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AN APPLICATION OF SINGLE TERM HAAR WAVELET SERIES SOLUTION TO NON-LINEAR STIFF DIFFERENTIAL EQUATIONS ARISING IN PHYSICS

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Abstract

The paper presents Single Term Haar Wavelet Series (STHWS) approach to the solution of nonlinear stiff differential equations arising in Physics. The properties of single term Haar wavelet series are given. The method of implementation is discussed. Numerical solutions of some model equations are investigated for their stiffness and stability and numerical solutions are obtained to demonstrate the suitability and applicability of the method. The results in the form of block-pulse and discrete solutions are given, for non-linear stiff systems. As compared with the TR BDF2 method of Shampine [12] and Gill's metheod. The STHWS turns out to be more effective in its ability to solve systems ranging from mildly to highly stiff equations. The method works even when higher order numerical methods for stiff systems fail to give solutions. The novel feature of the scheme is development of an algorithm exclusively for solving systems of non-linear algebraic equations resulting from STHWS. It reduces the computational effort enormously, at the same time, meeting accuracy requirements. We can see that the STHWS method takes substantially fewer steps and is about three times faster than Gill's method.

Keywords: *STHWS; Operational matrix; Block-pulse and discrete solutions; Gill's Method, Non-Linear Stiff systems.*

INTRODUCTION

Many physical systems such as nuclear reactors and laser oscillators etc. give rise to nonlinear ordinary stiff differential equations, the magnitudes of the eigenvalues of which vary greatly. Also, some differential equations are characterized by a property; of having solutions varying on completely different time scales. It is common to refer to such differential equations as stiff. The problem associated with stiff systems is twofold; stability and accuracy. The conventional and current methods, have, by and large, been proved to be stable [15]. However, in some cases, using a small step size can introduce enough round-off errors to cause instability, making the classical single-step methods unsuitable. The main focus in the problem of devising numerical methods of stiff differential equation is therefore, to reduce the computation time for a given level of accuracy. Stiff problems typically arise in Ionospheric physics, Chemical kinetics, Control theory, Biochemistry, Climatology, Electronics etc. where there is a slowly varying equilibrium solution with rapidly decaying transients. In practical solutions, the choice of step size, indeed, plays a crucial role. Although, non stiff methods can solve stiff problems, they just take a long time to do it. Integration of such equations, for instance, using traditional (Runge-Kutta) methods may consume too much time. Implicit methods might be the only option to reduce computation time. However, implicit methods are not very effective.

Carrol [4] presents a scheme for solving the stiff system of initial value problems which converge very fast. A long-standing work by Pavlov and Rodionova[8] retains non-linear terms of the original equation. Nevertheless, these methods have not succeeded in reducing the computation effort substantially. In this paper, an attempt is made to extend the Single Term Haar Wavelet series (STHWS) method proposed by Hsiao [6] to solve non-linear stiff systems. The method essentially, consists of choosing the single-segment approximation using Haar wavelets. To retain the specified level of accuracy, an interval $[0, 1/m)$ is stretched to unit length in which, only the first term of the Haar series expansion needs to be considered. Incidentally, one has the freedom to compute over any number of segments without the restriction of $m = 2^k$. This method was used with considerable success recently by various authors for solving linear stiff systems and their variants [6]. In this paper we follow a slightly a different approach. Following the works of Sepehrian *et al.* [10], we develop an algorithm to solve non-linear algebraic system of equations, resulting from the application of STHWS recursively. This innovative algorithm, in addition to providing block-pulse (piece-wise constant) solutions, gives discrete (point-wise) solutions as a by-product. The unique feature of this method is that, it avoids computation of operation matrices of integration and product matrices, thereby reducing the computation time to optimum level. It may be remarked that, while STHWS technique has long been applied to various problems of dynamic systems [6], to the author's knowledge, they have not been applied to non-linear stiff systems encountered in Physics. The major advantage of this approach is that, computations can be continued to any desired length of time after ensuring stability. Secondly, besides giving the option to choose between two types of solutions mentioned earlier, the recursive formula lends itself to the solution of non-linear stiff systems.

Survey of stiff solvers: We give here a brief survey of stiff solvers designed and tested, with comments on their relative merits and demerits. Classical numerical schemes do have some disadvantages in analyzing stiff differential equations. This can sometimes present severe problems and much effort has been expended to cope effectively with stiffness. The simplest remedy is to resort to implicit schemes. But, the nice features of implicit schemes hold only for linear systems. While implicit methods are first order accurate, most stiff problems benefit from higher order methods. There are three higher order methods for stiff systems: The classical Runge-Kutta method of 4^{th} 0rder, Generalization of Buliresh-Stoer method, and Predictor-Corrector method.

