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Remains in cliquish functions on GTSs *

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Abstract

In this paper, we give several interesting properties of cliquish and lower (upper) semi-continuous functions in a generalized topological space. Also, we give more examples for the reverse implications, that is, cliquish implies lower (upper) semi-continuous function need not be true in a generalized topological space.

1 Introduction

The notion of a generalized topological space was introduced by Császár in [3]. Let X be any nonempty set. A family $\mu \subset exp(X)$ is a *generalized topology* [6] in X if $\emptyset \in \mu$ and $\bigcup_{t\in T} G_t \in \mu$ whenever $\{G_t \mid t \in T\} \subset \mu$ where $exp(X)$ is a power set of X. We call the pair (X, μ) as a *generalized topological space* (GTS) [6]. If $X \in \mu$, then the pair (X, μ) is called a *strong generalized topological space* (sGTS) [6]. Let $Y \subset X$. Then the *subspace generalized topology* is defined by, $\mu_Y = \{Y \cap U \mid U \in \mu\}$ and (Y, μ_Y) is called the *subspace GTS*.

Let (X, μ) be a GTS and $A \subset X$. The *interior of A* [6] denoted by iA, is the union of all μ-open sets contained in A and the *closure of A* [6] denoted

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by cA, is the intersection of all μ-closed sets containing A when no confusion can arise. The elements in μ are called the μ-open sets, the complement of a μ-open set is called the μ-closed sets and the complement of μ is denoted by $\mu^{'}$. Denote $\{U \in \mu \mid U \neq \emptyset\}$ by $\tilde{\mu}$ [5] and denote $\{U \in \mu \mid x \in U\}$ by $\mu(x)$ [5].

Throughout this paper, $\mathbb{Z}, \mathbb{Q}, \mathbb{N}$ and $\mathbb{R}$ denote the set of all integers, rational numbers, natural numbers and real numbers respectively.

2 Preliminaries

In this section, we recall some basic definitions and lemmas for further developement of this paper.

A subset A of a generalized topological space (X, μ) is said to be a *μ-nowhere dense* [5] (resp. *μ-dense* [6], *μ-codense* [6]) set if $icA = \emptyset$ (resp. $cA = X$, $c(X - A) = X$). A is said to be a *μ-meager* set [5] if $A = \bigcup_{n \in \mathbb{N}} A_n$ where each A_n is μ-nowhere dense for all $n \in \mathbb{N}$.

Also, every subset of a μ-nowhere dense (resp. μ-meager) set is μ-nowhere dense (resp. μ-meager) [5]. A is said to be a *μ-second category* (μ-II category) set [5] if A is not a μ-meager set. A is a *μ-residual* [5] set if $X - A$ is a μ-meager set.

A GTS (X, μ) is said to be μ-II category if X is μ-II category as a subset. A space X is called a *Baire space* (BS) [5] if each set $V \in \tilde{\mu}$ is of μ-II category in X. A space (X, μ) is a *strong Baire space* (sBS) [5] if $V_1 \cap V_2 \cap ... \cap V_n$ is a μ-II category set for all $V_1, V_2, ..., V_n \in \mu$ such that $V_1 \cap V_2 \cap ... \cap V_n \neq \emptyset$.

Moreover, every sBS is a BS [5].

Define $\mu^{\star} = \{\bigcup_t (U_1^t \cap U_2^t \cap U_3^t \cap ... \cap U_{n_t}^t) \mid U_1^t, U_2^t, ..., U_{n_t}^t \in \mu\}$ and $\mu^{\star\star} = \{A \subset X \mid A \text{ is a } \mu\text{-II category set}\} \cup \{\emptyset\}$ [5]. Then $\mu \subset \mu^{\star}$ and $\mu \subset \mu^{\star\star}$ if (X, μ) is a BS [5].

A space (X, μ) is called *hyperconnected* [4] if every non-null μ-open set G of (X, μ) is μ-dense. A space (X, μ) is called *generalized submaximal* [4] if every μ-dense subset of X is μ-open.

Let (X, μ) be a GTS. A function $f : X \to \mathbb{R}$ is said to be *μ-lower semi-continuous* (μ-l.s.c.) (resp. *μ-upper semi-continuous* (μ-u.s.c.)) [5] at a point

$x_0 \in X$ if and only if for any real number $\alpha < f(x_0)$ (resp. $\alpha > f(x_0)$), there exists a set $U \in \mu(x_0)$ such that $f(U) \subset (\alpha, \infty)$ (resp. $f(U) \subset (-\infty, \alpha)$).

Let λ, τ be generalized topologies in X. A function $f : X \to \mathbb{R}$ is (λ, τ)-*lower semi-continuous* ((λ, τ)-l.s.c.) (resp. (λ, τ)-*upper semi-continuous* ((λ, τ)-u.s.c.)) [5] at a point $x_0 \in X$ if for any real number $\alpha < f(x_0)$ (resp. $\alpha > f(x_0)$), there exists a set $U \in \tau(x_0)$ being a λ-residual set such that $f(U) \subset (\alpha, \infty)$ (resp. $f(U) \subset (-\infty, \alpha)$).

