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ON THE $|\mathbf{N}, \mathbf{p_n}, \mathbf{q_n}|_\mathbf{k}$ SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract

In this paper a theorem on generalized Nörlund Summability Factors has been proved which generalizes some earlier factor theorems on $|N, p_n|_k$ ([5]), $|C,1|_k$ ([4]) *and* $|\overline{N}, p_n|_k$ ([1]) *for* $k > 1$.

1. INTRODUCTION

Given any series $\sum a_n$, if there exists a sequence $\{\lambda_n\}$ such that $\sum_{n=0}^{\infty} a_n \lambda_n$ is summable by a method A, then we say that $\{\lambda_n\}$ is a summability factor for the method A. Results establishing theorems on summability factors are called factor theorems.

In this section we introduce some notations, conventions and definitions which are to be used in this paper. Let $\sum_0^\infty a_n$ be an infinite series with sequences of partial sums $\{s_n\}$.

Let $p = \{p_n\}$ be a positive non-increasing sequence of real numbers such that

$$p_n = \sum_{i=0}^{n} p_i \to \infty \ \ as \ \ n \to \infty ; \tag{1.1}$$

and

$$p_{-i} = p_{-i} = 0 \ \ for \ \ i \geq 1.$$

For a positive real sequence $q = (q_n)$, we define an increasing sequence (r_n) by

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$$r_n = (p * q)_n = \sum_{i=0}^{n} p_{n-i} \; q_{i \to \infty} \;, \; as \; n \to \infty \tag{1.2}$$

where $q_n = 0(1)$ *and* $1 = 0(q_n)$ as $n \to \infty$ and $q_{-i} = Q_{-i} = r_{-i} = 0$ for $i \geq 1$, $*$ denotes the convolution product.

The $(N,\ p_n\ ,\ q_n)$-transform of the sequence (s_n) is defined by ([2]).

$$t_n = \frac{1}{r_n} \sum_{i=0}^{n} p_{n-i} \; q_i \; s_i \; . \tag{1.3}$$

The series $\sum a_n$ is said to be $\left|N,\ p_n\ , q_n\right|_k$ summable for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{r_n}{q_n} \right)^{k-1} \left| t_n - t_{n-1} \right|^k < \infty \, . \tag{1.4}$$

If (d_n) be the sequence of (C, 1)-transform of the sequence (n, a_n), then

$$d_n = \frac{1}{n+1} \sum_{k=0}^{n} k \; a_k = \frac{1}{n+1} \sum_{k=1}^{n} k \;\; a_k \;\; as \;\; a_0 = 0 \, . \tag{1.5}$$

2. NOTATIONS

We use the following notations

$$A_j^k = \sum_{j=i}^{k} p_{k-j} \; q_j = p_{k-i} \; q_i + p_{k-i-1} \; q_{i+1} + ... + p_0 \; q_k$$

so that

$$A_j^k = \sum_{j=i}^{k} p_{k-j} \; q_j , \;\; 0 \leq i < k \tag{2.1}$$

$$= A_i^n \, , \qquad k \geq n$$

$$= 0, \qquad i > k$$

and

$$B_i^k = \sum_{j=0}^{k} p_{n-j} \; q_{j-i} = p_{n-i} \; q_0 + p_{n-i-1} \; q_1 + ... + p_{n-k} \; q_{k-i}$$

so that

$$B_i^k = \sum_{j=i}^{k} p_{n-j}\, q_{j-i}\,, \quad 0 \le i < k < n \tag{2.2}$$

$$= B_i^n \,, \quad k \ge n$$

$$= 0 \,, \quad i > k$$

we find the relations as

$$B_k^n = r_{n-k} = A_0^{n-k}$$

and

$$B_i^k + A_{k+1-i}^{n-i} = (p * q)_{n-i} = r_{n-i}\,.$$

In particular, if i = 0, then

$$B_0^k + A_{k+1}^n = r_n\,.$$

3. The object of this note is to prove the following theorem on $|N, p_n, q_n|_k$ summability.

Theorem: Let (p_n) , (q_n) and (r_n) be the sequences satisfying (1.1) and (1.2). If (X_n) is a positive monotonic non-decreasing sequence and (λ_n) be any sequence such that