During 1970's Walsh functions and their cousins Haar wavelets received considerable attention in dealing with various problems of dynamical systems. Initially using orthogonal functions to construct operational matrices for solving optimization problems of dynamical systems was established. The pioneering work in system analysis via Haar Wavelets was initiated by Chen and Hsiao [3], who first derived a Haar Operational matrix for integration. Since then many operational matrices based on various orthogonal functions, such as Walsh, Block-pulse, Laguerre, Legendre, Chebyshev and Fourier have been developed. The main characteristic of this technique was to convert a differential equation into an algebraic equation, as a result of which, the solution procedures are greatly reduced or simplified. All the orthogonal functions mentioned above, however, are supported on the whole interval $[a, b]$. This kind of global support made them unsuitable for certain analyses, involving abrupt variations lasting for a very short duration. The operational matrix established for Haar wavelets eliminated all the drawbacks caused by the whole range support. A new approach called "single segment approximation" which avoids operational matrices of prohibitively large size and maximizes the reduction in the computational effort; was introduced by Rao et al. [9] using Walsh functions. The effectiveness of Single Term Walsh Series (STWS) method was further demonstrated by Balchandran and Muragesan [1]. A method for solving time-varying singular nonlinear systems by single-term Walsh series was proposed by Sepehrian and Razzaghi [10]. Hsiao [6] proposed a simple and effective algorithm based on the single-term Haar wavelet series (STHWS) for solving only linear stiff systems. The key idea behind STHWS is to represent the time-varying functions and their derivatives using only the first term of the Haar wavelet series and using the locality and orthonormality properties of Haar wavelets in transforming stiff systems into a system of algebraic equations. For many years the numerical solution of stiff ordinary differential equations has been an active field of research motivated by challenging real life applications. The presently most effective algorithm for this type of problems seems to be the TR BDF2 method of Shampine

[12] an implicit Runge – Kutta formula with a first stage that is a trapezoidal rule step and a second stage that is a backward differentiation formula of order two. By construction the same iteration matrix is used in evaluating both stages. As this method is based on an implicit discretization , it requires the iterative solution of a nonlinear algebraic system per each integration step. In view of this feature recent attention has focused on so called STHWS which uses the solution of nonlinear algebraic system using the initial values only once and then follows it up recursively. In this paper it is planned to look for a novel scheme so as to extend STHWS scheme to nonlinear stiff systems, that too, arising in Physics.

The paper is organized as follows. Section 2 is devoted to theoretical background of Haar wavelets and STHWS, which is relevant for the material that follows. Section 3 is about the method of solution and its implementation aspects. Section 4 is concerned with the connection between Stiffness and stability. Section 5 is mainly concerned with the application of the proposed scheme to the test problems to demonstrate the efficiency and effectiveness of the algorithm. Numerical findings are presented in Sec.5 in the form of Tables and graphs.

PROPERTIES OF HAAR WAVELET SERIES AND SINGLE TERM HAAR WAVELET SERIES

Haar wavelet series

The orthonormal basis $\{h_n\}$ of Haar wavelets for the Hilbert space $L_2[0,1]$ consists of $h_n = h_1(2^j t - k)$, $\mathrm{n} = 2^{\mathrm{j}} + k$. $\mathrm{j} \geq 0,\ 0 \leq \mathrm{k} \leq 2^{\mathrm{k}}$

where, $$h_0(t) = 1.\ 0 \leq t < 1, \quad h_1(t) = \begin{cases} 1. & 0 \leq t < 0.5 \\ -1. & 0.5 \leq t < 1 \end{cases} \tag{2.1}$$

Each Haar wavelet h_n has the support $\left(2^{-j}k, 2^{-j}(k+1)\right)$, so that it is zero elsewhere in the interval [0, 1). Interestingly, as n increases the Haar wavelets become more and more localized. Therefore $\{h_n\}$ forms a local basis. In contrast, Walsh functions which take only the values 1 and -1 form a global basis. They may be expressed as linear combinations of the Haar wavelets, so many results about the Haar wavelets carry over to the Walsh system easily [6]. Moreover, the Walsh functions are precisely the Haar wavelet packets [16].

