Communications on Stochastic Analysis
Vol. 14, No. 3-4 (2020)

FRACTIONAL STOCHASTIC EVOLUTION EQUATIONS: WHITE NOISE MODEL

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Abstract. The paper is dedicated to solvability of fractional order stochastic evolution equations perturbed with Balakrishnan white noise type additive terms. The results obtained in this paper can be applied in the analysis of various stochastic relaxation and diffusion processes in complicate systems or in fractal and porous media.

1. Introduction

The theory of stochastic evolution equations and it's applications has developed significantly in recent decades. On the one hand, this is due to the infinite dimensional analysis of semi-groups and evolution operators of solutions, but on the other hand, this is due to the fact that their finite dimensional realizations often occur as mathematical models in physics, technique, chemistry, mathematical biology, financial mathematics and other areas of the sciences. Generalization of the theory of Ito-–Stratonovich-–Skorokhod in the infinite dimensional case was originated in the works [1,2]. Within the framework of this theory in particular Ito's linear differential equation with multiplicative noise was investigated [3-6]. In [7-10] a different approach based on Nelson – Gliklikh derivative was offered to the analysis of stochastic differential equations. It was found in [10] that Nelson – Gliklikh derivative of Wiener process is well consistent with the predictions of theory of Brownian motion of Einstein-Smoluchowski. Therefore, the relevant stochastic process was called "white noise".

In this paper, we investigate a class of stochastic equations of fractional order perturbed with absolute random process or Balakrishnan type of white noise. This type of white noise was first introduced in the monograph [11].

We consider the following Cauchy problem

$$ {}^cD_t^\alpha + Au(t) = f(u(t)) + B\omega(t), u(0) = u_0 \tag{1.1}$$

where ${}^cD_t^\alpha$ - Caputo fractional derivative order $\alpha, 0 < \alpha < 1$, A - almost sectorial operator in separable Hilbert space H, $f : H \to H$ - given nonlinear mapping, $\omega(t)$ - absolute random process (white noise in the sense of Balkrishnan) in another

Date: Date of Submission June 02, 2020; Date of Acceptance July 20, 2020 , Communicated by Yuri E. Gliklikh .

2010 *Mathematics Subject Classification.* Primary 76S05; Secondary 35B06, 58J70.

Key words and phrases. white noise, Balakrishnan stochastic integral, fractional derivatives, Hilbert-Schmidt operator, resolvent families of operators.

separable Hilbert space H_n, B - linear operator, defined in H with values in the space of operators from H_n in H.

In analysis of Cauchy problem (1.1) the standard requirement is the generation of family of resolvent operators by operator A $\{S_\alpha(t)\}_{t\geq 0}$ and $\{Z_\alpha(t)\}_{t\geq 0}$. This condition guarantees the correctness of Cauchy problem for deterministic unperturbed homogeneous equation

$$^{c}D_t^\alpha + A(t)u(t) = 0.$$

Next we should require that nonlinear mapping $f(\cdot)$ satisfies a Lipschitz condition. Conditions imposed on operator B are closely related to the properties of an absolute random process (white noise). Next, everywhere in this paper by random process $\omega(t)$ we will understand white noise in the sense of Balakrishnan. We introduce the space $W = L_2((0,T),H)$, where $0 < T \leq \infty$, which will be a separable Hilbert space if H is a separable Hilbert space. We use the notations ω and μ for element of space W and standard Gaussian measure accordingly. The space W defined in this way is said to be a white noise and each element $\omega \in W$ is called a realization of white noise.

Let

$$W(t,\omega) = \int_0^t \omega(s)ds.$$

Function $W(t,\omega)$ is continuous with respect to t when ω is fixed and when $t > s$ the difference $[W(t,\omega) - W(s,\omega)]$ is a Gaussian random variable with expected value equal to zero and variance equal to $(t-s)\|h\|^2$, where $h \in W$. However $[W(t,\omega), h]$ cannot be the realization of Wiener process, because such realization has bounded variation at each finite interval when ω is fixed.

The space-time correlation Balakrishan model is one of the possible models based on delta function. Another approach is based on the theory of martingales and Wiener processes. If we limit ourselves to considering only linear operators both approaches will lead to the same results. When we use non-linear operators, there will be a profound difference between these approaches (see, for example, 12-14]).

The paper consists of an introduction, a list of literature and five paragraphs. The main definitions and results of the theory of integrals and fractional derivatives are outlined in the §2. §3 is dedicated to the study of the resolvent families of operators $S_\alpha(t)$ and $Z_\alpha(t)$. The main properties of Balakrishnan white noise and the corresponding stochastic integral are cited in the §4. Linear case of the Cauchy problem is studied in §5, and non-linear case in §6.