$$\lambda_m\, X_m = 0(1) \text{ as } m \to \infty \tag{3.1}$$

$$\sum_{n=1}^{m} n\, X_n \left|\Delta^2 \lambda_n\right| = 0(1),\ m \to \infty \tag{3.2}$$

$$\sum_{n=1}^{m} \frac{q_n}{r_n} \left|d_n\right|^k = 0(X_m),\ m \to \infty \tag{3.3}$$

and

$$r_n = 0\ (n, q_n) \text{ as } n \to \infty \tag{3.4}$$

where d_n is the n-th (C, 1) transform of the sequence (na_n), then the series $\sum a_n\ \lambda_n$ is summable $|N, p_n, q_n|_k$, $k \ge 1$.

It may be noticed that under the conditions of the theorem, we have that

$$\Delta \ \lambda_n = \lambda_n - \lambda_{n+1} \ \to \ 0 \ as \ \ n \to \infty. \tag{3.5}$$

4. We require the following Lemmas to prove the theorem.

It is easy to prove the following two equalities as

$$r_{n-1} \ A_i^n - r_n \ A_i^{n-1} = B_i^i \ A_i^n - B_0^{i-1} \ A_i^{n-1} \tag{4.1}$$

and

$$\left| B_1^i \ \ A_i^n \ + B_0^i \ \ A_{i+1}^{n-1} - B_0^{i-1} \ \ A_i^{n-1} \ - B_1^{i+1} \ \ A_{i+1}^n \right| \tag{4.2}$$

$$= \left| q_i \left(r_{n-1} \ \ p_{n-i} - r_n \ \ p_{n-i-1} \right) \right|$$

$$= q_i \left(r_n \ \ p_{n-i-1} \ - \ r_{n-1} \ \ p_{n-i} \right).$$

Lemma 1: Under the relations given above, we find

$$\frac{1}{q_n \ r_{n-1}} \ \sum_{i=1}^{n} \left| B_i^i \ \ A_i^n \ + \ B_0^i \ \ A_{i+1}^{n-1} \ - \ B_0^{i-1} \ \ A_i^{n-1} - B_i^{i+1} \ \ A_{i+1}^n \ \right| = 0(1), \tag{4.3}$$

as $n \to \infty$.

Proof: From (4.2)

$$\text{L.H.S} = \frac{1}{q_n \ r_{n-1}} \ \sum_{i=1}^{n-1} q_i \left(r_n \ \ p_{n-i-1} - r_{n-1} \ p_{n-i} \right) \tag{4.4}$$

$$= \frac{r_n}{q_n \ r_{n-1}} \ \sum_{i=1}^{n-1} q_i \ \ p_{n-i-1} - \frac{1}{q_n} \ \sum_{i=1}^{n-1} q_i \ \ p_{n-i}$$

$$= \frac{r_n}{q_n \ \ r_{n-1}} \left(r_{n-1} \ - \ p_{n-1} \ q_0 \right) - \frac{1}{q_n} \left(r_n \ - \ p_n \ q_0 \ - \ p_0 \ q_n \right)$$

$$= \frac{p_n \ q_0}{q_n} + \ p_0 \ - \frac{r_n \ \ p_{n-1} \ q_0}{q_n \ r_{n-1}}$$

$$= \frac{q_0}{q_n} \left(p_n \ - \frac{r_n}{r_{n-1}} \ p_{n-1} \right) + \ p_0$$

$$\leq \frac{q_0}{q_n}\left(p_n - p_{n-1}\right) + p_0 = 0(1) \quad as \;\; n \to \infty$$

= R.H.S.

Hence the Lemma.