Every (λ, τ)-l.(u.)s.c. function is τ-l.(u.)s.c. function and μ-l.(u.)s.c. function is $\mu^\star$-l.(u.)s.c. function [5]. Moreover, $f : X \to \mathbb{R}$ is μ-l.s.c. (resp. μ-u.s.c.) if and only if $f^{-1}((\alpha, \infty)) \in \mu$ (resp. $f^{-1}((-\infty, \alpha)) \in \mu$) for any $\alpha \in \mathbb{R}$ [2].

Throughout the paper, We will denote the set of μ-continuity (resp. μ-discontinuity) points of f, by $\mathcal{C}_\mu(f)$ (resp. $\mathcal{D}_\mu(f)$) where $f : X \to \mathbb{R}$.

A function $f : X \to \mathbb{R}$ is (μ, η)-*cliquish* if the set $\mathcal{C}_\eta(f)$ is μ-dense [5]. If (X, μ) is a BS and $\mathcal{D}_\eta(f)$ is a μ-meager set, then f is (μ, η)-cliquish [5].

Let (X, μ) be a generalized topological space. Then $x \in X$ is called μ-isolated [1] if $\{x\}$ is μ-open. If every point of X is μ-isolated, then X is called μ-discrete [1].

Lemma 2.1. [5, Theorem 3.2] If (X, μ) is a strong Baire sGTS, then
(i) each $(\mu, \mu^\star)$-l.(u.)s.c. function is $(\mu, \mu^\star)$-cliquish.
(ii) each μ-l.(u.)s.c. function is $(\mu, \mu^\star)$-cliquish.
(iii) each $(\mu^\star, \mu)$-l.(u.)s.c. function is $(\mu, \mu^\star)$-cliquish.

Lemma 2.2. [6, Lemma 2.2]. Let (X, μ) be a GTS and let $A \subset X$. Then $x \in cA$ if and only if $V \cap A \neq \emptyset$ for any $V \in \mu(x)$.

Lemma 2.3. [6, Proposition 4.7] Let (X, μ) be a GTS. If $F_n \in \mathcal{M}$ for each $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} F_n \in \mathcal{M}$ where $\mathcal{M}$ is the set of all μ-meager sets.

3 Functions on GTSs

In this section, we give some properties for lower (resp. upper) semi-continuous and cliquish functions in a generalized topological space. Also, we give the answer for the question raised by Korczak - Kubiak, et.al in [5].

In [5], Korczak-Kubiak et.al raised the question "Whether the order of topologies μ and $\mu^\star$ in Theorem 3.2 (i) or (iii) can be changed?". In a sBS-sGTS (X,μ), there exists a $(\mu,\mu^\star)$-l.(u.)s.c. function which is not a $(\mu^\star,\mu)$-cliquish as shown in Example 3.1.

Example 3.1. Consider the GTS (X,μ) as in Example 3.4 in [5] where $X=[0,3]$ and $\mu=\{\emptyset,[0,2),(1,3],X\}$. Then (X,μ) is a sBS and sGTS. Here $\mu^\star=\{\emptyset,[0,2),(1,2),(1,3],X\}$. Define a function $f:(X,\mu)\to(\mathbb{R},\tau_0)$ where τ_0 is the Euclidean topology on $\mathbb{R}$ by

$$f(x)=\begin{cases}1 & if \quad x\in[0,1]\cup[2,3],\\ 2 & if \quad x\in(1,2).\end{cases}.$$

For any real number $\alpha<f(x)$ for all $x\in X$, there exists $G\in\mu^\star(x)$ being a μ-residual set such that $f(G)\subset(\alpha,\infty)$. Therefore, f is a $(\mu,\mu^\star)$-l.s.c. function. Here $(1,2)\subset\mathcal{D}_\mu(f)$ so that $(1,2)\cap\mathcal{C}_\mu(f)=\emptyset$. Since $(1,2)\in\mu^\star,\mathcal{C}_\mu(f)$ is not a $\mu^\star$-dense set in X. Hence f is not $(\mu^\star,\mu)$-cliquish. Define a function $g:(X,\mu)\to(\mathbb{R},\tau_0)$ by

$$g(x)=\begin{cases}2 & if \quad x\in[0,1]\cup[2,3]\\ 1 & if \quad x\in(1,2).\end{cases}.$$

For any real number $\alpha>g(x)$ for all $x\in X$, there exists $G\in\mu^\star(x)$ being a μ-residual set such that $g(G)\subset(-\infty,\alpha)$. Therefore, g is a $(\mu,\mu^\star)$-u.s.c. function. Let $x=0$. Then $g(x)=2$. If $V=(1.5,2.5)$ is an neighbourhood of $g(x)$, then there is no $U\in\mu(x)$ such that $g(U)\subset V$. Thus, $x\in\mathcal{D}_\mu(g)$. Similarly, we can prove that $[0,2)\subset\mathcal{D}_\mu(g)$ so that $[0,2)\cap\mathcal{C}_\mu(g)=\emptyset$. Since $[0,2)\in\mu^\star$ we have $\mathcal{C}_\mu(g)$ is not a $\mu^\star$-dense set in X. Hence g is not $(\mu^\star,\mu)$-cliquish.