Any function $f \in L_2\left([0,1)\right)$ can be expanded in Haar series:

$$f(t)=\sum_{i=0}^{\infty} c_i h_i(t) \text{ where } \mathrm{c_i} = 2^j \int_0^1 f(t) h_i(t) dt \tag{2.2}$$

The convergence in (2.2) is in the L_2 sense i.e. "mean convergence". Accordingly, the *Haar coefficients* c_i are determined such that

$$\left| f(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right| \to 0, \; m = 2^j, \quad j \in \{0\} \cup N \tag{2.3}$$

In applications Haar series are always truncated to m terms, that is

$$f(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = \mathbf{c}_m^T \mathbf{h}_\mathrm{m}(t), \quad \mathrm{t} \in [0,1) \tag{2.4}$$

The coefficient vector $\mathbf{c}_m$ and Haar wavelet vector $\mathbf{h}_m(t)$ are defined as

$$\mathbf{c}_\mathrm{m} = [c_0, c_1, ---\mathrm{c}_{\mathrm{m-1}}]^T$$
$$\mathbf{h}_\mathrm{m}(t) = [h_0(t), h_1(t), ---h_{m-1}(t)]^T$$

The Haar transform matrix H_m is defined as

$$H_m(t) = \begin{bmatrix} h_0(t) \\ h_1(t) \\ - \\ - \\ h_{m-1}(t) \end{bmatrix} \text{ e.g. } H_1(t) = [1], \; \mathrm{H}_2(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.5}$$

In studying differential equation models of dynamical systems using Haar wavelets, it is frequently required to perform integrations in order to solve the problem. Since the differentiation of Haar wavelets results in generalized functions, which in any case should be avoided, the integration of Haar wavelets are preferred. Integration of Haar wavelets [3] should be expandable in Haar series:

$\int_0^t h_m(\tau)d\tau = \sum_{i=0}^{\infty} d_i h_i(t)$. If we truncate to $m = 2^n$ terms and use the above vector notation, then integration is performed by matrix-vector multiplication defined by [6]:

$$\int_0^t h_m(\tau)d\tau \approx P_{(m\times m)}h_m(t), \quad t\in[0,1) \tag{2.6}$$

where P is called the Operational matrix of integration which satisfies the following recursive formula:[3].

$$P_{(m\times m)} = \frac{1}{(2m)}\begin{bmatrix} 2mP_{\left(\frac{m}{2}\times\frac{m}{2}\right)} & -H_{\left(\frac{m}{2}\times\frac{m}{2}\right)} \\ H^{-1}_{\left(\frac{m}{2}\times\frac{m}{2}\right)} & 0_{\left(\frac{m}{2}\times\frac{m}{2}\right)} \end{bmatrix}$$

$P_{(1\times1)} = \frac{1}{2}$,

where

$H_{m\times m} = [h_{(m)}(x_0), h_{(m)}(x_i), \ldots\ldots\ldots\ldots\ldots\ldots h_{(m)}(x_{m\text{-}1})]$.

$$\frac{i}{m} \le x_i < \frac{(i+1)}{m}, \quad \text{and} \quad H^{-1}_{(m\times m)} = \left(\frac{1}{m}\right)H^T_{(m\times m)}\text{diag}(r),$$

$$r = [1, 1, 2, 2, 4, 4, 4, 4, \ldots\ldots\ldots, \underbrace{\frac{m}{2}, \frac{m}{2}, \ldots\ldots\frac{m}{2}}_{\frac{m}{2}\ elements}]^T, \ m > 2.$$

Single term Haar wavelet series (STHWS)

With the single Term Haar wavelet series approach, in the first interval, the given function is expanded as STHWS in the normalized interval $\tau \in [0,1]$ which corresponds to t$\in [0, \frac{1}{m})$ by taking $\tau = mt$, m being any integer. In STHWS, the matrix P in (2.6) becomes $P = \frac{1}{2}$ [6].

Let $y'(\tau)$ and $y(\tau)$ be expanded by STHWS in the first interval as

$$y'(\tau) = V_{(1)} h_0(\tau), y(\tau) = Y_{(1)} h_0(\tau) \quad (2.7)$$

and in the i^th interval as

$$y'(\tau) = V_{(i)} h_0(\tau), y(\tau) = Y_{(i)} h_0(\tau) \quad (2.8)$$

Integrating (2.7) with P = $\frac{1}{2}$, we get

$$Y_{(1)} = \frac{1}{2} V_{(1)} + y(0) \quad (2.9)$$

where y(0) is the initial condition and according to Sannuti[5],we have

$$V_1 = \int_0^1 y'(\tau) d\tau = y(1) - y(0) \quad (2.10)$$

In general for i = 1,2, ...,

we obtain , $Y_{(i)} = \frac{1}{2} V_{(i)} + y(i-1)$ (2.11)

$$y(i) = V_{(i)} + y(i-1) \quad (2.12)$$

In (2.11) and (2.12), $Y_{(i)}$ and $y(i)$ give the block pulse and the discrete values of the state, respectively[10].