We assume here that

$$\frac{1}{q_n \; r_{n-1}} \sum_{i=1}^{n-1} \frac{q_i\left(B_1^i \;\; A_i^n - B_0^{i-1} \;\; A_{i+1}^{n-1}\right)}{r_i} = 0(1) \tag{4.5}$$

and

$$\frac{1}{q_n \; r_{n-1}} \sum_{i=1}^{n-1} \frac{q_i\left(B_1^i \;\; A_{i+1}^n - B_0^i \;\; A_{i+1}^{n-1}\right)}{r_i} = 0(1) \text{ as } n \to \infty . \tag{4.6}$$

Lemma 2: ([1], Lemma)

Suppose (X_n) is a positive non-decreasing sequence such that (3.1) and (3.2) hold. Then

$$\sum_{n=1}^{\infty} X_n \left|\Delta \;\; \lambda_n\right| < \infty \tag{4.7}$$

and

$$n \;\; X_n \left|\Delta \; \lambda_n\right| = 0(1), \, as \;\; n \to \infty .$$

Lemma 3: As the notations defined above, we get

$$\sum_{n=i+1}^{m+1} \frac{B_1^i A_i^n - B_0^{i-1} \quad A_i^{n-1}}{r_n \;\; r_{n-1}} = 0(1), \;\; as \;\; m \to \infty .$$

Proof: We have

$$\sum_{n=i+1}^{m+1} \frac{B_1^i \;\; A_i^n - B_0^{i-1} \;\; A_i^{n-1}}{r_n \;\; r_{n-1}} = \sum_{n=i+1}^{m+1} \frac{r_n \;\; B_1^i - r_{n-1} \;\; B_0^{i-1}}{r_n \;\; r_{n-1}}$$

$$= \Lambda(m, i), \text{ say.}$$

Clearly $\Lambda(m,i)$ is one positive and decreasing sequence for $1 \leq i \leq m$, hence,

$$\sup_{1 \leq i \leq m} \Lambda(m.i) = \Lambda(m,1)$$

$$= \sum_{n=2}^{m+1} \frac{B_1^1 \ r_n - B_0^0 \ r_{n-1}}{r_n \ r_{n-1}} = \sum_{n=2}^{m+1} \left(\frac{B_1^1}{r_{n-1}} - \frac{B_0^0}{r_n} \right)$$

$$= \sum_{n=2}^{m+1} \left(\frac{p_{n-1} \ q_0}{r_{n-1}} - \frac{p_n \ q_0}{r_n} \right) = \left(\frac{p_1}{r_i} - \frac{p_{m+1}}{r_{m+1}} \right) q_0 \ \rightarrow \ \frac{p_1 \ q_0}{r_1} \ as \ m \rightarrow \infty .$$

Thus $\Lambda(m,i) = 0(1) \ as \ m \rightarrow \infty$ for each $i \leq m$; this completes the proof of Lemma.

Similarly we can prove for each i,

$$\sum_{n=i+1}^{m+1} \frac{B_1^i \ A_{i+1}^n - B_0^i \ A_{i+1}^{n-1}}{r_n \ r_{n-1}} = 0(1), \ as \ m \rightarrow \infty . \tag{4.9}$$

5. PROOF OF THE THEOREM

Let T_n be the n-th$(N, \ p_n \ , q_n)$ transform of the series $\sum a_n \ \lambda_n$, then

$$T_n = \frac{1}{r_n} \sum_{i=0}^{n} p_{n-i} \ q_i \ \sum_{k=0}^{i} a_k \ \lambda_k$$

$$= \frac{1}{r_n} \sum_{k=0}^{n} a_k \ \lambda_k \ \sum_{i=k}^{n} p_{n-i} \ q_i = \frac{1}{r_n} \sum_{k=0}^{n} a_k \ \lambda_k \ A_k^n .$$

Then for $n \geq 1$, we get, by Abel's Transformation

$$T_n - T_{n-1} = \frac{1}{r_n} \sum_{i=1}^{n} A_i^n \ a_i \ \lambda_i - \frac{1}{r_{n-1}} \sum_{i=1}^{n-1} A_i^{n-1} \ a_i \ \lambda_i$$

$$= \sum_{i=1}^{n} \left(\frac{A_i^n}{r_n} - \frac{A_i^{n-1}}{r_{n-1}} \right) a_i \ \lambda_i \quad \text{, since } A_n^{n-1} = 0$$

$$= \frac{1}{r_n \ r_{n-1}} \sum_{i=1}^{n} \left(r_{n-1} \ A_i^n - A_i^{n-1} \ r_n \right) a_i \ \lambda_i$$

