IJAMAA: *January-December 2015, Volume 10, Nos. 1-2, pp. 111-116*

RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF K-TH RECORD VALUES FROM POWER FUNCTION DISTRIBUTION, AND A CHARACTERIZATION

Sanjay Kumar Singh

ABSTRACT: In present study we give some recurrence relations satisfied by single and product moments of k-th upper record values from power function distribution using a recurrence relations for single moments we obtain a characterization of power function distribution.

Keywords and phrases : Record values; k-record values; moments; product moment; k-th upper record values; power function distribution.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with c.d.f. F(x)=p[X ≤ x] and p.d.f. f(x).For a fixed k ≥ 1 we define the sequence $\{U_n^{(k)}, n \geq 1\}$ of k-th upper record times of $\{X_n, n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\{ j > U_n^{(k)} : X_{j:j+k-1} > X_{u_n}^{(k)} : U_n^{(k)} + k - 1\}$$

For K=1 and n=1, 2,…, we write $U_n^{(1)} = U_n$. Then $\{U_n, n \geq 1\}$ is the sequence of record times of $\{X_n, n \geq 1\}$,where $y_n^{(1)} = X_{u_n}^{(k)}$ is called the sequence of k-th upper record values of $\{X_n, n \geq 1\}$.For convenience ,we shall also take $y_0^{(k)} = 0$. Note that for k=1 we have $y_n^{(1)} = y_{u_n}^{\backslash}$,ne",which are the record values of $\{X_n$,ne"1$\}$ (Ahsanullah(1995)).Moreover ,we see that $y_1^{(k)} = \min(X_1, X_2, \ldots X_k) = X_{1:k}$ then the pdf of $y_n^{(k)}$ and $(y_m^{(k)}, y_n^{(k)})$ are as follows:

$$f_{y_n}^{(k)}(x) = \frac{k^n}{(n-1)!}[-\ln(1-F(x)]^{n-1}(1-F(x))^{k-1} f(x); n \geq 1 \;, \tag{1.1}$$

$$f_{y_m}^{(k)}, y_n^{(k)}(x,y) = \frac{k^n}{(m-1)!(n-m-1)!}[\ln(1-F(x)) - \ln(1-F(y))]^{n-m-1}[-\ln(1-F(x))]^{m-1}$$

$$\frac{f(x)}{1-F(x)}[1-F(y)]^{k-1} f(y), x < y; 1 \leq m \leq n; n \geq 2 \tag{1.2}$$

We present recurrence relations for moment of $y_n^{(k)}$ and product moments of $(y_m^{(k)}, y_n^{(k)})$ when X has the three parameter power function distribution (Pearson's type I distribution) with the probability density function(p.d.f.) of the form

$$f(x;\alpha,\beta,\gamma) = \gamma\beta^{-\gamma}(\alpha+\beta-x)^{\gamma-1}, \ \alpha<x<\alpha+\beta,\ \beta>0,\ r>0, \tag{1.3}$$

=, otherwise

If shape parameter γ = 1 the p.d.f. in (1.3) reduces to uniform distribution on (α, α+β). Without loss of generality, we assume α=0 and β=1; i.e, the p.d.f. is

$$f(x;\gamma) = \gamma(1-x)^{\gamma-1}, \ 0<x<1 \tag{1.4}$$

$=0$, otherwise

And the c.d.f. is

$$F(x;\gamma) = 1-(1-x)^{\gamma}, \ 0<x<1 \tag{1.5}$$

Here, for γ = 1, the distribution in (1.4) and (1.5) becomes standard uniform distribution.

It is easy to note from (1.4) and (1.5) that

$$\gamma(1-F(x)) = (1-x)f(x)$$

Using these relations we give characterization of the power function distribution.

2. RELATIONS FOR SINGLE AND PRODUCT MOMENTS

Theorem 2.1. Fix a positive integer k ≥ 1. For n ≥ 1 and r=0,1,2,…,

$$E(y_n^{(k)})^{r+1} = \frac{1}{r+1+\gamma k}[(r+1)E(y_n^{(k)})^r + \gamma k(\gamma_{n-1}^{(k)})^{r+1}] \tag{2.1}$$

Proof. For n ≥ 1 and r=0,1,2,…, we have from (1.1) and (1.6)

$$E(y_n^{(k)})^r - E(y_n^{(k)})^{r+1} = \frac{k^n}{(n-1)!}\int_0^1 x^r(1-x)(-\ln(1-F(x))^{n-1}(1-F(x))^{k-1}f(x)dx$$

$$E(y_n^{(k)})^r - E(y_n^{(k)})^{r+1} = \frac{k^{n.\gamma}}{(n-1)!}\int_0^1 x^r(-\ln(1-F(x))^{n-1}(1-F(x))^{k}dx$$

Integrating by parts,taking x^r as the part to be integrated we get(2.1).