SOLUTIONS OF NONLINEAR STIFF SYSTEMS BY STHWS METHOD

A model of Physical system may be constructed from assumptions concerning the interaction and behavior of the components of the system. Some models may give rise to a set of differential equations of order greater than unity. However, such models lead to a set of differential equations, typically of the form:

$$\frac{d\,y_i}{dt} = f_i(y_i(t)) \tag{3.1}$$

each derivative being defined as known function of the concentrations. Here the functions depend on parameters which are assumed to be known. To complete the model, conditions are imposed $y_i(t) = \alpha_i$ (known) at $t = 0$, i = 1, 2,- -N, defining an initial value problem.

Here N is the number of equations and the nonlinear function $f_i \in R^n$. the state $y_i(t) \in R^n$. the response $y_i(t)$ is required to be found. The independent variable‘t’ does not appear explicitly in Equation. (3.1). Such a system is called autonomous.

Normalizing the time interval of (3.1) we let $\tau = mt$. then we get

$$m\,y'_i(\tau) = f_i(y_i(\tau)), \qquad y_i(0) = \alpha_i \tag{3.2}$$

Let $y'_i(\tau)$ be expressed by STHWS in the i^{th} interval as

$$y'_i(\tau) = V_{(i)}\,h_0(\tau) \tag{3.3}$$

By using (2.8) and (2.11), we get

$$y_i(\tau) = (\frac{1}{2}V_i + y(i-1))\,h_0(\tau) \tag{3.4}$$

To solve (3.2) we substitute (3.4) in $f_i(y_i(\tau))$.

We then express the resulting equation by STHWS as

$$f((\frac{1}{2}V_{(i)} + y(i-1))\,h_0(\tau)) = F^{(i)}\,h_0(\tau) \tag{3.5}$$

Using (3.3) and (3.4) in (3.2), we get

$$mV_i = F^{(i)} \tag{3.6}$$

Choosing a fixed value for '*m*', equation (3.6) becomes

$$\sum_{j=1}^{n} a_{ij} V_j = b_i, \quad \text{i} = 1, 2.....n. \tag{3.7}$$

where $a_i{}'s \; and \; b_i{}'s$ are constants

Equation (3.7) is a nonlinear system of algebraic equations for V_i (i = 1,2,....n). First we obtain one set of $V_i{}'s$ by solving Equation. (3.7) using Brown's method. Equation.(2.12) gives $y_i{}'s$ by using $V_i{}'s$ and the given initial conditions.

Later using known $y_i{}'s$ as a initial values, Equation (3.7) have to be solved by iteration using the following recurrence relation

$$V_i = \frac{bi - \sum_{i \neq j} a_{ij} \; V_j}{a_{ii}}, \qquad \text{i} = 1,2,\text{- - n} \tag{3.8}$$

together with Equations. (2.11) and (2.12) recursively, to get block-pulse and discrete solutions at any specified level of accuracy.

THE CONNECTION BETWEEN STIFFNESS AND STABILITY

The fast change of y_i' in (3.1) occurs over a very short time scale. We therefore have a situation where the step size is controlled by the maximum eigenvalue (i.e., very small step size), whereas the full evolution is controlled by the smallest eigenvalue. The values of the eigenvalues are therefore crucial in assessing the behavior of the system. This is one of the features of stiff systems. This occurs when small and large time constants occur in the same system. The small time constant controls earlier response, whereas the large one controls tailing.

To measure the degree of stiffness, one can introduce the following stiffness ratio:

$$SR = \frac{\max|\lambda|}{\min|\lambda|} \tag{4.1}$$

When $SR < 20$ the problem is not stiff, up to $SR = 1000$ the problem is classified stiff, and when $SR => 1{,}000{,}000$ the problem is very stiff (Finlayson [5]).

For a nonlinear problem of the type (3.1), we linearize the above equation around time $t_n(y_n)$ using a Taylor expansion. Retaining only the first two terms

$$\frac{dy_i}{dt} \approx f(y_n) + J(t_n).(y - y_n) \tag{4.2}$$

where

$$J(t_n) = \left\{ J_{ij} = [\frac{\partial f_i(y)}{\partial y_j}]t_n \right\} \tag{4.3}$$

which is the Jacobin matrix for the problem at $t = t_n$.

Here, J_{it} is the element of J in row i , column j. The definition of stiffness in Eq. (4.2) utilizes the eigenvalues obtained from the Jacobin, and since this Jacobin matrix changes with time, the stiffness of the problem also changes with time.