$$= \frac{1}{r_n \ r_{n-1}} \sum_{i=1}^{n} \left(B_1^i \ A_i^n - B_0^{i-1} \ A_1^{n-1} \right) a_i \ \lambda_i \quad \text{(by (4.1))}$$

$$= \frac{1}{r_n \; r_{n-1}} \sum_{i=1}^{n} \frac{\left(B_1^i \; A_i^n - B_0^{i-1} \; A_i^{n-1}\right) \lambda_i}{i} \; i \; a_i$$

$$= \frac{1}{r_n \; r_{n-1}} \left\{ \sum_{i=1}^{n-1} \Delta \frac{\left(B_1^i \; A_i^n - B_0^{i-1} \; A_i^{n-1}\right) \lambda_i}{i} \; \sum_{k=1}^{i} k \; a_k \right.$$

$$\left. + \frac{\left(B_1^n \; A_n^n - B_0^{n-1} \; A_n^{n-1}\right) \lambda_n}{n} \; \sum_{k=1}^{n} k \cdot a_k + 0 \right\}$$

$$= \frac{1}{r_n \; r_{n-1}} \sum_{i=1}^{n-1} \left[\frac{\left(B_1^i \; A_i^n - B_0^{i-1} \; A_i^{n-1}\right) \lambda_i}{i} - \frac{\left(B_1^{i+1} \; A_{i+1}^n - B_0^i \; A_{i+1}^{n-1}\right) \lambda_{i+1}}{i+1} \right] (i+1) \, d_i$$

$$+ \frac{1}{r_n \; r_{n-1}} \; \frac{\left(B_1^n \; A_n^n - B_0^{n-1} \cdot 0\right)}{n} (n+1) d_n .$$

As $\Delta \lambda_i = \lambda_i - \lambda_{i+1}$, we have, by using (4.2)

$$T_n - T_{n-1} = \frac{(n+1) \, p_0 \; q_n \; \lambda_n \; d_n}{n \; r_n} + \frac{1}{r_n \; r_{n-1}} \sum_{i=1}^{n-1} \left\{ \frac{\left(B_1^i \; A_i^n - B_0^{i-1} \; A_i^{n-1}\right) \lambda_i}{i} \right.$$

$$\left. - \frac{B_1^{i+1} \; A_{i+1}^n - B_0^i \; A_{i+1}^{n-1}}{(i+1)} \left(\lambda_i - \Delta \lambda_i\right) \right\} (i+1) \; d_i$$

$$= \frac{(n+1) \; p_0 \; q_n \; \lambda_n \; d_n}{n \; r_n} + \frac{1}{r_n \; r_{n-1}} \sum_{i=1}^{n-1} \left\{ \left(B_1^i \; A_i^n - B_0^{i-1} \; A_i^{n-1}\right) \right.$$

$$\left. - \left(B_1^i \; A_{i+1}^n - B_0^i \; A_{i+1}^{n-1}\right) \right\} \lambda_i \; d_i + \frac{1}{r_n \; r_{n-1}} \sum_{i=1}^{n-1} \frac{1}{i} \; \left(B_1^i \; A_i^n - B_0^{i-1} \; A_i^{n-1}\right) \lambda_i \; d_i$$

$$+ \frac{1}{r_n \; r_{n-1}} \sum_{i=1}^{n-1} \left(B_1^{i+1} \; A_{i+1}^n - B_0^i \; A_{i+1}^{n-1}\right) \Delta \lambda_i \; d_i$$

$$= \frac{(n+1) \; p_0 \; q_n \; \lambda_n \; d_n}{n \; r_n} + \frac{1}{r_n \; r_{n-1}} \sum_{i=1}^{n-1} q_i \left(r_{n-1} \; p_{n-i} - r_n - p_{n-i-1}\right) \lambda_i \; d_i$$

$$+ \frac{1}{r_n \ r_{n-1}} \sum_{i=1}^{n-1} \frac{1}{i} \left(B_1^i \ A_i^n - B_0^{i-1} \ A_i^{n-1} \right) \lambda_i \ d_i \ +$$

$$+ \frac{1}{r_n \ r_{n-1}} \sum_{i=1}^{n-1} \left(B_1^{i+1} \ A_{i+1}^n - B_0^i \ A_{i+1}^{n-1} \right) \Delta \lambda_i \ d_i$$

$$= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \text{ say.}$$

To prove the Theorem, it is sufficient to show, by Minkowski's inequality, that

$$\sum_{n=1}^{\infty} \left(\frac{r_n}{q_n} \right)^{k-1} \left| T_{nj} \right|^k < \infty \quad for \quad j = 1,2,3,4 .$$

