right]. \tag{3.2}$$

Proof.

For convenience, we denote the left-hand side of the result (3.2) by $\mathcal{B}$, then using the definition of Beta transform, we have

$$\mathcal{B}=\int_0^1 z^{l-1}(1-z)^{m-1}\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}F_{p}^{(\delta,\zeta;\kappa,\mu)}(a,b;c;tz)\right)(x)dz, \tag{3.3}$$

now using the definition of family of extended Gauss hypergeometric function given in (1.12) and then changing the order of integration and summation, we have

$$\mathcal{B}=\sum_{n=0}^{\infty}(a)_n\frac{B_{p}^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{1}{n!}\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda+n-1}\right)(x)\int_0^1 z^{l+n-1}(1-z)^{m-1}dz, \tag{3.4}$$

applying the result in equation (2.9) and the integral formula after simplification the equation (3.4) reduced to the following form:

$$\mathcal{B}=\Gamma(m)\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}\sum_{n=0}^{\infty}(a)_n\frac{B_{p}^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{\Gamma\left(\frac{\lambda}{\rho}+1+\frac{1}{\rho}n\right)}{\Gamma\left(\frac{\lambda}{\rho}+\sigma+1+\frac{1}{\rho}n\right)}\frac{\Gamma(l+n)}{\Gamma(l+m+n)}\frac{x^n}{n!} \tag{3.5}$$

Interpreting the above the above (3.5) in the view of the Hypergeometric function given in equation (1.12) and Wright function following the definition of Hadmard product given in equation (1.15) for two series of function, we have the required result.

THEOREM 4.

Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$,

$\min\{\Re(\delta),\Re(\zeta),\Re(\kappa),\Re(\mu)\}>0;\ \Re(c)>\Re(b)>0;\Re(p)\geq 0$,

$\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, for $x<0$, we have the following formulae:

$$B\left\{\left({}^{\rho}I_{0-}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;tz)\right)(x):l,m\right\}$$

$$=(-1)^{\sigma}\,\Gamma(m)\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;x)*{}_2\Psi_2\left[\begin{matrix}(l,1),\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\(l+m,1),\left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right]\quad(3.6)$$

Proof.

The proof of the Theorem 4 is same as that of Theorem 3. Therefore, it is omitted.

4. IMAGE FORMULAE ASSOCIATED WITH LAPLACE TRANSFORM

The Laplace transform f(z) is defined as [13] :

$$L\{f(z)\}=\int_0^{\infty}e^{-sz}f(z)dz\quad(4.1)$$

THEOREM 5.

Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$,
$\min\{\Re(\delta),\Re(\zeta),\Re(\kappa),\Re(\mu)\}>0;\ \Re(c)>\Re(b)>0;\Re(p)\geq 0$,
$\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, for $x>0$, we have the following formulae:

$$L\left\{z^{l-1}\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;tz)\right)(x)\right\}$$

$$=\frac{x^{\lambda+\sigma\rho}}{s^l\rho^{\sigma}}F_p^{(\delta,\zeta;\kappa,\mu)}\left(a,b;c;\frac{x}{s}\right)*{}_2\Psi_1\left[\left.\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right),(l,1)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\right|\frac{x}{s}\right] \tag{4.2}$$

Proof.

For convenience, we denote the left-hand side of the result (4.2) by $\mathcal{L}$. After using the definition of Laplace transform, we have:

$$\mathcal{L}=\int_0^{\infty}e^{-sz}z^{l-1}\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;tz)\right)(x)dz \tag{4.3}$$

using the result from (2.9), then changing the order of integration and summation, the above equation (4.3) reduces to the following form:

$$\mathcal{L}=\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{x^n}{n!}\frac{\Gamma\left(\frac{\lambda}{\rho}+1+\frac{1}{\rho}n\right)}{\Gamma\left(\frac{\lambda}{\rho}+\sigma+1+\frac{1}{\rho}n\right)}\int_0^{\infty}e^{sz}z^{l+n-1}dz, \tag{4.4}$$

after simplification the above equation (4.4) reduces to the following the form:

$$\mathcal{L}=\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{x^n}{n!}\frac{\Gamma\left(\frac{\lambda}{\rho}+1+\frac{1}{\rho}n\right)}{\Gamma\left(\frac{\lambda}{\rho}+\sigma+1+\frac{1}{\rho}n\right)}\frac{\Gamma(l+n)}{s^{l+n}}, \tag{4.5}$$

Interpreting the above (4.5) in the view of the Hypergeometric function given in equation (1.12) and Wright function following the definition of Hadmard product given in equation (1.15) for two series of function, we have the required result.

THEOREM 6.

Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$,

$\min\{\Re(\delta),\Re(\zeta),\Re(\kappa),\Re(\mu)\}>0;\ \Re(c)>\Re(b)>0;\Re(p)\geq 0$,

$\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, for $x<0$, we have the following formulae:

$$L\left\{z^{l-1}\left({}^{\rho}I_{0-}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta;\kappa,\mu)}(a,b;c;tz)\right)(x)\right\}$$

$$=(-1)^{\sigma}\frac{x^{\lambda+\sigma\rho}}{s^l\rho^{\sigma}}\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}*{}_2\Psi_1\left[\left.\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}n\right),(l,1)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\right|\frac{x}{s}\right] \tag{4.6}$$

Proof.