2. Integrals and derivatives of fractional order

Let $J = [0,T]$, X - Banach space, $\mathbb{N}$ - set of natural numbers, $\mathbb{R}$ - real number line, $L(J,X)$ - space of Lebesgue integrable functions defined on J with output values from $X, AC^m(J,X)$ - space of $(m-1)$ times continuously differentiable functions on J X - valued functions with absolutely continuous derivative of order m, $W^{m.1}(J,X)$ - Sobolev space.

Definition 2.1. Fractional integral of order $\alpha > 0$ of function $f \in L^1(J, X)$ is

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{f(s)ds}{(t-s)^{1-\alpha}},\ t > 0 \tag{2.1}$$

if, the right side is a pointwise defined function on J, where $\Gamma(\cdot)$ - Euler gamma function of second type.

Definition 2.2. Liouville fractional derivative of order $\alpha > 0$ of function f is

$$^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int\limits_0^t \frac{f(s)ds}{(t-s)^(\alpha+1-n)},\ t > 0,\ n-1 < \alpha < n \tag{2.2}$$

For convenience integral (2.1) will be written in the form

$$I_t^\alpha f(t) = (g_\alpha * f)(t)$$

for function

$$g_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, \text{ if } t > 0, \\ \quad 0 \text{ if } t \leq 0. \end{cases}$$

Here is a brief summary of the main properties of fractional order integral (2.1) and derivative (2.2) in the form of the following statements (see, for example. [15-18]).

Lemma 2.3. *Let $\beta > 0$ and $m = [\beta]$.*

(1) If $f \in L^1(J, X)$, then $g_{m-\beta} * (I_t^\beta f) \in W^{m,1}(J, X)$ and $^{RL}D_t^\beta I_t^\beta f(t) = f(t)$ almost everywhere.

(2) If $\alpha > \beta$ and $f \in L^1(J, X)$, then $^{RL}D_t^\beta I_t^\alpha f(t) = I_t^{\alpha-\beta} f(t)$ almost everywhere, in particular, if $\alpha > k$, then $D^k I_t^\beta f(t) = I_t^{\beta-k} f(t)$ almost everywhere.

(3) If $p > \frac{1}{q}$ and $f \in L^p(J, X)$, then $I^\alpha f$ is continuous on J.

Riemann-–Liouville derivative is not very convenient for modeling real physical processes. Therefore, a modified fractional derivative associated with the name M. Caputo is introduced.

Definition 2.4. Caputo fractional derivative order $\alpha > 0$ of function $f : J \to X$ is

$$^cD_t^\alpha f(t) =^{RL} D^\alpha[f(t) - \sum_{k=0}^{n=1} \frac{t^k}{k!} f^k(0)],\ t > 0,\ n-1 < \alpha < n. \tag{2.3}$$

It is obvious that Caputo derivative of constant function is equal to zero. Note also that integrals in all definitions (2.1), (2.2) and (2.3) are understood in the sense of Bochner.

Lemma 2.5. *Let $\alpha \in (0,1)$ and $h : J \to X$ is continuous functions. Function $u \in C(Y,\mathbb{R})$ given in the form*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} h(s) ds$$

is a unique solution of the following fractional Cauchy problem

$$^{c}D_t^{\alpha} u(t) = h(t),\ t \in J$$

$$u(0) = u_0$$

Corollary 2.6. *If instead of a continuous function h in the lemma 2.5 we take the integrable function, the result of the lemma will remain true.*

3. Almost sectorial operators and main properties

Let X be a Banach space with the norm $\|\cdot\|$, A closed linear operator, $D(A), R(A)$ and $\sigma(A)$ domain, range and spectrum of operator A accordingly.

Definition 3.1. (see. [19]). Let $-1 < \gamma < 0$ and $0 \leq \delta \leq \pi$. By $\Theta_\delta^\gamma(X)$ we denote the set of closed linear operators $A : D(A) \subseteq X \to X$ such that: (1) spectrum $\sigma(A)$ belongs to the sector

$$\Sigma_\mu = \{\lambda \in C \setminus \{0\} : |arg\lambda| \leq \mu\} U\{0\};$$

(2)for every $\delta < \mu < \pi$ exists a constant $c_\mu > 0$ such that

$$\|(\lambda - A)^{-1}\| \leq c_\mu |\lambda|^\gamma \text{ for all } \lambda \bar{\in} \Sigma^o_{\pi/2-\delta}.$$

If $A \in \Theta_\delta^\gamma(X)$, then A named almost sectorial operator.