Now, by using Abel's transformation

$$I_1 = \sum_{n=1}^{m+1} \left(\frac{r_n}{q_n} \right)^{k-1} \left| T_{n1} \right|^k$$

$$= \sum_{n=1}^{m+1} \frac{p_0^k \ q_n \ |\lambda_n|^{k-1} \ |\lambda_n| \ |d_n|^k}{r_n} \left(\frac{n+1}{n} \right)^k$$

$$= 0(1) \sum_{n=1}^{m+1} \frac{q_n |\lambda_n| \ |d_n|^k}{r_n}$$

$$= 0(1) \left[\sum_{n=1}^{m} \Delta |\lambda_n| \sum_{i=1}^{n} \frac{q_i \ |d_i|^k}{r_i} + |\lambda_m| \sum_{i=1}^{m} \frac{q_i |d_i|^k}{r_i} \right]$$

$$= 0(1) \sum_{n=1}^{m} \Delta |\lambda_n| \ X_n + 0(1) \ |\lambda_m| \ X_m$$

$$= 0(1) \sum_{n=1}^{m} \Delta |\lambda_n| \ X_n + 0(1)$$

$$= 0(1) \ as \ m \to \infty$$

by virtue of Lemma 2 and hypothesis.

Next, by using Holder's inequality and (4.4), we get

$$I_2 = \sum_{n=1}^{m+1} \left(\frac{r_n}{q_n}\right)^{k-1} |T_{n2}|^k$$

$$\leq \sum_{n=2}^{m+1} \frac{1}{q_n^{k-1}\ r_n\ r_{n-1}^k} \left\{\sum_{i=1}^{n-1} q_i \left(r_n\ p_{n-i-1} - r_{n-1}\ p_{n-i}\right) |\lambda_i|\ |d_i|\right\}^k$$

$$\leq \sum_{n=2}^{m+1} \frac{1}{r_n\ r_{n-1}} \sum_{i=1}^{n-1} q_i \left(r_n\ p_{n-i-1} - r_{n-1}\ p_{n-i}\right) |\lambda_i|^k\ |d_i|^k$$

$$\times \left\{\frac{1}{q_n\ r_n} \sum_{i=1}^{n-1} q_i \left(r_n\ p_{n-i-1} - r_{n-1}\ p_{n-i}\right)\right\}^{k-1}$$

$$= 0(1) \sum_{i=1}^{m} q_i |\lambda_i|^k\ |d_i|^k \sum_{n=i+1}^{m+1} \frac{r_n\ p_{n-i-1} - r_{n-1}\ p_{n-i}}{r_n\ r_{n-1}}$$

$$= 0(1) \sum_{i=1}^{m} q_i |\lambda_i|^{k-1}\ |\lambda_i|\ |d_i|^k \sum_{n=i+1}^{m+1} \left(\frac{p_{n-i-1}}{r_{n-1}} - \frac{p_{n-i}}{r_n}\right)$$

$$= 0(1) \sum_{i=1}^{m} q_i |\lambda_i| |d_i|^k \left(\frac{p_0}{r_i} - \frac{p_{m+1-i}}{r_{m+1}}\right)$$

$$= 0(1) \sum_{i=1}^{m} \frac{|\lambda_i|\ q_i\ |d_i|^k}{r_i} + 0(1) \sum_{i=1}^{m} \frac{|\lambda_i|\ q_i\ |d_i|^k}{r_{m+1}}$$

$= 0(1)$ *as* $m \to \infty$, from I_1.