Let us briefly review how (Byrne & Hindmarsh [7], Shampine & Gear [13]) investigate stability. Suppose we have two distinct solutions y and w. Then

$$y' - w' = f(t, y) - f(t, w)$$

If we neglect higher order terms, then

$$y' - w' = \mathrm{f}_{\mathrm{y}}(t, w)(y - w)$$

If we assume that *y-w* is sufficiently small, in an approximate sense, then

$$y' - w' = J(y - w)$$

We assume that J is locally a constant. If J is a stable matrix (all eigenvalues of J have negative real parts) then $(y - w) \to 0$ as $t \to \infty$. If we reevaluate J as t increases, requiring that each J be locally constant and a stable matrix, then it follows that y and w tend to the same finite function as $t \to \infty$. That is, (3.1) is stable. By stable, we mean that given any two particular solutions y and w of (3.1), they tend to the same finite function as $t \to \infty$. (Other kinds of stability are also important, but this is the one needed here.) The connection between stiff ODEs and stable ODEs is this: Stiff ODEs are extremely stable, in that there is at least one eigenvalue with a large negative real part. In fact, they can be called super-stable (Shampine & Gear[13]).

APPLICATIONS

We apply in this section, the STHWS Method to three nonlinear stiff differential equations, which arise in physics. We present them here mainly with a view to using the algorithm given in section 3.

A Ruby Laser Oscillator Model

This pair of coupled non linear stiff system (Byrne and Hindmarsh[2])represents a model of a ruby laser oscillator. If we let ϕ denote photon density and η denote dimensionless population inversion, then we can write

$$\frac{d\eta}{dt} = -\eta(\alpha\phi + \beta) + \gamma \tag{5.1}$$

$$\frac{d\phi}{dt} = \phi(\rho\eta - \sigma) + \tau(1+\eta) \tag{5.2}$$

where the parameters are as follows:

$$\alpha = 1.5\times 10^{-18},\ \beta = 2.5\times 10^{-6}, \gamma = 2.1\times 10^{-6},\ \rho = 0.6,\ \sigma = 0.18, \tau = 0.016.$$

The initial conditions are:

$$\eta(0) = -1,\ \phi(0) = 1 \tag{5.3}$$

This problem is challenging because it is stiff initially, but mildly damped and oscillatory later.

with the initial conditions (5.3). The Jacobian at time t = 0,

$$J = \begin{bmatrix} -2.5*10^{-6} & -1.5*10^{-18} \\ 0.6 & -0.78 \end{bmatrix}$$

The eigenvalues of this Jacobian are, $\lambda = [-0.78,\ -2.5*10^{-6}]$

Stiff Ratio, $SR = 312000$, which indicates that the present problem is very stiff at time t = 0. Further this, problem is super stable since there is at least one eigenvalue with a negative real part.

The solution is presented in Table 1. and Fig. 1. with m = 1000, The STHWS solution agrees with the TR BDF2 method of Shampine[12]. But Gill's Method gives least accuracy.

Table 1
Comparison of STHWS solution with TR BDF2 method of Shampine [12] and Gill's Method solutions of Eq. (5.1)-(5.3)

T	*STHWS (m = 1000)*				*TR BDF2[12]*		*Gill's Method*	
	$\eta(t)$		$\phi(t)$		$\eta(t)$	$\phi(t)$	$\eta(t)$	$\phi(t)$
	DISC	*BP*	*DISC*	*BP*				
0	-1	-1	1	1	-1	1	-1	1
0.1	-1	-1	0.9401	0.9409	-1	0.9401	-1	1
0.2	-1	-1	0.8837	0.8849	-1	0.8837	-1	1
0.3	-1	-1	0.8307	0.8320	-1	0.8307	-1	1
0.4	-1	-1	0.7809	0.7822	-1	0.7809	-1	1
0.5	-1	-1	0.7341	0.7352	-1	0.7341	-1	1
0.6	-1	-1	0.6901	0.6911	-1	0.6901	-1	1
0.7	-1	-1	0.6487	0.6495	-1	0.6487	-1	1
0.8	-1	-1	0.6098	0.6105	-1	0.6098	-1	1
0.9	-1	-1	0.5732	0.5738	-1	0.5732	-1	1
1	-1	-1	0.5388	0.5394	-1	0.5388	-1	1
1.1	-1	-1	0.5065	0.5070	-1	0.5065	-1	1
1.2	-1	-1	0.4762	0.4766	-1	0.4762	-1	1
1.3	-1	-1	0.4476	0.4481	-1	0.4476	-1	1
1.4	-1	-1	0.4208	0.4212	-1	0.4208	-1	1
1.5	-1	-1	0.3955	0.3960	-1	0.3955	-1	1
1.6	-1	-1	0.3718	0.3723	-1	0.3718	-1	1
1.7	-1	-1	0.3495	0.3500	-1	0.3495	-1	1
1.8	-1	-1	0.3286	0.3290	-1	0.3286	-1	1
1.9	-1	-1	0.3088	0.3093	-1	0.3088	-1	1
2	-1	-1	0.2903	0.2907	-1	0.2903	-1	1

