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A NOTE ON WAVELET PACKET COEFFICIENTS AND FRAMES

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Abstract: In this paper, we study the action of double infinite matrix on wavelet packet coefficients. Some results related to wavelet packet frame and wavelet packet coefficients are obtained.

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1. INTRODUCTION

Introduced by Duffin and Schaeffer [12], theory of frames has got into prominence because of its ability of redundant representation of signals. The redundant property of frames is useful for recovering information from corrupted signals. Many authors have studied and extended the general concept of multiresolution analysis (MRA) to frame multiresolution analysis (FMRA), see especially [3, 4]. The related concept of general frame theory has been extended to Gabor and wavelet frames, for literatures and results we refer to Chui and Shi [6], Daubechies [11], Heil and Walnut [14], Ahmad and Iqbal [2] and the references therein.

2. PRELIMINARIES

A system of elements $\{f_n\}_{n \in \Lambda}$ in a Hilbert space H is called a frame for H if there exists two +ve numbers A and B such that for any $f \in H$,

$$A \|f\|_2^2 \leq \sum_{n \in \Lambda} |\langle f, f_n \rangle|^2 \leq B \|f\|_2^2,$$

The numbers A and B are called frame bounds. If $A = B$, the frame is said to be tight. The frame is called exact if it ceases to be a frame whenever any single element is deleted from the frame.

The connection between frames and numerically stable reconstruction from discretized wavelet coecients was pointed out by Grossmann *et al*. [13]. A wavelet function $\psi \in L^2(\mathbb{R})$, constitutes a wavelet frame with frame bounds A and B, if for any $f \in L^2(\mathbb{R})$,

$$A\|f\|_2^2 \leq \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \leq B\|f\|_2^2,$$

where A and B are some positive numbers and $\psi_{j,k} = 2^{j/2}\psi(2^jx - k), j, k \in \mathbb{Z}$. Again, if $A = B$, the frame is said to be tight.

The continuous wavelet transformation of a L^2-function f with respect to the wavelet ψ, satisfying admissibility condition, is defined as,

$$(T^{wav}f)(a, b) = |a|^{-1/2}\int_{-\infty}^{\infty} f(t)\,\overline{\psi\left(\frac{t-b}{a}\right)}\,dt, \ \ a, b \in \mathbb{R}, a \neq 0\,.$$

The term wavelet denotes a family of functions of the form $\psi_{a,b} = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right)$, obtained from a single function ψ by the operation of dilation and translation. The wavelet coefficients at discrete points $a = \frac{k}{2^j}$, $b = \frac{1}{2^j}$, are given by $\psi_{j,k} = (T^{wav}f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right)$.

Wavelet Packets. Though the wavelet basis has good localization in time-frequency domain but in order to get better localization for high frequency components in the wavelet decomposition, Coifman *et al*. [8] introduced another kinds of bases called wavelet packets. We have the following sequence of functions due to Wickerhauser [19]. For $l = 0, 1, 2, \ldots,$

$$\psi_{2l}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} a_k\psi_l(2x-k) \quad \text{and} \quad \psi_{2l+1}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} b_k\psi_l(2x-k)\,, \qquad (*)$$

where $a = \{a_k\}$ is the filter such that $\Sigma_{n\in\mathbb{Z}}a_{n-2k}\,a_{n-2l} = \delta_{kl}$, $\Sigma_{n\in\mathbb{Z}}a_n = \sqrt{2}$ and $b_k = (-1)^k a_{1-k}$. For $l = 0$ in $(*)$, we get

$$\psi_0(x) = \psi_0(2x) + \psi_0(2x-1),\ \ \psi_1(x) = \psi_0(2x) - \psi_0(2x-1),$$

where ψ_0 is a scaling function and may be taken as a characteristic function. If we increase l, we get the following

$$\psi_2(x) = \psi_1(2x) + \psi_1(2x-1),\ \psi_3(x) = \psi_1(2x) - \psi_1(2x-1)$$

$$\psi_4(x) = \psi_1(4x) + \psi_1(4x-1) + \psi_1(4x-2) + \psi_1(4x-3)$$

and so on.

Here ψ_l's have a fixed scale but different frequencies. They are Walsh functions in [0, 1[. The functions $\psi_l(t-k)$, for integers k, l with $l \geq 0$, form an orthonormal basis of $L^2(\mathbb{R})$.