Let's introduce family of operators

$$\{T(t)\}_{t\in\Sigma^o_{\pi/2-\delta}}, \{S_\alpha(t)\}_{t\in\Sigma^o_{\pi/2-\delta}}, \{Z_\alpha(t)\}_{t\in\Sigma^o_{\pi/2-\delta}}$$

associated with operator A as follows

$$T(t) = e^{-t\lambda}(A) =$$
$$= \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{-t\lambda} (\lambda - A)^{-1} d\lambda,\ t \in \Sigma^o_{\pi/2-\delta},\ \lambda \in \mathbb{C} \setminus (-\infty, 0];$$

$$S_\alpha(t) = E_\alpha(-\lambda t^\alpha)(A)^\theta =$$
$$= \frac{1}{2\pi i} \int_{\Gamma_\theta} E_\alpha(-\lambda t^\alpha)(\lambda - A)^{-1} d\lambda,\ t \in \Sigma^o_{\pi/2-\delta},\ \lambda \in \mathbb{C} \setminus (-\infty, 0];$$

$$Z_\alpha(t) = E_{\alpha,\alpha}(-\lambda t^\alpha)(A) =$$
$$= \frac{1}{2\pi i} \int_{\Gamma_\theta} E_{\alpha,\alpha}(-\lambda t^\alpha)(\lambda - A)^{-1} d\lambda,\ t \in \Sigma^o_{\pi/2-\delta},\ \lambda \in \mathbb{C} \setminus (-\infty, 0], \quad (3.1)$$

where $E_{\alpha,\alpha}(\lambda) = \sum\limits_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha(k+1))},\ \alpha > 0,\ \lambda \in \mathbb{C}$ - two-parametric Mittag-Leffler function, $E_\alpha(\lambda) = E_{1,\alpha}(z)$, contour $\Gamma_\theta = \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\} (0 < \theta < \pi)$ directed in a way that open sector $\sum_{\theta}^{0} = \{\lambda \in \mathbb{C} \setminus \{0\} : |arg\lambda| < \theta\} \cup \{0\}$ is on the right side of Γ_θ.

Note that the family $\{T(t)\}_t \in \Sigma^o_{\pi/2-\delta}$ is a semigroup due to the property

$$T(s+t) = T(s)T(t)$$

for all $\Sigma^o_{\pi/2-\delta}$, in addition operator $T(t)$ characterized as resolvent $(\lambda + A)^{-1}$ of operator A, which is the Laplace transformation of semigroup $T(t)$, i.e.

$$(\lambda + A)^1 = \int_0^\infty e^{-\lambda t} T(t)dt,\ \lambda \in \mathbb{C}, Re\lambda > 0. \tag{3.2}$$

From (3.1) and (3.2) it follows that there is one to one mapping between operator A and semigroup $T(t)$. From Definition (3.1) it follows that operators $S_\alpha(t)$ and $Z_\alpha(t)$ can be presented through $T(t)$ in the following way

$$S_\alpha(t)x = \int_0^\infty \Phi_\alpha(s)T(st^\alpha)xds,\ t \in \Sigma^o_{\pi/2-\delta}, x \in D(S_\alpha(t)), \tag{3.3}$$

$$Z_\alpha(t)x = \int_0^\infty \alpha\Phi_\alpha(s)T(st^\alpha)xds,\ t \in \Sigma^o_{\pi/2-\delta}, x \in D(Z_\alpha(t)), \tag{3.4}$$

where function $\Phi_\alpha(s)$ (see for example [19]) is defined as

$$\Phi_\alpha(\lambda) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-\lambda)^k}{(k-1)!}\Gamma(k\alpha)\sin(k\pi\alpha),\ 0 < \alpha < 1, \lambda \in \mathbb{C}.$$

The main properties of operators $S_\alpha(t)$ and $Z_\alpha(t)$ will be given in the form of following statements.

Lemma 3.2. *Let $A \in \Theta^\gamma_\delta$, $-1 < \gamma < 0$, $0 < \delta < \pi/2$. There are statements:*

(1) For every $t \in \sum^0_{\pi/2-\delta}$, $S_\alpha(t)$ and $Z_\alpha(t)$ are bounded linear operators on X. Moreover, there exist constants $C_S = C(\alpha,\gamma) > 0$ and $C_Z = C(\alpha,\gamma) > 0$ such that, for all $t \geq 0$

$$\|S_\alpha(t)\| \leq C_S t^{-\alpha(1+\gamma)} \text{ and } \|Z_\alpha(t)\| \leq C_z t^{-\alpha(1+\gamma)}.$$

(2) For $t > 0$, the families $S_\alpha(t)$ and $Z_\alpha(t)$ are continuous in uniform operator topology. Moreover, for every $r > 0$ uniformly continuous on $[r,\infty)$.

(3) For every fixed $t \in \Sigma_{\pi/2-\delta}$ and all $x \in D(A)$,

$$(S_\alpha(t) - I)x = \int_0^t -s^{\alpha-1} A Z_\alpha(z)xds.$$

(4) For all $x \in D(A)$ and $t > 0$

$${}^cD^\alpha_t S_\alpha(t)x = -AS_\alpha(t)x$$

(5) For all $t > 0$ $S_\alpha(t)I^\alpha_t(t^{\alpha-1}Z_\alpha(t))$.

(6) Let $\beta > 1 + \gamma$. For all $x \in D(A^\beta)$, $\lim\limits_{t\to 0+} S_\alpha(t)x{=}x$.