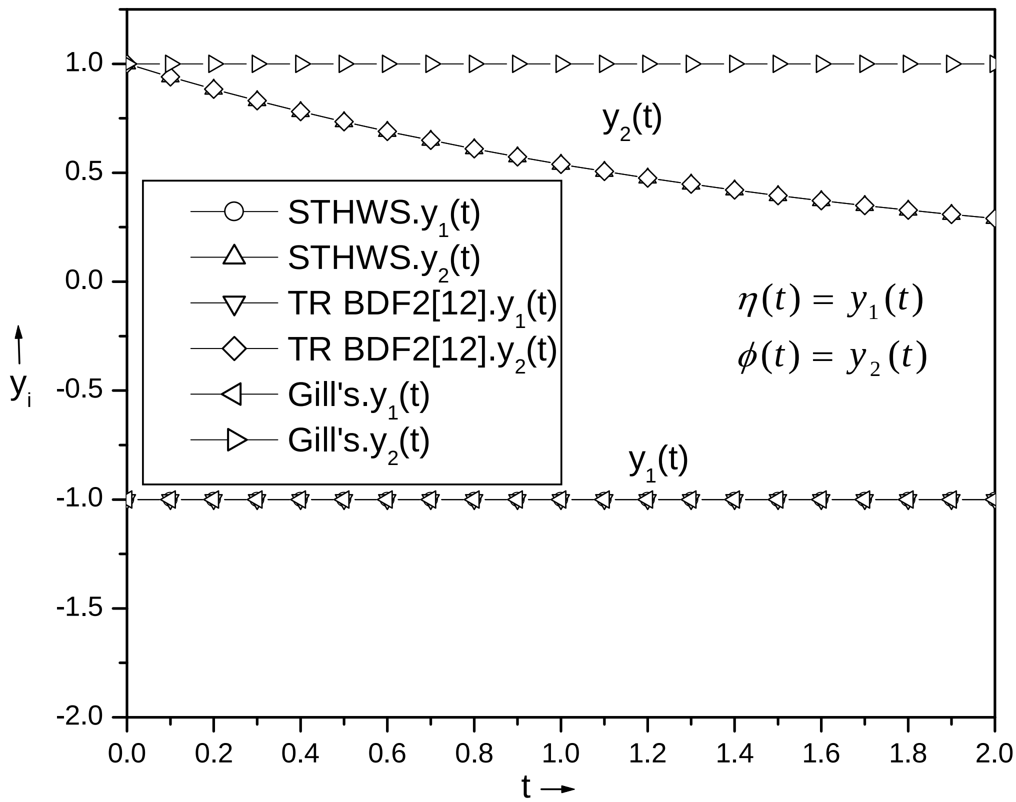

Fig. 1: Comparision of STHWS with TR BDF2 method of Shapine [12] and Gill's solutions of Eq. (5.1)-(5.3)

Next consider a nuclear reactor model (Scraton, [14]).

$$y'_1 = 0.01 - (0.01 + y_1 + y_2) \times [1 + (y_1 + 1000)(y_1 + 1)] \qquad (5.4)$$

$$y'_2 = 0.01 - (0.01 + y_1 + y_2)(1 + y_2^2) \qquad (5.5)$$

with $$y_1(0) = 0, \quad y_2(0) = 0. \qquad (5.6)$$

This is a nonlinear very stiff system that arises in nuclear reactor theory. with the initial conditions

The Jacobian at time $t = 0$, $J = \begin{bmatrix} -1011.01 & -1001 \\ -1 & -1 \end{bmatrix}$

The eigenvalues of this Jacobian are, $\lambda = [-1012,\ -0.0098913]$

Stiff Ratio, $SR = 102312$, which indicates that the present problem is very stiff at time t = 0. Further this, problem is super stable since there is at least one eigenvalue with a negative real part.

The solution is presented in Table 2. and Fig.2. with m =1000 , the STHWS solution agrees with the TR BDF2 method of Shampine[12] solution up to four decimal places. But Gill's method fails to give solution.