Theorem 3.2. *Let (X,μ) be a GTS, $f:X\to(0,1)$ and $g:X\to(0,1)$. If f and g are μ-l.s.c. functions on X, then $f.g:X\to\mathbb{R}^+$ is a $\mu^\star$-l.s.c. function on X.*

Proof. Suppose f and g are μ-l.s.c. functions on X. Let $x_0\in X$ and $\alpha<(f.g)(x_0)$ where $\alpha\in\mathbb{R}^+$. Then $\alpha<f(x_0).g(x_0)$ and so $\frac{\alpha}{f(x_0)}<g(x_0)$. Take $t=\frac{\alpha}{f(x_0)}$. By assumption, there exists a set $H_1\in\mu(x_0)$ such that $g(H_1)\subset(t,\infty)$

which implies that $g(H_1) \subset (\alpha, \infty)$. Also, $\frac{\alpha}{g(x_0)} < f(x_0)$. Take $s = \frac{\alpha}{g(x_0)}$. By assumption, there exists a set $H_2 \in \mu(x_0)$ such that $f(H_2) \subset (s, \infty)$ which implies that $f(H_2) \subset (\alpha, \infty)$. Take $H = H_1 \cap H_2$. Then $H \in \mu^\star(x_0)$. Let $l \in (f.g)(H)$. Then there is an element $m \in H$ such that $l = (f.g)(m)$. Since $m \in H$ we have $f(m), g(m) \in (\alpha, \infty)$. Thus, $l \in (\alpha, \infty)$. Therefore, $(f.g)(H) \subset (\alpha, \infty)$. Hence $f.g$ is a $\mu^\star$-l.s.c. function on X. □

Theorem 3.3. *Let (X, μ) be a GTS, $f : X \to (0, 1)$ and $g : X \to (0, 1)$. If f and g are (μ, μ)-l.s.c. functions on X, then $f.g : X \to \mathbb{R}^+$ is a $(\mu^\star, \mu)$-l.s.c. function on X.*

Theorem 3.4. *Let (X, μ) be a GTS, $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. If f and g are μ-l.(u.)s.c. functions on X, then $min(f, g) : X \to \mathbb{R}$ is a $\mu^\star$-l.(u.)s.c. function on X.*

Proof. Suppose f and g are μ-l.s.c. functions on X. Let $x_0 \in X$ and $\alpha < min(f, g)(x_0)$ where $\alpha \in \mathbb{R}$. Then $\alpha < min(f(x_0), g(x_0))$ and so $\alpha < g(x_0)$. By assumption, there exists a set $U_1 \in \mu(x_0)$ such that $g(U_1) \subset (\alpha, \infty)$. Also, $\alpha < f(x_0)$. By assumption, there exists a set $U_2 \in \mu(x_0)$ such that $f(U_2) \subset (\alpha, \infty)$. Take $U = U_1 \cap U_2$. Then $U \in \mu^\star(x_0)$. Let $a \in min(f, g)(U)$. Then there is an element $b \in U$ such that $a = min(f, g)(b)$. Since $b \in U$ we have $f(b), g(b) \in (\alpha, \infty)$. Thus, $a \in (\alpha, \infty)$. Therefore, $min(f, g)(U) \subset (\alpha, \infty)$. Hence $min(f, g)$ is a $\mu^\star$-l.s.c. function on X.
Suppose that, f and g are μ-u.s.c. functions on X. Let $x_0 \in X$ and $\alpha > min(f, g)(x_0)$ where $\alpha \in \mathbb{R}$. Then $\alpha > min(f(x_0), g(x_0))$ and so $\alpha > f(x_0)$ or $\alpha > g(x_0)$ or $\alpha > f(x_0); \alpha > g(x_0)$.
Case-1: Suppose $\alpha > f(x_0)$. Then there is $U_1 \in \mu(x_0)$ such that $f(U) \subset (-\infty, \alpha)$. Take $U = U_1$. Then $U \in \mu^\star(x_0)$. Let $s \in min(f, g)(U)$. Then there is an element $t \in U$ such that $s = min(f(t), g(t))$. Since $t \in U$ we have $f(t) < \alpha$. This implies $min(f(t), g(t)) < \alpha$ which implies that $s \in (-\infty, \alpha)$. Thus, $min(f, g)(U) \subset (-\infty, \alpha)$.
Case-2: Similar considerations in Case-1, we get $U \in \mu^\star(x_0)$ such that $min(f, g)(U) \subset (-\infty, \alpha)$.
Case-3: Suppose $\alpha > f(x_0); \alpha > g(x_0)$. Then there exist $U_1 \in \mu(x_0)$ such that $f(U_1) \subset (-\infty, \alpha)$ and $U_2 \in \mu(x_0)$ such that $g(U_2) \subset (-\infty, \alpha)$. Take $U =$

$U_1 \cap U_2$. Then $U \in \mu^\star(x_0)$. Let $p \in min(f,g)(U)$. Then there is an element $q \in U$ such that $p = min(f(q), g(q))$. Since $q \in U$ we have $f(q) < \alpha; g(q) < \alpha$. This implies $min(f(q), g(q)) < \alpha$ which implies that $p \in (-\infty, \alpha)$. Thus, $min(f,g)(U) \subset (-\infty, \alpha)$. From all the cases, we get for any $\alpha \in \mathbb{R}$ with $\alpha > min(f,g)(x_0)$ there is $G \in \mu^\star(x_0)$ such that $min(f,g)(G) \subset (-\infty, \alpha)$. Hence $min(f,g)$ is a $\mu^\star$-u.s.c. function on X. $\square$