Lemma 3.3. *([19]). Let $A \in \Theta_\delta^\gamma(X)$, $-1 < \gamma < 0$, $0 < \delta < \pi/2$, and let $0 < \beta < 1-\gamma$, then:*

(1) The range $R(Z_\alpha(t))$ of family $Z_\alpha(t)$ for $t > 0$ is contained in $D(A^B)$.

(2) $S'_\alpha(t)x = -t^{\alpha-1}AZ_\alpha(t)x$, and $S'_\alpha(t)x$ for $x \in D(A)$ locally integrable on $(0,\infty)$.

(3) For all $x \in D(A)$ and $t > 0$, $\|AS^\alpha(t)x\| \le ct^{-\alpha(1+\gamma)}\|Ax\|$, where n is a constant depending on γ, α.

(4) For every fixed $t \in \Sigma^o_{\pi/2-\delta}$, $Z_\alpha(t)$ is a bounded linear operator on X^β. Moreover, there exist a positive constant C_0 such that, for all $t \ge 0$

$$\|A^\beta Z_\alpha(t)x\| \le \alpha C_0 \frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha(\gamma+\beta))} t^{-\alpha(\gamma+\beta+1)}\|x\|.$$

Lemma 3.4. *If the resolvent $(\lambda + A)^{-1}$ is compact for every $\lambda > 0$, then $S_\alpha(t)$ and $Z_\alpha(t)$ are compact for every $t > 0$*

Proof. Let $\varepsilon > 0$ arbitrary number. We will demonstrate that

$$\zeta_\varepsilon(t) = \int\limits_\varepsilon^\infty \Phi_\alpha(s)T(st^\alpha - \varepsilon t^\alpha)ds, \ \xi_\varepsilon(t) = \int\limits_t^\infty \Phi_\alpha(s)T(st^\alpha)ds,$$

Then $\xi_\varepsilon(t) = T(\varepsilon t^\alpha)\zeta_\varepsilon(t)$ and it is easy to prove that for every $t > 0$, $\xi_\varepsilon(t)$ is a bounded linear operator on X. Next, with the use of compactness of $T(t), t > 0$ we can demonstrate that $\xi_\varepsilon(t)$ is compact for every $t > 0$.

Notice that

$$\|\xi_\varepsilon(t) - S_\alpha(t)\| \le \|\int\limits_0^\infty \Phi_\alpha(s)T(st^\alpha)ds\| \le c_0 t^{-\alpha(1+\gamma)} \int\limits_0^\infty \Phi_\alpha(s)s^{-1-\gamma}ds.$$

Therefore, from compactness of $\xi_\varepsilon(t)$, $t > 0$ it follows that $S_\alpha(t)$ is compact for every $t > 0$. Using similar reasoning we can conclude that $Z_\alpha(t)$ is compact for every $t > 0$. Lemma is proven.

4. White noise and Balakrishnan stochastic integral

Let H be a separable Hilbert space and $W = L_2((0,T), H)$ Hilbert space of all measurable functions

$$f : [0,T] \to H$$

such that

$$\int_0^T \|f(t)\|^2 < \infty,$$

where $\|\cdot\|$ is a norm in H.

Let's define the cylindrical measure μ on (W, Σ) (Σ - algebra of cylindrical sets) as

$$\mu(E) = \int\limits_{\tilde{B}} G(x)dx,$$

where $\tilde{B}$ is a Borel set in $\mathbb{R}^n$ isomorphically to B- base of cylinder E, and G(x) - n - dimensional Gaussian density with the zero mean and covariance equal to one. Such measure μ is said to be Gaussian measure on W.

Let $\mathbb{H}$ be a separable Pre-Hilbert space. It is obvious that $H \subset \mathbb{H}$. Borel measurable mapping $f : W \to \mathbb{H}$ is said to be random vector if the Gaussian measure μ can be extended to countably additive on σ - algebra of sets of type ($f^{-1}(B)$, A - Borel set on $\mathbb{H}$).

If $H = R^1$, then f is said to be a random variable.

Under the finitely additive white noise in W we understand the process with trajectory in $\omega(\cdot)$ in W with Gaussian measure μ and with the characteristic function

$$C(h) = E[exp(i \int_0^T [N(t), h(t)]dt)] = exp(-\frac{1}{2} \int_0^T [h(t), h(t)]dt).$$

This measure cannot be extended to the countably additive on W. Let $f(\cdot)$ stands for any Borel measurable function mapping W to another (separable) Hilbert space H_r. In general, there is no need to determine the distribution over H_r. Let P_N be any sequence of finite dimensional projectors on W such that P_N strongly converges to the identity operator. Then for every $N, f(P_N f)$ is a random variable. Assume that

$$\{f(P_N\omega)\}$$

is a Cauchy sequence of probability measures. Then

$$C(h) = \lim_N C_N(h), h \in H_r, \tag{4.1}$$

where

$$C_N(h) = E[exp(i[f(P_N\omega), h])]$$

determines the countably additive measure on Borel sets of space H_r. If the limit characteristic function $C(h)$ is independent of the partial sequence of projectors P_N, then we call $f(\cdot)$ physical random variable (PRV) and will determine the limit measure as distribution $f(\cdot)$. In this regard we recall the famous Prokhorov limit theorem [12]: for any Borel set A in H_r

$$\mu_N(B) \to \mu_f(B) \text{ if } \mu_f(\partial B) = 0, \tag{4.2}$$

where $\mu_N(\cdot)$ is a distribution function $f(P_N\omega)$.