Theorem 2.1: For every partition P of the non negative integers into the sets of the form $I_{lj} = \{2^jl, \ldots, 2^j(l+1) - 1\}$, the collection of functions $\psi_{l;jk} = 2^{j/2}\,\psi_l(2^jx - \mathrm{k})$, $I_{lj} \in P$, $k \in \mathbb{Z}$, is an orthonormal basis of $L^2(\mathbb{R})$.

The collection of functions give rise to many bases including Walsh, wavelet and subband basis. A wavelet packet basis of $L^2(\mathbb{R})$ is an orthonormal basis selected from among the functions $\psi_{l,jk}$.

The space $f \in L^2(\mathbb{R})$ of measurable function f is defined on the real line $\mathbb{R}$, that satisfies $\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty$.

Here, we have used the inner product of functions $f, g \in L^2(\mathbb{R})$ as $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx$, the Fourier transform of $f \in L^2(\mathbb{R})$ as $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx$ and the relationship between functions and their Fourier transform as $2\pi \langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$. For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform $\hat{f}$ of f is in $L^2(\mathbb{R})$ and satisfies the Parseval identity $\|\hat{f}\|_2^2 = 2\pi \|f\|_2^2$.

Let $A = (a_{smjk})$ be a double infinite matrix of real numbers. Then, A-transform of a double sequence $\chi = (\chi_{jk})$ is

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{smjk}\, \chi_{jk},$$

which is called A-transform of the sequence $\chi = (\chi_{jk})$ [15, 17]. A double matrix $A = (a_{smjk})$ is said to be regular (see [16]) if the following conditions hold:

(i) $\lim_{s, m \to \infty} \Sigma_{j, k=0}^{\infty} a_{smjk} = 1$,

(ii) $\lim_{s, m \to \infty} \Sigma_{j=0}^{\infty} |a_{smjk}| = 0$, $(k = 0, 1, 2, ...)$,

(iii) $\lim_{s, m \to \infty} \Sigma_{k=0}^{\infty} |a_{smjk}| = 0$, $(j = 0, 1, 2, ...)$,

(iv) $\|A\| = \sup_{s, m > 0} \Sigma_{j, k=0}^{\infty} |a_{s, m}| < \infty$.

Either of condition (ii) and (iii) implies that

$$\lim_{s, m \to \infty} a_{smjk} = 0 .$$

3. MAIN RESULTS

Theorem 3.1: Let $A = (a_{smjk})$ be a double nonnegative regular matrix. If

$$f(x) = \sum_{l} \sum_{j, k \in \mathbb{Z}} c_{l;\, jk}\, \psi_{l;\, jk}(x),$$

is a wavelet packet expansion of $f \in L^2(\mathbb{R})$ with wavelet packet coefficients

$$c_{l;\, jk} = \int_{-\infty}^{\infty} f(x)\, \overline{\psi_{l;jk}(x)}\, dx = \left\langle f, \psi_{l;jk} \right\rangle,$$

then the wavelet packet frame condition for A-transform of $f \in L^2(\mathbb{R})$ is

$$a\,\|f\|_2^2 \le \sum_{l} \sum_{s,m\in\mathbb{Z}} |\langle Af, \psi_{l;\,sm}\rangle|^2 \le b\,\|f\|_2^2,$$

where Af is the A-transform of f, $0 < a \le b < \infty$ and $l \in \mathbb{Z}_0^+$.

Proof: Since

$$f(x) = \sum_{l} \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{l;\,jk}\rangle\, \psi_{l;\,jk}.$$

If we take A-transform of f, we get

$$Af(x) = \sum_{l} \sum_{s,m\in\mathbb{Z}} \langle Af, \psi_{l;\,sm}\rangle\, \psi_{l;\,sm},$$

and then

$$\sum_{l} \sum_{s,m\in\mathbb{Z}} |\langle Af, \psi_{l;\,sm}\rangle|^2 \le \sum_{l} \sum_{s,m\in\mathbb{Z}} \int_{-\infty}^{\infty} |Af(x)|^2\, |\overline{\psi_{l;sm}}|^2\, dx$$

$$\le \|A\|^2\, \|f(x)\|_2^2 \sum_{l} \sum_{s,m\in\mathbb{Z}} \|\psi_{l;\,sm}\|_2^2.$$

Since A is regular matrix and $\|\psi_{l;\,sm}\|_2 = 1$, therefore

$$\sum_{l} \sum_{s,m\in\mathbb{Z}} |\langle Af, \psi_{l;\,sm}\rangle|^2 \le b\,\|f\|_2^2, \tag{3.1}$$

where b, is +ve constant. Now, for any arbitrary $f \in L^2(\mathbb{R})$, define

$$\tilde{f} = \left[\sum_{l} \sum_{s,m\in\mathbb{Z}} |\langle Af, \psi_{l;sm}\rangle|^2\right]^{-1/2} f.$$