Table 2
Comparison of STHWS Solution with TR BDF2 Method of Shampine [12]
Solution of Eq. (5.4)-(5.6)

t	*STHWS (m = 1000)*				*TR BDF2 [12]*	
	$y_1(t)$		$y_1(t)$		$y_1(t)$	$y_2(t)$
	DISC	*BP*	*DISC*	*BP*		
0	0	0	0	0	0	0
0.1	-0.0110	-0.0109	0.0010	0.0009	-0.0110	0.0010
0.2	-0.0120	-0.0119	0.0020	0.0019	-0.0120	0.0020
0.3	-0.0130	-0.0129	0.0030	0.0029	-0.0130	0.0030
0.4	-0.0140	-0.0139	0.0040	0.0039	-0.0140	0.0040
0.5	-0.0150	-0.0149	0.0050	0.0049	-0.0150	0.0050
0.6	-0.0160	-0.0159	0.0060	0.0059	-0.0160	0.0060
0.7	-0.170	-0.168	0.0070	0.0069	-0.170	0.0070
0.8	-0.0180	-0.0178	0.0080	0.0079	-0.0180	0.0080
0.9	-0.0190	-0.0189	0.0090	0.0089	-0.0190	0.0090
1	-0.0199	-0.0199	0.0100	0.0100	-0.0199	0.0100
1.1	-0.0209	-0.0208	0.0110	0.0109	-0.0209	0.0110
1.2	-0.0219	-0.0218	0.0120	0.0119	-0.0219	0.0120
1.3	-0.0229	-0.0228	0.0130	0.0129	-0.0229	0.0130
1.4	-0.0239	-0.0239	0.0140	0.0139	-0.0239	0.0140
1.5	-0.0249	-0.0248	0.0150	0.0149	-0.0249	0.0150
1.6	-0.0259	-0.0258	0.0160	0.0159	-0.0259	0.0160
1.7	-0.0269	-0.0268	0.0170	0.0169	-0.0269	0.0170
1.8	-0.0279	-0.0279	0.0180	0.0179	-0.0279	0.0180
1.9	-0.0289	-0.0289	0.0190	0.0190	-0.0289	0.0190
2	-0.0299	-0.0299	0.0199	0.0199	-0.0299	0.0199

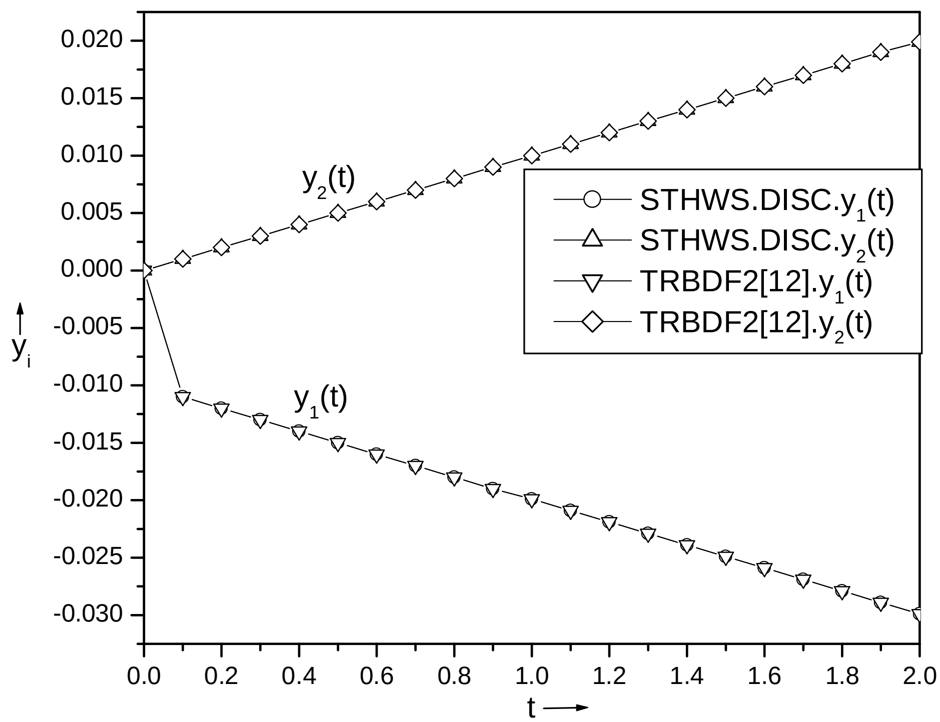

Fig. 2: Comparision of STHWS with TR BDF2 method of Shampine [12] solutions of Eq. (5.4)-(5.6)

Lastly consider fourth order non-linear stiff system, **Ionospheric physics problem**. (David s.Watkins [17])

$$y'_1 = -y_1 + y_2 y_4 + 1 \tag{5.7}$$

$$y'_2 = -2y_2 + y_4^2 + 1 \tag{5.8}$$

$$y'_3 = -3y_3 + y_1 y_2 \tag{5.9}$$

$$y'_4 = -1000y_4 + y_1^2 + y_2^2 + y_3^2 \tag{5.10}$$

with the initial values

$$y_1(0) = 1,\ \ y_2(0) = 1,\ \ y_3(0) = 34.4975,\ y_4(0) = 0.003006. \tag{5.11}$$

For a pair of coupled differential equations, the Jacobian will be a 4×4 matrix. To find the Jacobian at time t = 0, we simply replace y_i by their values at t = 0.

$$\text{i.e, } J = \begin{bmatrix} -1 & 0.003006 & 0 & 1 \\ 0 & -2 & 0 & 6.012*10^{-3} \\ 1 & 1 & -3 & 0 \\ 2 & 2 & 62.995068 & -100 \end{bmatrix}$$

The associated eigenvalues are $\lambda = [-1000, -3.03144, -1.99963, -0.966975]$ therefore $SR = 1034.15$, which indicates that the present problem is classified stiff at time t = 0. Further this, problem is super stable since there is at least one eigenvalue with a negative real part.