Theorem 3.5. *Let (X, μ) be a GTS, $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. If f and g are μ-l.(u.)s.c. functions on X, then $max(f,g) : X \to \mathbb{R}$ is a $\mu^\star$-l.(u.)s.c. function on X.*

Theorem 3.6. *Let (X, μ) be a GTS, $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. If f and g are (μ, μ)-l.(u.)s.c. functions on X, then $min(f,g) : X \to \mathbb{R}$ is a $(\mu^\star, \mu)$-l.(u.)s.c. function on X.*

Proof. Suppose f and g are (μ, μ)-l.s.c. functions on X. Let $x_0 \in X$ and $\alpha < min(f,g)(x_0)$ where $\alpha \in \mathbb{R}$. Then $\alpha < min(f(x_0), g(x_0))$ and so $\alpha < g(x_0)$. By assumption, there exists a set $A_1 \in \mu(x_0)$ being a μ-residual set such that $g(A_1) \subset (\alpha, \infty)$. Also, $\alpha < f(x_0)$. By assumption, there exists a set $A_2 \in \mu(x_0)$ being a μ-residual set such that $f(A_2) \subset (\alpha, \infty)$. Take $A = A_1 \cap A_2$. Then $A \in \mu^\star(x_0)$ and A is a μ-residual set, by Lemma 2.3. Let $k \in min(f,g)(A)$. Then there is an element $l \in A$ such that $k = min(f,g)(l)$. Since $l \in A$ we have $f(l), g(l) \in (\alpha, \infty)$. Thus, $k \in (\alpha, \infty)$. Therefore, $min(f,g)(A) \subset (\alpha, \infty)$. Hence $min(f,g)$ is a $(\mu^\star, \mu)$-l.s.c. function on X. $\square$

Corollary 3.7. *Let (X, μ) be a GTS, $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. If f and g are (μ, μ)-l.(u.)s.c. functions on X, then $max(f,g) : X \to \mathbb{R}$ is a $(\mu^\star, \mu)$-l.(u.)s.c. function on X.*

Theorem 3.8. *Let (X, μ_1) be a generalized submaximal space and μ_1 satisfy the codition: if $V_1, V_2 \in \mu_1$ and $V_1 \cap V_2 \neq \emptyset$, then $i_{\mu_1}(V_1 \cap V_2) \neq \emptyset$. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are (μ_1, μ_2)-cliquish functions where μ_2 is any GT in X, then $f + g$ is (μ_1, μ_2)-cliquish.*

Proof. Suppose f and g are (μ_1, μ_2)-cliquish functions. Then $\mathcal{C}_{\mu_2}(f)$ and $\mathcal{C}_{\mu_2}(g)$ are μ_1-dense subsets of X. By hypothesis, $\mathcal{C}_{\mu_2}(f)$ and $\mathcal{C}_{\mu_2}(g)$ are μ_1-open sets in X. Let $G \in \tilde{\mu}_1$. Then $G \cap \mathcal{C}_{\mu_2}(g) \neq \emptyset$ and so $i_{\mu_1}(G \cap \mathcal{C}_{\mu_2}(g)) \neq \emptyset$, by hypothesis. This implies $\mathcal{C}_{\mu_2}(f) \cap i_{\mu_1}(G \cap \mathcal{C}_{\mu_2}(g)) \neq \emptyset$ which implies that $\mathcal{C}_{\mu_2}(f) \cap G \cap \mathcal{C}_{\mu_2}(g) \neq \emptyset$. Let $a \in \mathcal{C}_{\mu_2}(f) \cap G \cap \mathcal{C}_{\mu_2}(g)$. Then $a \in G$ and also $a \in \mathcal{C}_{\mu_2}(f+g)$. Thus, $G \cap \mathcal{C}_{\mu_2}(f+g) \neq \emptyset$. Therefore, $\mathcal{C}_{\mu_2}(f+g)$ is μ_1-dense. Hence $f+g$ is (μ_1, μ_2)-cliquish. □

Corollary 3.9. *Let (X, μ_1) be a generalized submaximal space and μ_1 satisfy the codition: if $V_1, V_2 \in \mu$ and $V_1 \cap V_2 \neq \emptyset$, then $i_{\mu_1}(V_1 \cap V_2) \neq \emptyset$. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are (μ_1, μ_2)-cliquish functions where μ_2 is any GT in X, then the following hold.*
(a) $f - g$ is (μ_1, μ_2)-cliquish.
(b) $f.g$ is (μ_1, μ_2)-cliquish.
(c) $min(f, g)$ is (μ_1, μ_2)-cliquish.
(d) $max(f, g)$ is (μ_1, μ_2)-cliquish.

Theorem 3.10. *Let (X, μ) be a GTS and $f : X \to \mathbb{R}$ be a map. If X has a μ-isolated point, then the following hold.*
(a) f is a μ-l.(u.)s.c. function at that point in X.
(b) f is a $\mu^\star$-l.(u.)s.c. function at that point in X.
(c) f is a $\mu^{\star\star}$-l.(u.)s.c. function at that point in X.