Clearly,

$$\langle A\tilde{f}, \psi_{l;\,sm}\rangle = \left[\sum_{l} \sum_{s,m\in\mathbb{Z}} |\langle Af, \psi_{l;sm}\rangle|^2\right]^{-1/2} \langle Af, \psi_{l;sm}\rangle,$$

then

$$\sum_{l} \sum_{s,m\in\mathbb{Z}} |\langle A\tilde{f}, \psi_{l;\,sm}\rangle|^2 \le 1.$$

Hence, if there exists α, *a* +ve constant, then

$$\|A\tilde{f}\|_2^2 \le \alpha,$$

$$\left[\sum_{l} \sum_{s,m\in\mathbb{Z}} |\langle Af, \psi_{l;sm}\rangle|^2\right]^{-1} \|Af\|_2^2 \le \alpha.$$

Since A is regular, we have

$$\left[\sum_{l}\sum_{s,m\in\mathbb{Z}}|\langle Af,\psi_{l;sm}\rangle|^2\right]^{-1}\|f\|_2^2\le\alpha_1,$$

where, $\alpha_1=\frac{\alpha}{\|A\|^2}$, is another +ve constant. Therefore,

$$a\|f\|_2^2\le\sum_{l}\sum_{s,m\in\mathbb{Z}}|\langle Af,\psi_{l;sm}\rangle|^2, \tag{3.2}$$

where, $a=\alpha>0$. Combining equations (1) and (2), we have

$$a\|f\|_2^2\le\sum_{l}\sum_{s,m\in\mathbb{Z}}|\langle Af,\psi_{l;sm}\rangle|^2\le b\|f\|_2^2.$$

This completes the proof.

Theorem 3.2: If $c_{l;jk}$ are the wavelet packet coefficients of $f\in L^2(\mathbb{R})$, i.e. $c_{l;jk}=\langle f,\psi_{l;jk}\rangle$. Then the $d_{l;sm}$ are the wavelet packet coecients of Af, where $d_{l;sm}$ is defined as the A-transform of $c_{l;jk}$ by

$$d_{l;sm}=\sum_{l}\sum_{j,k=-\infty}^{\infty}a_{smjk}\,c_{l;jk}.$$

Proof: We can write

$$\langle Af,\psi_{l;jk}\rangle=\int_{-\infty}^{\infty}Af(x)\,\overline{\psi_{l;sm}(x)}\,dx$$

$$=\int_{-\infty}^{\infty}\sum_{j,k=-\infty}^{\infty}a_{smjk}\,c_{l;jk}\,\psi_{l;jk}(x)\,\overline{\psi_{l;sm}(x)}\,dx$$

Now,

$$\sum_{l}\sum_{s,m=-\infty}^{\infty}\langle Af,\psi_{l;sm}\rangle\,\psi_{l;sm}$$

$$=\sum_{l}\sum_{s,m=-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{j,k=-\infty}^{\infty}a_{smjk}\,c_{l;jk}\,\psi_{l;jk}(x)\,\psi_{l;sm}(x)\,\overline{\psi_{l;sm}(x)}\,dx$$

$$=\sum_{l}\sum_{s,m=-\infty}^{\infty}d_{l;sm}\,\psi_{l;sm}\int_{-\infty}^{\infty}\|\psi_{l;sm}\|_2^2$$

$$=\sum_{l}\sum_{s,m=-\infty}^{\infty}d_{l;sm}\,\psi_{l;sm}.$$

Therefore,

$$\sum_{l} \sum_{s, m=-\infty}^{\infty} d_{l;\, sm}\, \psi_{l;\, sm} = \sum_{l} \sum_{s, m=-\infty}^{\infty} \langle Af, \psi_{l;\, sm}\rangle\, \psi_{l;\, sm}.$$

This yields that $d_{l;sm}$ are wavelet packet coefficients of Af. Thus,

$$d_{l;sm} = \langle f, \psi_{l;\, sm}\rangle.$$

This completes the proof.