Remark: If γ = 1 we get the recurrence relation for standard uniform distribution.

$$E(y_n^{(k)})^{r+1} = \frac{1}{(r+1+k)}[(r+1)E(y_n^{(k)})^r + vkE(y_{n-1}^{(k)})^{r+1}]$$

Theorem 2.2. For 1 ≤ m ≤ n-2 and r, s=0, 1, 2,…,

$$E[(y_m^{(k)})^r (y_n^{(k)})^{s+1}] = \frac{k\gamma}{(k\gamma+s+1)} E[(y_m^{(k)r} (y_{n-1}^{(k)})^{s+1}] + \frac{(s+1)}{k\gamma+s+1} E[(y_m^{(k)})^r (y_n^{(k)})^s] \tag{2.2}$$

And for m ≥ 1 , γ, s = 0, 1, 2,…,

$$E[(y_m^{(k)})^r (y_{m+1}^{(k)})^{s+1}] = \frac{k\gamma}{(k\gamma+s+1)} E[(y_m^{(k)})^{r+s+1}] + \frac{(s+1)}{k\gamma+s+1} E[(y_m^{(k)})^r (y_{m+1}^{(k)})^s] \tag{2.3}$$

Proof: from (1.2) for 1 ≤ m ≤ n-1 and γ, s = 0, 1, 2,…, and on using (1.6) we get,

$$E[(y_m^{(k)})^r (y_n^{(k)})^s] - E[(y_m^{(k)})^r (y_n^{(k)})^{s+1}] = \frac{k^n}{(m-1)!(n-m-1)!} \int_0^1 x^r \frac{f(x)}{1-F(x)} (-\ln(1-F(x))^{m-1} I(x)dx, \text{ x<y} \tag{2.4}$$

Where $I(x) = \gamma \int\limits_{g>x} y^s [\ln(1-F(x)) - \ln(1-F(y)]^{n-m-1} (1-F(y))^k dy$

Integrating I(x) by parts, taking y^s as the part to be integrated

$$I(x) = \frac{\gamma k}{(s+1)} \int y^{s+1} [\ln(1-F(x)) - \ln(1-F(y))]^{n-m-1} (1-F(y))^k dy$$

$$-\frac{(n-m-1)\gamma}{(s+1)} \int y^{s+1} (\ln(1-F(x)) - \ln(1-F(y)]^{n-m-2} (1-F(y))^{k-1} f(y)dy$$

Substituting this expression into (2.4) and simplifying, we obtain

$$E[(y_m^{(k)})^r (y_m^{(k)})^s] - E[(y_m^{(k)})^r (y_n^{(k)})^{s+1}] = \frac{k\gamma}{(s+1)} [E\{(y_m^{(k)})^r (y_n^{(k)})^{s+1}\} - E\{(y_m^{(k)})^r (y_{n-1}^{(k)})^{s+1}\}]$$

Hence we have (2.2) when n=m+1 we obtain (2.3)

3. CHARACTERIZATION

This section contains characterization of the power function distribution. We start with the following result of Lin(1986)

Proposition: Let n_0 be any fixed non-negative integer, $-\infty \le a < b \le \infty$, and g(x)≥0 an absolutely continuous function with g'(x) ≠ 0 a.e on (a, b). Then the sequence of function $\{[g(x)]^n e^{-g(x)}, n \ge n_0\}$ is complete in L(a, b) iff g(x) is strictly monotone on (a,b).

Using the above proposition we get a stronger version of theorem 2.1.

Theorem 3.1 Fix a positive integer k ≥ 1 and let r be a non-negative integer. A necessary and sufficient condition for a random variable x to be distributed with pdf given by (1.4) is that

$$E(y_n^{(k)})^{r+1} = \frac{1}{r+1+\gamma k}[(r+1)E(y_n^{(k)})^r + \gamma k E(\gamma_{n-1}^{(k)})^{r+1}] \quad (3.1)$$

Proof: The necessary part follows immediately from (2.1) on the other hand if the recurrence relation(3.1) is satisfied, then

$$(r+1+\gamma k)E(y_n^{(k)})^{r+1} = (r+1)E(y_n^{(k)})^r + \gamma k E(y_{n-1}^{(k)})^{r+1}$$

$$= (r+1)\frac{k^{n.}}{(n-1)!}\int_0^1 x^r(-\ln(1-F(x))^{n-1}(1-F(x))^{k-1} f(x)dx$$

Integrating last integral by parts we have

$$(r+1+\gamma k)E(y_n^{(k)})^{r+1} = (r+1)\frac{k^n}{(n-1)!}\int x^r(-\ln(1-F(x))^{n-1}(1-F(x))^{k-1} f(x)dx$$

$$-\frac{\gamma k^n}{(n-1)!}(r+1)\int x^r(-\ln(1-F(x))^{n-1}(1-F(x))^k dx$$

$$+\frac{\gamma k^{r+1}}{(n-1)!}\int x^{r+1}(-\ln(1-F(x))^{n-1}(1-F(x))^{k-1} f(x)dx$$