Proposition 3.11. *Let μ and η be two generalized topologies in X, $f : X \to \mathbb{R}$ be a map. If f is (μ, η)-cliquish, then the following hold.*
(a) f is η-l.(u.)s.c. function at one point in X.
(b) f is η-l.(u.)s.c. function at some point in G for every $G \in \tilde{\mu}$.
(c) f is η-l.(u.)s.c. function at every μ-isolated point in X.

Proof. It is enough to prove (a) only. Suppose that f (μ, η)-cliquish. Then $\mathcal{C}_\eta(f)$ is a μ-dense set in X. Let $a \in X$. Then $a \in c_\mu(\mathcal{C}_\eta(f))$ and so $V \cap c_\mu(\mathcal{C}_\eta(f)) \neq \emptyset$ for every $V \in \mu(a)$, by Lemma 2.2. Take $t \in V \cap c_\mu(\mathcal{C}_\eta(f))$. Then $t \in c_\mu(\mathcal{C}_\eta(f))$ and so f is continuous at t with respect to η. Hence f is η-l.(u.)s.c. function at $t \in X$. □

In the rest of this section, we give more examples for the reverse implications, that is, cliquish implies l.(u.)s.c. function need not be true in a generalized topological space.

Example 3.12. Consider the generalized topological space (X,μ) where $X = [0,4]$ and $\mu = \{\emptyset, [0,2), \{\frac{3}{2}\} \cup \{\frac{7}{4}\}, (1,3], [0,3]\}$. Then (X,μ) is a sBS. Here $\mu^\star = \{\emptyset, [0,2), \{\frac{3}{2}\} \cup \{\frac{7}{4}\}, (1,2), (1,3], [0,3]\}$. Define a function $f : (X,\mu) \to (\mathbb{R}, \tau_0)$ where τ_0 is the Euclidean topology on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & if \quad x \in [0,1), \\ 1 & if \quad x \in [1,2), \\ 2 & if \quad x \in [2,3), \\ 3 & if \quad x \in [3,4]. \end{cases}.$$

Here $(1,2) \subseteq \mathcal{C}_{\mu^\star}(f)$. Hence $\mathcal{C}_{\mu^\star}(f)$ is a μ-dense set in X. Therefore, f is $(\mu,\mu^\star)$-cliquish and also f is a $(\mu^\star,\mu^\star)$-cliquish function. But f is not a $\mu^\star$-l.s.c. function and also f is not a μ-l.s.c. function. For, let $x = 1 \in X$. Then $f(x) = 1$. Take $\alpha = 0.9$ which implies that $\alpha < f(x)$. Here there is no $G \in \mu^\star(x)$ (resp. $U \in \mu(x)$) such that $f(G) \subset (\alpha,\infty)$ (resp. $f(U) \subset (\alpha,\infty)$). Define a function $g : (X,\mu) \to (\mathbb{R},\tau_0)$ by

$$g(x) = \begin{cases} 3 & if \quad x \in [0,1), \\ 2 & if \quad x \in [1,2), \\ 1 & if \quad x \in [2,3), \\ 0 & if \quad x \in [3,4]. \end{cases}.$$

Hence $\mathcal{C}_{\mu^\star}(g)$ is a μ-dense set in X. Therefore, g is $(\mu,\mu^\star)$-cliquish and also g is a $(\mu^\star,\mu^\star)$-cliquish function. But g is not a $\mu^\star$-u.s.c. function and also g is not a μ-u.s.c. function. For, let $x = 1 \in X$. Then $g(x) = 2$. Take $\alpha = 2.1$ which implies that $\alpha > g(x)$. Here there is no $G \in \mu^\star(x)$ (resp. $U \in \mu(x)$) such that $g(G) \subset (-\infty,\alpha)$ (resp. $g(U) \subset (-\infty,\alpha)$).
Here $\mu^{\star\star} = \{\emptyset\} \cup \{A \mid \text{either } \{\frac{3}{2}\} \in A \text{ or } \{\frac{7}{4}\} \in A\}$. Moreover, f is $(\mu^\star,\mu^{\star\star})$-cliquish, $(\mu,\mu^{\star\star})$-cliquish, $(\mu^{\star\star},\mu^{\star\star})$-cliquish, $(\mu^{\star\star},\mu^\star)$-cliquish. But f is not a $\mu^\star$-l.s.c. function and also f is not a μ-l.s.c. function. Also, g is $(\mu^\star,\mu^{\star\star})$-cliquish, $(\mu,\mu^{\star\star})$-cliquish, $(\mu^{\star\star},\mu^{\star\star})$-cliquish, $(\mu^{\star\star},\mu^\star)$-cliquish. But g is not a $\mu^\star$-u.s.c. function and also g is not a μ-u.s.c. function.