Theorem 3.3: Let $A = (a_{smjk})$ be a double nonnegative matrix whose elements are $\langle \psi_{l;jk}, \psi_{l;sm}\rangle$. Then, $\{\psi_{l;jk}\}$ constitutes a wavelet packet frame of $f \in L^2(\mathbb{R})$ iff $\{\psi_{l;sm}\}$ constitutes a wavelet packet frame of $f \in L^2(\mathbb{R})$, where $c_{l;\, jk} = \langle f, \psi_{l;jk}\rangle$ and $d_{l;sm} = \langle f, \psi_{l;\, sm}\rangle$.

Proof: We observe that

$$\begin{aligned}
a_{smjk}\, c_{l;\, jk} &= \langle \psi_{l;\, jk}, \psi_{l;\, sm}\rangle \langle f, \psi_{l;\, jk}\rangle \\
&= \int_{-\infty}^{\infty} \psi_{l;\, jk}(x)\, \overline{\psi_{l;sm}(x)}\, dx \int_{-\infty}^{\infty} f(x)\, \overline{\psi_{l;jk}(x)}\, dx \\
&= \int_{-\infty}^{\infty} f(x)\, \overline{\psi_{l;sm}(x)}\, dx \int_{-\infty}^{\infty} \psi_{l;\, jk}(x)\, \overline{\psi_{l;jk}(x)}\, dx \\
&= \int_{-\infty}^{\infty} f(x)\, \overline{\psi_{l;sm}(x)}\, dx \\
&= \langle f, \psi_{l;\, sm}\rangle.
\end{aligned}$$

i.e., $a_{smjk}\, c_{l;\, jk} = d_{l;\, sm}$.

Now,

$$\begin{aligned}
\sum_{l} \sum_{s,m} | d_{l;\, sm}|^2 &= \sum_{l} \sum_{s,m} | a_{smjk}\, c_{l;\, jk}|^2 \\
&= \sum_{l} \sum_{s,m} |\langle f, \psi_{l;\, sm}\rangle|^2 \\
&= \frac{1}{(2\pi)^2} \sum_{l} \sum_{s,m} |\langle \hat{f}, \hat{\psi}_{l;sm}\rangle|^2
\end{aligned}$$

$$\frac{1}{(2\pi)^2} \sum_{l} \sum_{s,m} \left| \int_0^{2\pi} \sum_{p=-\infty}^{\infty} \hat{f}(\omega + 2\pi p)\, \overline{\hat{\psi}_l(\omega + 2\pi p)}\, e^{ism\omega} d\omega \right|^2$$

$$\begin{aligned}
&= \frac{1}{2\pi} \sum_{l} \int_0^{2\pi} \left| \sum_{p=-\infty}^{\infty} \hat{f}(\omega + 2\pi p)\, \overline{\hat{\psi}_l(\omega + 2\pi p)}\, d\omega \right|^2 \\
&= T \text{ (say)}.
\end{aligned}$$

By Parseval's formula for trigonometric Fourier series.

Let

$$f(\omega) = \sum_{q=-\infty}^{\infty} \overline{\hat{f}(\omega + 2\pi q)}\, \hat{\psi}_l(\omega + 2\pi q) \ .$$

Therefore,

$$T = \frac{1}{2\pi} \sum_l \left(\int_0^{2\pi} \sum_{p=-\infty}^{\infty} \hat{f}(\omega + 2\pi p)\, \overline{\hat{\psi}_l(\omega + 2\pi p)}\, d\omega\, f(\omega)\, d\omega \right)$$

$$= \frac{1}{2\pi} \sum_l \left(\int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{\psi}_l(\omega)}\, f(\omega)\, d\omega \right)$$

$$= \frac{1}{2\pi} \sum_l \left(\sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{\psi}_l(\omega)}\, \overline{\hat{f}(\omega + 2\pi p)}\, \hat{\psi}_l(\omega + 2\pi p)\, d\omega \right)$$

$$= \frac{1}{2\pi} \sum_l \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, |\hat{\psi}_l(\omega)|^2\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega$$

$$= \|f\|^2,$$

that is,

$$\sum_l \sum_{s,m} |d_{l;\, sm}|^2 = \|f\|^2, f \in L^2(\mathbb{R}).$$

Therefore, for a regular matrix $A = (a_{smjk})$, we have

$$a\,\|f\|_2^2 \le \sum_l \sum_{s,m \in \mathbb{Z}} |d_{l;\, sm}|^2 \le b\,\|f\|_2^2$$

if and only if

$$a'\,\|f\|_2^2 \le \sum_l \sum_{s,m \in \mathbb{Z}} |c_{l;\, jk}|^2 \le b'\,\|f\|_2^2,$$

where, $0 \le a', b' < \infty$. This completes the proof.

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