Which reduces to

$$\int x^r(-\ln(1-F(x))^{n-1}(1-F(x))^{k-1}\{(r+1+\gamma k)xf(x)-(r+1)f(x)+\gamma(r+1)(1-F(x))-\gamma kx(fx)\}dx = 0$$

$$\int x^r(-\ln(1-F(x))^{n-1}(1-F(x))^{k-1}\{(r+1)(x-1)f(x)+\gamma(r+1)(1-F(x))\}dx = 0$$

It now follows from the above proposition with

$$G(x) = -l_n(1-F(x))$$

That f(x) (1-x) = γ (1-F(x))

Which proves that f(x) has the form (1.4)

Now we shall show that Theorem 3.1 can be used as in a characterization of the power function distribution in terms of minimal order statistics

Letting n = 1 in theorem (3.1) and we get

$$E(x_{1:k}^{r+1}) = \frac{r+1}{r+1+\gamma k} E(x_{1:k}^{r}),$$ for fixed integer k ≥ 1

From the above result we can make the following observation.

Theorem 3.2. Let r be a non negative integer.A necessary and sufficient condition for a random variable X to be distributed with the p.d.f. (1.3) is that

$$E(X_{1:k}^{r+1}) = \frac{r+1}{r+1+r_k} E(X_{1:k}^{r}), \text{ for k=1,2,…,} \tag{3.2}$$

Proof: The necessary part follows from (2.1) on the other hand if (3.2) is satisfied then,

$$(r+1+r_k)\int_0^1 x^{r+1}(1-F(x))^{k-1} f(x)dx = (r+1)\int_0^1 x^r (1-F(x))^{k-1}.f(x)dx$$

$$(r+1)\int_0^1 x^r (1-F(x)^{k-1} f(x)dx + \gamma k\int_0^1 x^{r+1}(1-F(x)^{k-1}.f(x)dx = (r+1)\int x^r (1-F(x))^{k-1} f(x)dx$$

$$\int_0^1 x^r (1-F(x)^{k-1}.\{\gamma(r+1)(1-F(x)) + (r+1)xf(x) - (r+1)f(x)\}dx = 0$$

k=1,2,…, (3.3)

Now applying a generalization of the Muntz-szasz theorem (cf.hwang and Lin(1988)) to (3.3) we obtain

$$\gamma(1\text{-}F(x))=f(x)(1\text{-}x)$$

which proves $F(x)=1-(1-x)^{\gamma}$, 0<x<1

CONCLUSION

In this study some recurrence relations for single and product moments of k-th upper record values from the power function distribution have been established, which generalize the correspondirig results for upper 1-record values from the power function distribution. Further, these recurrence relations have been utilized to obtain a characterization of the power function distribution by using a result of Lin (1986). Similar results for Weibull distribution have been obtained by Pawlas and Szynal (2000).

References

Ahsanullah, M.(1995), Record statistics .Nova Science Publishers, Inc. Commack, NY, USA.

Balakrishnan, R. and Ahsanullah, M. (1994). Recurrence relations for single and product moments of record values from generalized aPareto distribution.Commun. Statist. Theory Meth., 23(10), 2841-2852.

Balakrishnan, R. Ahsanullah, M. (1995). Relations for single and product moments of record values from exponential distribution. J .Appl.Statist.Sc., 2(1), 73-87.

Dziubdziela, W. and Kopocinski, B. (1976). Limiting properties of k-th record value. Appl. Math. 15, 187-190.

Glick, N., 1978. Breaking records and breaking boards. Am. Math. MontNy, 85: 2-26.

Grudzien, Z., 1982. Characterization of Distribution of Time Limits in Record Statistics as Well as Distnbutioris and Moments of Linear Record Statistics from the Samples of Random Numbers. Praca Doktorska, UMCS, Lublin.

Hwang, J.S. and G.D. Lin, 1988. On a generalized moment problem II. Proc. Amer. Math. Soc., 91: 577-580.

Lin, G.D., 1986. On a moment problem. Tohoku Math. J., 38: 595-598. Nevzorov, V. B., 1987. Records. Theor. Prob. Appl., 32: 201-228.

Pawlas, P. and D. Szynal, 2000. Recurrence relations for single and product moments of k-th record values from Weibull distributions and a characterization. J. Appl. Stat. Sci., 10: 17-25.

Resnick, S.I., 1987. Extreme Values, Regular Variation and Point Processes. 1st Edu., Springer-Verlag, New York.

Sanjay Kumar Singh
Department of Statistics,
PGDAV College
University of Delhi,
India