Example 3.13. Consider the generalized topological space (X,μ) where $X = [0,4]$ and $\mu = \{\emptyset, [0,2), (1,3], [0,3]\}$. Then (X,μ) is a sBS. Here $\mu^{\star\star} = \{\emptyset\} \cup \{A, B \mid A \subseteq exp((1,2)) - \{\emptyset\}, A \subset B\}$. Define a function $f : (X,\mu) \to (\mathbb{R},\tau_0)$ where τ_0 is the Euclidean topology on $\mathbb{R}$ by

$$f(x)=\begin{cases}1 & if \quad x\in[0,1),\\ 2 & if \quad x\in[1,2),\\ 3 & if \quad x\in[2,3),\\ 4 & if \quad x\in[3,4].\end{cases}.$$

Here $(1,2)\subseteq \mathcal{C}_{\mu^{\star\star}}(f)$. Hence $\mathcal{C}_{\mu^{\star\star}}(f)$ is a μ-dense set in X. Therefore, f is $(\mu,\mu^{\star\star})$-cliquish and also f is a $(\mu^{\star\star},\mu^{\star\star})$-cliquish function. But f is not a $\mu^{\star\star}$-l.s.c. function. For, let $x=2\in X$. Then $f(x)=3$. Take $\alpha=2.9$ which implies that $\alpha<f(x)$. Here there is no $G\in\mu^{\star\star}(x)$ such that $f(G)\subset(\alpha,\infty)$. Define a function $g:(X,\mu)\to(\mathbb{R},\tau_0)$ by

$$g(x)=\begin{cases}4 & if \quad x\in[0,1),\\ 3 & if \quad x\in[1,2),\\ 2 & if \quad x\in[2,3),\\ 1 & if \quad x\in[3,4].\end{cases}.$$

Hence $\mathcal{C}_{\mu^{\star\star}}(g)$ is a μ-dense set in X. Therefore, g is $(\mu,\mu^{\star\star})$-cliquish and also g is a $(\mu^{\star\star},\mu^{\star\star})$-cliquish function. But g is not a $\mu^{\star\star}$-u.s.c. function. For, let $x=2.5\in X$. Then $g(x)=2$. Take $\alpha=2.1$ which implies that $\alpha>g(x)$. Here there is no $G\in\mu^{\star\star}(x)$ such that $g(G)\subset(-\infty,\alpha)$.
Here $\mu^{\star}=\{\emptyset,[0,2),(1,2),(1,3],[0,3]\}$. Moreover, f is $(\mu^{\star},\mu^{\star\star})$-cliquish, $(\mu,\mu^{\star})$-cliquish, $(\mu^{\star\star},\mu^{\star})$-cliquish. But f is not a $\mu^{\star\star}$-l.s.c. function. Also, g is $(\mu^{\star},\mu^{\star\star})$-cliquish, $(\mu,\mu^{\star})$-cliquish, $(\mu^{\star\star},\mu^{\star})$-cliquish. But g is not a $\mu^{\star\star}$-u.s.c. function.

Example 3.14. Consider the generalized topological space (X,μ) where $X=[0,3]$ and $\mu=\{\emptyset,[0,2),[2,3],(1,3],X\}$. Then (X,μ) is a sBS and sGTS. Here $\mu^{\star}=\{\emptyset,[0,2),(1,2),[2,3],(1,3],X\}$. Define a function $f:(X,\mu)\to(\mathbb{R},\tau_0)$ where τ_0 is the Euclidean topology on $\mathbb{R}$ by

$$f(x)=\begin{cases}1 & if \quad x\in[0,1),\\ 2 & if \quad x\in[1,2),\\ 3 & if \quad x\in[2,3].\end{cases}.$$

Here $(1,3]\subseteq\mathcal{C}_{\mu^{\star}}(f)$. Hence $\mathcal{C}_{\mu^{\star}}(f)$ is a $\mu^{\star}$-dense set in X. Therefore, f is $(\mu^{\star},\mu^{\star})$-cliquish and also f is a $(\mu,\mu^{\star})$-cliquish function. But f is not a $(\mu^{\star},\mu^{\star})$-l.s.c. function and also f is not a $(\mu,\mu^{\star})$-l.s.c. function. For, let

$x = 1.5 \in X$. Then $f(x) = 2$. Take $\alpha = 1.9$ which implies that $\alpha < f(x)$. Take $U = (1, 2)$. Then $U \in \mu^\star(x)$ and $f(U) \subset (\alpha, \infty)$. But U is not a $\mu^\star$-residual set and also U is not a μ-residual set. Thus, there is no $G \in \mu^\star(x)$ being $\mu^\star$-residual (resp. being μ-residual) such that $f(G) \subset (\alpha, \infty)$. Let $x = 2 \in X$. Then $f(x) = 3$. Take $\alpha = 2.9$ which implies that $\alpha < f(x)$. Take $U = [2, 3]$. Then $U \in \mu(x)$ and $f(U) \subset (\alpha, \infty)$. But U is not a μ-residual set and also U is not a $\mu^\star$-residual set. Thus, there is no $G \in \mu(x)$ being μ-residual (resp. being $\mu^\star$-residual) such that $f(G) \subset (\alpha, \infty)$. Hence f is not a (μ, μ)-l.s.c. function and also f is not a $(\mu^\star, \mu)$-l.s.c. function.
Define a function $g : (X, \mu) \to (\mathbb{R}, \tau_0)$ by

$$g(x) = \begin{cases} 3 & if \quad x \in [0, 1), \\ 2 & if \quad x \in [1, 2), \\ 1 & if \quad x \in [2, 3]. \end{cases} .$$