Now using relations (4.1) and (4.2) will give the characteristic of the PRV. To that end, let's use the following concept of continuity.

Definition 4.1. Let H^1, H^2 are real separable Hilbert spaces. $F : H^1 \to H^2$ is continuous in $x \in H^1$ with respect to S - topology if for any $\varepsilon > 0$ exist Hilbert-–Schmidt operator $L_\varepsilon(x) : H^1 \to H^1$ such that

$$\|L_\varepsilon(x)(x - x')\| < 1 \tag{4.3}$$

follows

$$\|F(x) - F(x')\| < \varepsilon \tag{4.4}$$

F is uniformly S - continuous on $U \subset H^1$ if Hilbert-Schmidt operator in (4.3) is independent of $x \in U$

Definition 4.2. $F : H^1 \to H^2$ is said to be uniformly S - continuous in the neighborhood of origin if F is uniformly S - continuous on sets

$$U_n = \{u \in H^1 : \|L_n u\| \le 1\},$$

where $\{L_n\}_1$ is the sequence of Hilbert-Schmidt operators such that

$$\{L_n\}_{HS} \to 0 \text{ and } \bigcup_{n=1}^{\infty} U_n = H^1.$$

It is obvious that uniformly S - continuous mapping is also uniformly S - continuous in the neighborhood of the origin.

Taking into account inequalities (4.3), (4.4) and definitions (4.2) we can give a criterion characterizing the PRV.

Lemma 4.3. *Uniformly S - continuity of $F : H^1 \to H^2$ in the neighborhood of the origin is the sufficient condition that F is PRV.*

5. Stochastic evolution equation with linear drift

Let H be a separable Hilbert space, A - almost sectorial operator, i.e. $A \in \Theta_\sigma^\gamma(H)$, $-1 < \gamma < 0, 0 < \sigma < \pi/2$. According to the results from section 4 we determine the functional space $W = L_2((0,T), H)$, where $T \le \infty$. Let H_n - another separable Hilbert space (here n - stands for white noise) and let $W_n = L_2((0,T), H_n)$. Let A be a bounded linear operator from H_n to H.

We will consider evolution equation

$$^cD_t^\alpha u(t) + Au(t) = B\omega(t), t > 0, u(0) = u_0, \tag{5.1}$$

where $^cD_t^\alpha$ - fractional Caputo derivative order $\alpha, 0 < \alpha < 1$, $A \in Q_\sigma^\gamma(H), -1 < \gamma < 0, 0 < \sigma < \pi/2, \omega(t)$ - white noise with output values from another separable Hilbert space H_n, B - linear operator from H to H_n.

Since we want to emphasize the dependence of the solution on the random entry of ω, we will use the notation $u(t,\omega)$ for it.

The equation

$$[u(t,\omega), v] = [u(0), v] + \int_0^t (t-\tau)^{\alpha-1}[u(t,\omega), A^*v]d\tau +$$

$$\int_0^t (t-\tau)^{\alpha-1}[B\omega(\tau), v]d\tau, v \in D(A^*) \tag{5.2}$$

has a solution given by

$$u(t,\omega) = S_\alpha(t)u_0 + \int\limits_0^t (t-\tau)^{\alpha-1} Z_\alpha(t-\tau)B\omega(\tau)d\tau \tag{5.3}$$

and for every ω this is a unique solution in the class of weakly continuous functions which satisfies equation (5.2). Here $S_\alpha(t)$, $Z_\alpha(t)$ are family of resolvent operators introduced in section 3. The statement directly follows from theorem 4.8.3 of monograph [11] and paper [20]. Next, using the formula (5.3), we calculate the correlation operator corresponding to the process $u(t,\omega)$. Since $u(t,\omega)$ defined for every t, it can be found for every moment t. Suppose that u_0 is given, we obtain

$$E([u(t,\omega) - S_\alpha u_0 v][u(\tau,\omega) - S_\alpha u_0, z]) =$$

$$E(\int\limits_0^t [(t-\tau)^{\alpha-1} Z_\alpha(t-\tau)B\omega(\tau), v])d\tau \int_0^\tau [(\tau-\sigma)^{\alpha-1} Z_\alpha(\tau-\sigma)B\omega(\tau), z]d\sigma) =$$