Also, from the fact that $r_n = 0(nq_n)$, we find

$$I_3 = \sum_{n=1}^{m+1} \left(\frac{r_n}{q_n}\right)^{k-1} |T_{n3}|^k$$

$$= \sum_{n=2}^{m+1} \left(\frac{r_n}{q_n}\right)^{k-1} \left|\frac{1}{r_n\ r_{n-1}} \sum_{i=1}^{n-1} \frac{1}{i} \left(B_1^i\ A_i^n - B_0^{i-1}\ A_i^{n-1}\right) \lambda_i\ d_i\right|$$

$$= 0(1) \sum_{n=2}^{m+1} \frac{1}{q_n^{k-1} \ r_n \ r_{n-1}^k} \left\{ \sum_{i=1}^{n-1} \frac{q_i \left(B_1^i \ A_i^n \ - B_0^{i-1} \ A_i^{n-1} \right)}{r_i} \ |\lambda_i| \ |d_i| \right\}^k$$

$$= 0(1) \sum_{n=2}^{m+1} \frac{1}{r_n \ r_{n-1}} \ \sum_{i=1}^{n-1} \frac{q_i \left(B_1^i \ A_i^n - B_0^{i-1} \ A_i^{n-1} \right)}{r_i} \ |\lambda_i|^k \ |d_i|^k$$

$$\times \left\{ \frac{1}{q_n \ r_{n-1}} \sum_{i=1}^{n-1} \frac{q_i \left(B_1^i \ A_i^n \ - \ B_0^{i-1} \ A_i^{n-1} \right)}{r_i} \right\}^{k-1}$$

This is due to Hölder’s inequality. Now from (4.5) and Lemma 3, we get

$$I_3 = 0(1) \sum_{n=2}^{m+1} \frac{1}{r_n \ r_{n-1}} \sum_{i=1}^{n-1} \frac{q_i \left(B_1^i \ A_i^n \ - B_0^{i-1} \ A_i^{n-1} \right)}{r_i} |\lambda_i|^k \ |d_i|^k$$

$$= 0(1) \sum_{n=1}^{m} \frac{q_i \ |\lambda_i|^k \ |d_i|^k}{r_i} \ \sum_{n=i+1}^{m+1} \frac{\left(B_1^i \ A_i^n \ - B_0^{i-1} \ A_i^{n-1} \right)}{r_n \ r_{n-1}}$$

$$= 0(1) \sum_{i=1}^{m} \frac{q_i |\lambda_i| \ |\lambda_i|^{k-1} \ |d_i|^k}{r_i}$$

$$= 0(1) \sum_{i=1}^{m} \frac{|\lambda_i| \ q_i \ |d_i|^k}{r_i}$$

$= 0(1)$ as $m \to \infty$, from I_1.

Lastly, from (4.6) and (4.9) and by applying Hölder’s inequality, we have that

$$I_4 = \sum_{n=1}^{m+1} \left(\frac{r_n}{q_n} \right)^{k-1} |T_{n4}|^k$$

$$= \sum_{n=2}^{m+1} \left(\frac{r_n}{q_n} \right)^{k-1} \left| \frac{1}{r_n \ r_{n-1}} \sum_{i=1}^{n-1} \left(B_1^{i+1} \ A_{i+1}^n \ - B_0^i \ A_{i+1}^{n-1} \right) \Delta \lambda_i \ d_i \right|^k$$

$$\leq \sum_{n=2}^{m+1} \frac{1}{q_n^{k-1} \ r_n \ r_{n-1}^k} \left\{ \sum_{i=1}^{n-1} \frac{i \left(B_1^{i+1} \ A_{i+1}^n \ - B_0^i \ A_{i+1}^{n-1} \right)}{i} \ |\Delta \lambda_i| \ |d_i| \right\}^k$$

$$= 0(1) \sum_{n=2}^{m+1} \frac{1}{q_n^{k-1}\; r_n\; r_{n-1}^k} \left\{ \sum_{i=1}^{n-1} \frac{i\left(B_1^{i+1}\; A_{i+1}^n - B_0^i\; A_{i+1}^{n-1}\right) q_i}{r_i} \,|\Delta \lambda_i|\; |d_i| \right\}^k$$