Hence $\mathcal{C}_{\mu^\star}(g)$ is a $\mu^\star$-dense set in X. Therefore, g is $(\mu^\star, \mu^\star)$-cliquish and also g is a $(\mu, \mu^\star)$-cliquish function. But g is not a $(\mu^\star, \mu^\star)$-u.s.c. function and also g is not a $(\mu, \mu^\star)$-u.s.c. function. For, let $x = 3 \in X$. Then $g(x) = 1$. Take $\alpha = 1.1$ which implies that $\alpha > g(x)$. Take $U = [2, 3]$. Then $U \in \mu^\star(x)$ and $g(U) \subset (-\infty, \alpha)$. But U is not a $\mu^\star$-residual set and also U is not a μ-residual set. Thus, there is no $G \in \mu^\star(x)$ being $\mu^\star$-residual (resp. being μ-residual) such that $g(G) \subset (-\infty, \alpha)$. Let $x = 2 \in X$. Then $g(x) = 1$. Take $\alpha = 1.1$ which implies that $\alpha > g(x)$. Take $U = [2, 3]$. Then $U \in \mu(x)$ and $g(U) \subset (-\infty, \alpha)$. But U is not a μ-residual set and also U is not a $\mu^\star$-residual set. Thus, there is no $G \in \mu(x)$ being μ-residual (resp. being $\mu^\star$-residual) such that $g(G) \subset (-\infty, \alpha)$. Thus, g is not a (μ, μ)-u.s.c. function and also g is not a $(\mu^\star, \mu)$-u.s.c. function.
Here $\mu^{\star\star} = \{\emptyset\} \cup \{A, B \mid A \in exp((1, 3]) - \{\emptyset\}, A \subset B\}$. Thus, $(1, 3] \subseteq \mathcal{C}_{\mu^{\star\star}}(f)$ which implies that $\mathcal{C}_{\mu^{\star\star}}(f)$ is a $\mu^{\star\star}$-dense set in X. Therefore, f is $(\mu^{\star\star}, \mu^{\star\star})$-cliquish, $(\mu, \mu^{\star\star})$-cliquish, $(\mu^\star, \mu^{\star\star})$-cliquish. Moreover, f is $(\mu^{\star\star}, \mu^\star)$-cliquish. But f is not a $(\mu, \mu^{\star\star})$-l.s.c. function, f is not a $(\mu^\star, \mu^{\star\star})$-l.s.c. function, f is not a $(\mu^{\star\star}, \mu^{\star\star})$-l.s.c. function, f is not a $(\mu^{\star\star}, \mu^\star)$-l.s.c. function, f is not a $(\mu^{\star\star}, \mu)$-l.s.c. function. Also, g is $(\mu, \mu^{\star\star})$-cliquish, $(\mu^\star, \mu^{\star\star})$-cliquish, $(\mu^{\star\star}, \mu^{\star\star})$-cliquish, $(\mu^{\star\star}, \mu^\star)$-cliquish. But g is not a $(\mu, \mu^{\star\star})$-u.s.c. function, g is not a $(\mu^\star, \mu^{\star\star})$-u.s.c. function, g is not a $(\mu^{\star\star}, \mu^{\star\star})$-u.s.c. function, g is not a $(\mu^{\star\star}, \mu^\star)$-u.s.c. function, g is not a $(\mu^{\star\star}, \mu)$-u.s.c. function.

Example 3.15. Consider the generalized topological space (X, μ) where $X = [0, 4]$ and $\mu = \{\emptyset, [0, 2), \{\frac{4}{3}\} \cup \{\frac{3}{2}\}, (1, 3], [0, 3]\}$. Then (X, μ) is a sBS. Here $\mu^\star = \{\emptyset, [0, 2), \{\frac{4}{3}\} \cup \{\frac{3}{2}\}, (1, 2), (1, 3], [0, 3]\}$. Define a function $f : (X, \mu) \to (\mathbb{R}, \tau_0)$ where τ_0 is the Euclidean topology on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & if \quad x \in [0, 1), \\ 1 & if \quad x \in [1, 2), \\ 2 & if \quad x \in [2, 3), \\ 3 & if \quad x \in [3, 4]. \end{cases}.$$

Here $(\{\frac{4}{3}\} \cup \{\frac{3}{2}\}) \subseteq \mathcal{C}_\mu(f)$. Hence $\mathcal{C}_\mu(f)$ is a μ-dense set in X. Therefore, f is (μ, μ)-cliquish and also f is a $(\mu^\star, \mu)$-cliquish function. But f is not a μ-l.s.c. function and also f is not a $\mu^\star$-l.s.c. function. For, let $x = 1 \in X$. Then $f(x) = 1$. Take $\alpha = 0.9$ which implies that $\alpha < f(x)$. Here there is no $G \in \mu(x)$ (resp. $U \in \mu^\star(x)$) such that $f(G) \subset (\alpha, \infty)$ (resp. $f(U) \subset (\alpha, \infty)$). Define a function $g : (X, \mu) \to (\mathbb{R}, \tau_0)$ by

$$g(x) = \begin{cases} 3 & if \quad x \in [0, 1), \\ 2 & if \quad x \in [1, 2), \\ 1 & if \quad x \in [2, 3), \\ 0 & if \quad x \in [3, 4]. \end{cases}.$$