$$= E(\int\limits_0^t [\omega(\tau), B^*(t-\tau)^{\alpha-1} Z^*_\alpha(t-\tau)v]d\tau \int\limits_0^\tau [\omega(\sigma), B^*(\tau-\sigma)^{\alpha-1} Z^*_\alpha(\tau-\sigma)]d\tau) =$$

$$\int\limits_0^t [\omega(\sigma), B^*(t-\sigma)^{\alpha-1} Z^*_\alpha(\tau-\sigma)v, B^*(t-\sigma)^{\alpha-1} Z^*_\alpha(\tau-\sigma)z]d\sigma =$$

$$[v, (t-\tau)^{\alpha-1} R_\alpha(\tau,\tau)z], t \geq \tau.$$

Hence, the correlation operator $R_{\alpha(t,\tau)}$ is determined by the formula

$$R_\alpha(t,\tau) = S_\alpha(t-\tau)R_\alpha(\tau,\tau), t \geq \tau, \tag{5.4}$$

where

$$R_\alpha(\tau,\tau)u_0 = \int\limits_0^\tau (\tau-\sigma)^{\alpha-1} Z_\alpha(\tau-\sigma)BB^* Z^*_\alpha(\tau-\sigma)u_0 d\tau. \tag{5.5}$$

From the last formulas directly follows the equation

$${}^cD^\alpha_t [R_\alpha(\tau,\tau)u_0, v] = [R_\alpha(\tau,\tau)A^* u_0, v] +$$

$$+[R_\alpha(\tau,\tau)u_0, A^* v] + [B^* u_0, A^* v]R_\alpha(0,0) = 0 \tag{5.6}$$

for all u_0, v from the domain of $D(A^*)$. And vice versa, the equation (5.6) has unique solution and the solution determines by formula (5.5).

Next, for every u_0 the following equation is true

$$[R_\alpha(\tau,\tau)u_0, u_0] = \int\limits_0^\tau \|B^* S^*(\tau-\sigma)u_0\|^2 d\sigma,$$

and therefore this value does not decrease when $\tau \geq 0$.

Let

$$[R_{\alpha\infty}u_0, u_0] = \lim_{\tau\to\infty}[R_\alpha(\tau,\tau)u_0, u_0].$$

The limit on the right side is finite if the following conditions are satisfied

$$\lim_{t\to\infty}\frac{1}{t}log\|S(t)\| = \omega_0 < 0 \tag{5.7}$$

In this case the operator $R_{\alpha\infty}$ will be a bounded linear operator determined by the equation

$$[R_\infty u_0, v] = \lim_{\tau\to\infty}[R_\alpha(\tau,\tau)u_0, v], u_0, v \in H \tag{5.8}$$

Taking into account equations (5.6) we can say that $R_{\alpha\infty}$ satisfies the equation

$$[R_{\alpha\infty}A^*u_0, v] + [R_{\alpha\infty}u_0, A^*v] + [B^*u_0, B^*v] = 0 \tag{5.9}$$

for all $u_0, v \in H$ from the domain of operator A^*.

It is obvious that condition (5.7) is not necessary for the operator $R_{\alpha\infty}$ to be defined correctly. As an example, we might consider the case of a compact semigroup such that $S(t)Bx = \exp(-\zeta t)x, \zeta > 0$. Notice that condition

$$[R_\alpha(\tau,\tau)u_0, u_0] = 0 \text{ implies } u_0$$

does not mean that zero belongs to the resolvent set. This fact reflects the specifics of the infinite case.

The fact is that in finite dimensional case operator $R(\tau,\tau)$ is always compact.

Next, let

$$\tilde{u}(t,\omega) = \int_0^t (t-s)^{\alpha-1}Z_\alpha(t-s)B\omega(s)ds.$$

Then the equation

$$L\omega = \tilde{u}(\cdot,\omega) \tag{5.10}$$

determines the bounded linear operator, which maps space W_n to W. The formula $\chi(c) = \mu[\omega : L\omega \in C\}]$ (see section 4) determines Gaussian cylindrical measure on the class of cylindrical sets from W. It will be countably additive τ algebra of Borel sets of this space if an only if when L is a Hilbert-Schmidt operator, or if and only if when$Z_\alpha(t)B$ is Hilbert-Schmidt operator for almost all $t \in (0,T)$ and

$$\int_0^t\int_0^t \|(t-\tau)^{\alpha-1}Z_\alpha(t-\tau)B\|^2 d\tau dt < \infty.$$

This condition will be satisfied if $Z_\alpha(t)$ is a Hibert-Schmidt operator, or if B is Hilbert- Schmidt operator. Notice that

$$E((L\omega)(L\omega)^*) = LL^*.$$

The above reasoning makes it possible to formulate the following statement.