The proof of the Theorem 6 is same as that of Theorem 5, therefore, it is omitted.

5. SPECIAL CASES OF THE MAIN FORMULAE

By assigning the different particular values to the parameters we have the following special cases of our main results established in Section 2.

1. Choose $\mu=1$, the result in equations (2.5) and (2.10) reduce to the following form:

Corollary 1. Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$,
$\min\{\Re(\delta),\Re(\zeta),\Re(\kappa)\}>0;\ \Re(c)>\Re(b)>0;\Re(p)\geq 0$,
$\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$;

for $x>0$, we have the following formulae:

$$\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta;\kappa)}(a,b;c;t)\right)(x)$$

$$=\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p^{(\delta,\zeta;\kappa)}(a,b;c;x)*{}_1\Psi_1\left[\left.\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\right|x\right] \tag{5.1}$$

Corollary 2. Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$, $\min\{\Re(\delta),\Re(\zeta),\Re(\kappa)\}>0$; $\Re(c)>\Re(b)>0;\Re(p)\geq 0$, $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$,

for $x<0$, we have the following formulae:

$$\left({}^{\rho}I_{0-}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta;\kappa)}(a,b;c;t)\right)(x)$$

$$=(-1)^{\sigma}\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p^{(\delta,\zeta;\kappa)}(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right] \tag{5.2}$$

2. Choose $\kappa=\mu=1$, the results in equations (2.5) and (2.10) reduces to the following form:

Corollary 3. Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$,

$\min\{\Re(\delta),\Re(\zeta)\}>0$; $\Re(c)>\Re(b)>0;\Re(p)\geq 0$, $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$,

for $x>0$, we have the following formulae:

$$\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta)}(a,b;c;t)\right)(x)$$

$$=\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p^{(\delta,\zeta)}(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right] \tag{5.3}$$

Corollary 4. Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$, $\min\{\Re(\delta),\Re(\zeta)\}>0$; $\Re(c)>\Re(b)>0;\Re(p)\geq 0$, $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, for $x<0$, we have the following formulae:

$$\left({}^{\rho}I_{0-}^{\sigma}t^{\lambda-1}F_p^{(\delta,\zeta)}(a,b;c;t)\right)(x)$$

$$=(-1)^{\sigma}\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p^{(\delta,\zeta;\kappa)}(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right]. \tag{5.4}$$

3. If we select $\delta=\zeta=\kappa=\mu=1$ then the established results in equations (2.5) and (2.10) reduces to the following form:

Corollary 5. Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$; $\Re(c)>\Re(b)>0;\Re(p)\geq 0$,$\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$.for $x>0$, we have the following formulae:

$$\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}F_p(a,b;c;t)\right)(x)=\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right] \tag{5.5}$$

Corollary 6. Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$ $\Re(c)>\Re(b)>0;\Re(p)\geq 0$,$\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$.for $x<0$, we have the following formulae:

$$\left({}^{\rho}I_{0-}^{\sigma}t^{\lambda-1}F_p(a,b;c;t)\right)(x)$$

$$=(-1)^{\sigma}\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}F_p(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right] \tag{5.6}$$

4. If we select $p=0;\delta=\zeta=\kappa=\mu=1$ the established results in equations (2.5) and (2.10) reduces to the following form:

Corollary 7. Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$; $\Re(c)>\Re(b)>0;$, $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, for $x>0$, we have the following formulae:

$$\left({}^{\rho}I_{0+}^{\sigma}t^{\lambda-1}{}_2F_1(a,b;c;t)\right)(x)=\frac{x^{\rho+\sigma\rho}}{\rho^{\sigma}}{}_2F_1(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\ \left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\middle|x\right] \tag{5.7}$$

Corollary 8. Let $\lambda,\sigma,\vartheta,\rho,\delta,\zeta,\kappa,\mu,a,b,c,p\in\mathbb{C}$ be such that $\Re(\lambda)>0$; $\Re(c)>\Re(b)>0$; $\Re(\rho)>\max\left[0,\Re(\sigma-\vartheta)\right]$, for $x<0$, we have the following formulae:

$$\left({}^{\rho}I_{0-}^{\sigma}t^{\lambda-1}\,{}_2F_1(a,b;c;t)\right)(x)=(-1)^{\sigma}\frac{x^{\lambda+\sigma\rho}}{\rho^{\sigma}}\,{}_2F_1(a,b;c;x)*{}_1\Psi_1\left[\begin{matrix}\left(\frac{\lambda}{\rho}+1,\frac{1}{\rho}\right)\\\left(\frac{\lambda}{\rho}+\sigma+1,\frac{1}{\rho}\right)\end{matrix}\Big|x\right]\quad(5.8)$$

6. CONCLUSION

In the present paper we establish numerous fractional integral formulae involving the family of extended Gauss hypergeometric function by using the generalized form of Riemann-Liouville fractional integrals.

The special cases of the main results are also presented in Section 5. Also by assigning the particular values to the parameters, we can obtained more result from image formulae associated with Beta transform and Laplace transform.

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