Hence $\mathcal{C}_\mu(g)$ is a μ-dense set in X. Therefore, g is (μ, μ)-cliquish and also g is a $(\mu^\star, \mu)$-cliquish function. But g is not a μ-u.s.c. function and also g is not a $\mu^\star$-u.s.c. function. For, let $x = 1 \in X$. Then $g(x) = 2$. Take $\alpha = 2.1$ which implies that $\alpha > g(x)$. Here there is no $G \in \mu(x)$ (resp. $U \in \mu^\star(x)$) such that $g(G) \subset (-\infty, \alpha)$ (resp. $g(U) \subset (-\infty, \alpha)$).
Here $\mu^{\star\star} = \{\emptyset\} \cup \{A \mid \text{either } \{\frac{4}{3}\} \in A \text{ or } \{\frac{3}{2}\} \in A\}$. Moreover, f is $(\mu^{\star\star}, \mu)$-cliquish. But f is not a μ-l.s.c. function and also f is not a $\mu^\star$-l.s.c. function. Also, g is $(\mu^{\star\star}, \mu)$-cliquish. But g is not a μ-u.s.c. function and also g is not a $\mu^\star$-u.s.c. function.

Example 3.16. Consider the generalized topological space (X, μ) where $X = [0, 3]$ and $\mu = \{\emptyset, [0, 2), \{\frac{3}{2}\} \cup \{\frac{5}{3}\}, [2, 3], (1, 3], X\}$. Then (X, μ) is a sBS and sGTS. Here $\mu^\star = \{\emptyset, [0, 2), (1, 2), \{\frac{3}{2}\} \cup \{\frac{5}{3}\}, [2, 3], (1, 3], X\}$. Define a function $f : (X, \mu) \to (\mathbb{R}, \tau_0)$ where τ_0 is the Euclidean topology on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 & if \quad x \in [0,1), \\ 2 & if \quad x \in [1,2), \\ 3 & if \quad x \in [2,3]. \end{cases}.$$

Here $(\{\frac{3}{2}\} \cup \{\frac{5}{3}\}) \cup [2,3] \subseteq \mathcal{C}_\mu(f)$. Hence $\mathcal{C}_\mu(f)$ is a μ-dense set in X. Therefore, f is (μ,μ)-cliquish and also f is a $(\mu^\star,\mu)$-cliquish function. But f is not a (μ,μ)-l.s.c. function and also f is not a $(\mu^\star,\mu)$-l.s.c. function. For, let $x = 2 \in X$. Then $f(x) = 3$. Take $\alpha = 2.9$ which implies that $\alpha < f(x)$. Take $U = [2,3]$. Then $U \in \mu(x)$ and $f(U) \subset (\alpha,\infty)$. But U is not a μ-residual set and also U is not a $\mu^\star$-residual set. Thus, there is no $G \in \mu(x)$ being μ-residual (resp. being $\mu^\star$-residual) such that $f(G) \subset (\alpha,\infty)$. Let $x = 3 \in X$. Then $f(x) = 3$. Take $\alpha = 2.9$ which implies that $\alpha < f(x)$. Take $U = [2,3]$. Then $U \in \mu^\star(x)$ and $f(U) \subset (\alpha,\infty)$. But U is not a $\mu^\star$-residual set and also U is not a μ-residual set. Thus, there is no $G \in \mu^\star(x)$ being $\mu^\star$-residual (resp. being μ-residual) such that $f(G) \subset (\alpha,\infty)$. Hence f is not a $(\mu^\star,\mu^\star)$-l.s.c. function and also f is not a $(\mu,\mu^\star)$-l.s.c. function.
Define a function $g : (X,\mu) \to (\mathbb{R},\tau_0)$ by

$$g(x) = \begin{cases} 3 & if \quad x \in [0,1), \\ 2 & if \quad x \in [1,2), \\ 1 & if \quad x \in [2,3]. \end{cases}.$$

Hence $\mathcal{C}_\mu(g)$ is a μ-dense set in X. Therefore, g is (μ,μ)-cliquish and also g is a $(\mu^\star,\mu)$-cliquish function. Here, there is no $G \in \mu(x)$ being μ-residual (resp. being $\mu^\star$-residual) such that $g(G) \subset (-\infty,\alpha)$. Hence g is not a (μ,μ)-u.s.c. function and also g is not a $(\mu^\star,\mu)$-u.s.c. function. Also, there is no $G \in \mu^\star(x)$ being $\mu^\star$-residual (resp. being μ-residual) such that $g(G) \subset (-\infty,\alpha)$. Thus, g is not a $(\mu^\star,\mu^\star)$-u.s.c. function and also g is not a $(\mu,\mu^\star)$-u.s.c. function.
Here $\mu^{\star\star} = \{\emptyset\} \cup \{A,B,C \mid A \in exp([2,3]) - \{\emptyset\}, A \subset B$, either $\{\frac{3}{2}\} \in C$ or $\{\frac{5}{3}\} \in C\}$. Hence f is $(\mu^{\star\star},\mu)$-cliquish. But f is not a (μ,μ)-l.s.c. function, f is not a $(\mu^\star,\mu)$-l.s.c. function, f is not a $(\mu^\star,\mu^\star)$-l.s.c. function, f is not a $(\mu,\mu^\star)$-l.s.c. function. Also, g is $(\mu^{\star\star},\mu)$-cliquish. But g is not a (μ,μ)-u.s.c. function, g is not a $(\mu^\star,\mu)$-u.s.c. function, g is not a $(\mu^\star,\mu^\star)$-u.s.c. function, g is not a $(\mu,\mu^\star)$-u.s.c. function.

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