Theorem 5.1. *Let $f(\omega) = L(\omega)$, where L is any bounded linear transformation from H to H_n. $f(\cdot)$ is physical random variable if and only if when L is Hilbert-Schmidt operator.*

6. Stochastic evolution equations with nonlinear drift

In this section we will consider nonlinear fractional Cauchy problem

$$^{c}D_t^{\alpha}u(t) + Au(t) = f(u(t)) + B\omega(t), t > 0, u(0) = u_0, \tag{6.1}$$

where $A \in \Theta_{\sigma}^{\gamma}, -1 < \gamma < 0, 0 < \sigma < \pi/2$. Under weak solution of problem (6.1) we understand function $u(t,\omega) : W_n = L_2((0,T), H_n) \to H$ which satisfies the equation

$$u(t,\omega) = S_{\alpha}(t)u_0 + \int_0^t (t-\tau)^{\alpha-1}Z_{\alpha}(t-\tau)d\tau[f(u(\tau,\omega)) + B\omega(\tau)]d\tau, \tag{6.2}$$

μ_G - Gaussian measure on W_n, ω point in H_n, $B : H \to H_n$ - bounded operator and $f : H \to H$ Lipschitz uniformly mapping such that

$$\|f(u_1) - f(u_2)\| \le L(f)\|u_1 - u_2\|, \tag{6.3}$$

where $L(f)$ is a constant which depends on f.

We will determine the integral contour Γ_{σ} directed in a way that open sector Σ_{σ}^{0} is on the right side of Γ_{σ}. Let $\beta \in \mathbb{C}$ and $A \in \Theta_{\sigma}^{\gamma}(H), -1 < \gamma < 0, 0 < \sigma < \pi/2$. Then the complex power A^{β} of operator A determines in the following way

$$A^{\beta} = \lambda^{\beta}(A) = \frac{1}{2\pi i}\int_{\Gamma_{\sigma}} \lambda^{\beta}(\lambda - A)^{-1}d\lambda, \lambda \in \mathbb{C} \setminus (-\infty, 0]$$

In the case $Re(\beta) > 1 + \gamma, A^{-\beta}$ belongs to $\mathfrak{L}(H)$ - space of all bounded linear operators from H to H. Linear space $H^{\beta} = D(A^{\beta})$ is a Banach space with the norm

$$\|u\|_{\beta} = \|A^{\beta}u\|, u \in H^{\beta}.$$

Let's formulate and prove the following statement.

Theorem 6.1. *Suppose (6.3) and $u(t,\omega)$ satisfies the problem (6.1). Then for every $u^0 \in H^{\beta}$ function $u(t,\omega)$ satisfies the equation*

$$u(t,\omega) = S_{\alpha}(t) + \int_0^t (t-\tau)^{\alpha-1}Z_{\alpha}(t-\tau)f(u(t,\omega))d\tau +$$

$$\int_0^t (t-\tau)^{\alpha-1}Z_{\alpha}(t-\tau)B\omega(\tau)d\tau \tag{6.4}$$

Proof. It is obvious that in the conditions of theorem the problem (6.1) is equivalent to the solution of integral stochastic equation

$$u(t,\omega) = u_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} Au(\tau,\omega)d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(u(\tau,\omega))d\tau +$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} B\omega(\tau)d\tau, t \geq 0. \tag{6.5}$$

Applying Laplace transformation to both sides of (6.5), we obtain

$$U(s) = S^{\alpha-1}(S^\alpha + A)^{-1}u_0 + (S^\alpha + A)^{-1}F(s) + (S^\alpha + A)^{-1}G\omega(s), \tag{6.6}$$

where

$$U(s) = \int_0^\infty e^{-st}u(t)dt, F(s) = \int_0^\infty e^{-st}f(u(t))dt,$$
$$G\omega(s) = \int_0^\infty e^{-ts}G\omega(t)dt, Res > \gamma.$$

Using lemmas from section 3, integration by parts and taking into consideration (6.5) we obtain

$$S^{\alpha-1}(S^\alpha + A)^{-1}u_0 + (S^\alpha + A)^{-1}G\omega(s) =$$
$$= \int_0^\infty e^{-st}S_\alpha(t)u_0dt + \int_0^\infty e^{-st}(\int_0^t (t-\tau)^{\alpha-1}Z_\alpha(t-\tau)f(u(\tau)dt))dt +$$
$$+\int_0^\infty e^{-st}(\int_0^t (t-\tau)^{\alpha-1}Z_\alpha(t-\tau)B\omega(\tau)d\tau)dt. \tag{6.7}$$

Combining (6.6) and (6.7) and taking into consideration theorem about uniqueness of Laplace transformation, we obtain

$$u(t,\omega) = S_\alpha(t)u_0 \int_0^t (t-\tau)^{\alpha-1}Z_\alpha(t-\tau)f(u(\tau,\omega))dt +$$
$$\int_0^t (t-\tau)^{\alpha-1}Z_\alpha(t-\tau)B\omega(\tau)d\tau.$$