The solution is presented in Fig.3.with m = 1000. The STHWS solution agrees with the TR BDF2 method of Shampine[12] solution up to four decimals. But Gill's method fails to give solution.

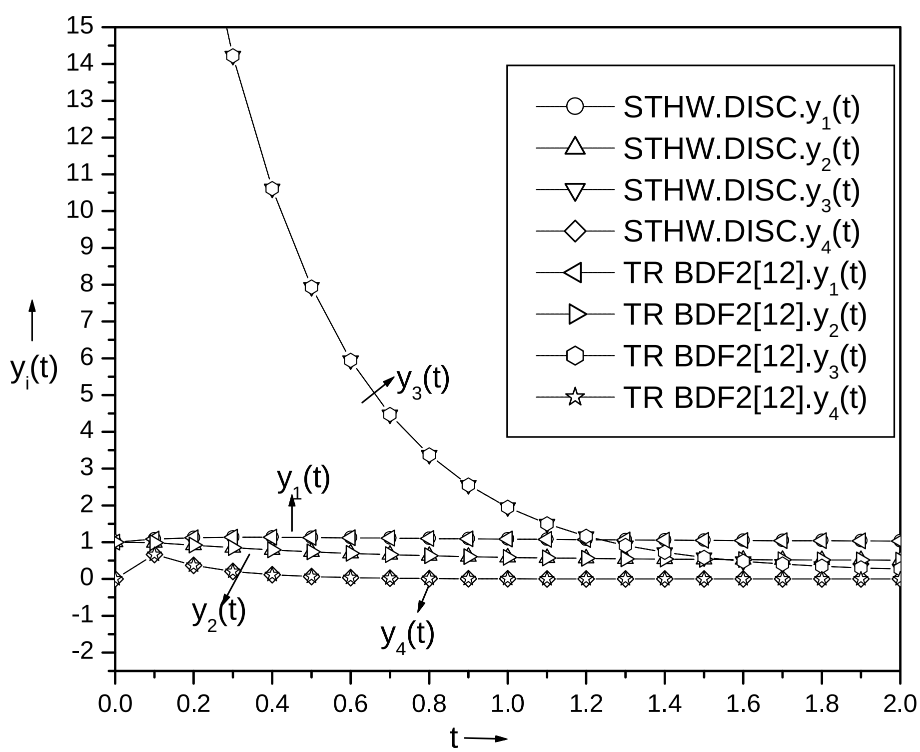

Fig. 3: Comparision of STHWS with TR BDF2 method of Shampine [12] solution of Eq. (5.7)-(5.11)

For the above three problems, we can see that the STHWS method takes substantially fewer steps and is about three times faster than Gill's method. Therefore STHWS technique is suitable for studying the behavior of this type of physical systems. The recurrence relationship give the discrete time solution for any length of time and it is easily amenable to digital computer.

CONCLUSION

In this paper, a novel efficient STHWS method based on a Haar wavelet series has been presented. From theoretical point of view the elegance of the Haar wavelets is appreciated from their simplicity and compact derivations and proofs. From practical standpoint, the accuracy and overall performance of the method is demonstrated by applying it to nonlinear stiff differential equations arising from Physics. It is observed that, it is the most reliable method; especially for non-linear stiff systems. In the ultimate analysis, an attempt is made to demonstrate the built-in features of STHWS in achieving accuracy without causing instability, no matter how small the step-size is chosen.

In general, the degree of stiffness (if any) of a problem is usually unknown. Normally, a conventional explicit method (such as Runge-kutta 4) is chosen to carry out the integration. It is only when such a method starts to take very small step sizes that stiffness is suspected and an STHWS method is chosen instead. Since the STHWS method has better stability properties than the classical explicit methods, it would be better to use first. If the Gill's method is taking an excessive amount of computing time (as for above problems) or fails to give a solution, then we would have to switch to an STHWS method. Hence regardless of the degree of stiffness, using the STHWS method instead of conventional explicit methods would generally save time. The numerical results show that the new stiff integrator compares favorably to existing integrators–even in large scale stiff ODE systems.

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