Theorem 6.2. *Let the conditions of theorem 6.1. are satisfied. Then the solution $u(t,\omega)$ of problem (6.1)is a physical random variable, if A is Hilbert- Schmidt operator and/or B is Hilbert-Schmidt operator*

Proof. Let's write down the equation (6.1) in an integral form

$$u(t,\omega) = u_0 + \int_0^t (t-\tau)^{\alpha-1}Au(\tau,\omega)d\tau +$$
$$\int_0^t (t-\tau)^{\alpha-1}f(u(\tau,\omega))d\tau + \int_0^t (t-\tau)^{\alpha-1}B\omega(\tau)d\tau \tag{6.8}$$

or equivalently

$$u(t,\omega) = S_\alpha(t)u_0 + \int_0^t (t-\tau)^{\alpha-1} Z_\alpha(t-\tau) f(u(\tau,\omega))d\tau +$$
$$\int_0^t (t-\tau)^{\alpha-1} Z_{\alpha(t-\tau)} B\omega(\tau)d\tau \tag{6.9}$$

Solution (6.9) is a unique weak continuous solution of problem (6.1)

$$[u(t,\omega),v] = [u_0,v] + \int_0^t [(t-\tau)^{\alpha-1}u(t,\omega),v]d\tau +$$
$$+ \int_0^t [(t-\tau)^{\alpha-1} f(u(\tau,\omega)),v]d\tau + \int_0^t [(t-\tau)^{\alpha-1}B\omega(\tau),v]d\tau$$

for every $v \in D(A^*)$, where $D(A^*)$ is the domain of conjugate almost sectorial operator A^*. We would like to demonstrate that $u(\cdot,\omega)$ is a physical random variable.

From (6.9) we obtain

$$\|u(\cdot,\omega_1) - u(\cdot,\omega_2)\|_W^2 \leq$$
$$\leq 2\int_0^T \| \int_0^t (t-\tau)^{\alpha-1} Z_\alpha(t-\tau)[f(u(\tau,\omega_1)) - f(u(\tau,\omega_2))]d\tau\|^2 dt +$$
$$2\int_0^T \| \int_0^t (t-\tau)^{\alpha-1} B[\omega_1(\tau) - \omega_2(\tau)]d\tau\|^2 dt \leq$$
$$\leq 2\int_0^T \| \int_0^t (t-\tau)^{\alpha-1} Z_\alpha(t-\tau)\|^2 \|[f(u(\tau,\omega_1)) - f(u(\tau,\omega_2))]\|^2 d\tau dt +$$
$$2\int_0^T \| \int_0^t (t-\tau)^{\alpha-1} Z_\alpha(t-\tau) B[\omega_1(\tau) - \omega_2(\tau)]d\tau\|^2 dt \leq$$
$$2M(\alpha)e^{kT}L(f)\int_0^T \| \int_0^t \|u(\tau,\omega_1)) - f(u(\tau,\omega_2)\|_H^2 d\tau dt +$$
$$2\int_0^T \| \int_0^t (t-\tau)^{\alpha-1} Z_\alpha(t-\tau) B[\omega_1(\tau) - \omega_2(\tau)]d\tau\|_H^2 dt. \tag{6.10}$$

Let $m(t) = \int_0^t \|u(\tau,\omega_1)) - f(u(\tau,\omega_2)\|_H^2 d\tau$,

$$P(\alpha) = 2M(\alpha)e^{kT}L(f), \beta(T) =$$
$$= 2\int_0^T \| \int_0^t (t-\tau)^{\alpha-1} Z_\alpha(t-\tau) \times BL[\omega_1(\tau) - \omega_2(\tau)]d\tau\|_H^2 dt,$$

then (6.10) can be written in the form

$$m(T) \leq \int_0^T Nm(t)dt + \beta(T) \tag{6.11}$$

and from Gronwall inequality it follows that

$$m(t) \leq e^{NT}\beta(T) \tag{6.12}$$

or

$$\|u(\cdot,\omega_1)-u(\cdot,\omega_2)\|_W^2 \le 2e^{NT}\|\int_0^t (t-\tau)^{\alpha-1}Z_\alpha(t-\tau)B[\omega_1(\tau)-\omega_2(\tau)]d\tau\|_W^2.$$

Now, since $Z_\alpha(t-\tau)(t-\tau)^{\alpha-1}$ is Hilbert-Schmidt operator, we can determine operator $L: W^1 \to W^2$ such that

$$Lf = g, g = \int_0^t (t-\tau)^{\alpha-1}Z_\alpha(t-\tau)Bfd\tau.$$

then introducing $\gamma(N) = 2e^{NT}$, we obtain

$$\|u(\cdot,\omega_1)-u(\cdot,\omega_2)\|_W^2 \le \gamma(N)\|L(\omega_1-\omega_2)\|_W^2$$

or $u(\cdot,\omega)$ is uniformly continuous in S - topology and $u(\cdot,\omega)$ is physical random variable.

The theorem is proven.

Notice that similar problems were considered also in the papers [23-26].

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