$$= 0(1) \sum_{n=2}^{m+1} \frac{1}{r_n\; r_{n-1}} \sum_{i=1}^{n-1} i^k \frac{\left(B_1^{i+1}\; A_{i+1}^n - B_0^i\; A_{i+1}^{n-1}\right)}{r_i} |\Delta\lambda_i|^k\; |d_i|^k$$

$$\times \left\{ \frac{1}{q_n\; r_{n-1}} \sum_{i=1}^{n-1} \frac{B_1^{i+1}\; A_{i+1}^n - B_0^i\; A_{i+1}^{n-1}}{r_i} \right\}^{k-1}$$

$$= 0(1) \sum_{i=1}^{m} \frac{q_i}{r_i}\, i^k\; |\Delta \lambda_i|^k\; |d_i|^k \sum_{n=i+1}^{m+1} \frac{\left(B_1^{i+1}\; A_{i+1}^n - B_0^i\; A_{i+1}^{n-1}\right)}{r_n\; r_{n-1}}$$

$$= 0(1) \sum_{i=1}^{m} \left(i|\Delta\lambda_i|\right)^{k-1} \left(i|\Delta\lambda_i|\right) \frac{q_i}{r_i}\; |d_i|^k$$

$$= 0(1) \sum_{i=1}^{m} i|\Delta\lambda_i|\; \frac{q_i |d_i|^k}{r_i}$$

$$= 0(1) \left[\sum_{i=1}^{m-1} \Delta\left(i|\Delta\lambda_i|\right) \sum_{v=1}^{i} \frac{q_v |d_v|^k}{r_v} + m|\Delta\lambda_m| \sum_{i=1}^{m} \frac{q_i |d_i|^k}{r_i} \right]$$

$$= 0(1) \sum_{i=1}^{m-1} \Delta\left(i|\Delta\lambda_i|\right) X_i + 0(1)\; m|\Delta\lambda_m|\; X_m$$

$$= 0(1) \sum_{i=1}^{m-1} i\; X_i \left|\Delta^2\, \lambda_i\right| + 0(1) \sum_{i=1}^{m-1} |\Delta\lambda_{i+1}|\; X_i + 0(1)$$

$$= 0(1),\; as\;\; m \to \infty$$

by virtue of the hypothesis and Lemma 2.

This completes the proof of the theorem.

Corollary 1 ([1]): Let (q_n) be a sequence of positive number such that

$$Q_n = 0(n, q_n)\;\; as\;\; n \to \infty.$$

If (x_n) is a positive monotonic non decreasing sequence such that

$$\lambda_m \;\; X_m = 0(1) \;\; as \;\; m \to \infty,$$

$$\sum_{n=1}^{\infty} n X_i \left|\Delta^2 \lambda_n\right| = 0(1)$$

and

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} \left|t_n\right|^k = 0(X_m) \;\; as \;\; m \to \infty,$$

where t_n is the n-th (C, 1) transform of (na_n), then the series $\sum a_n \lambda_n$ is summable $\left|\overline{N}, q_n\right|_k$, $k \geq 1$.

Taking $p_n = 1$ for all n in the theorem we get the corollary. In addition to this if we take $q_n = 1$ for all values of n we find,

Corollary 2([4]): If (X_n) is a positive monotonic non-decreasing sequence such that

$$\lambda_m \; X_m = 0(1) \;\; as \;\; m \to \infty$$

$$\sum_{n=1}^{m} n \; X_n \left|\Delta^2 \lambda_n\right| = 0(1)$$

and

$$\sum_{n=1}^{m} \frac{1}{n} \left|t_n\right|^k = 0(X_m) \;\; as \;\; m \to \infty,$$

then the series $\sum a_n \lambda_n$ is summable $\left|C, 1\right|_k$, $k \geq 1$.

Corollary 3 ([5]): Let (p_n) be a monotonic decreasing sequence such that

$$P_n = 0(n) \; as \;\; n \to \infty.$$

If (X_n) is a positive monotonic non-decreasing sequence such that

$$\lambda_m \; X_m = 0(1) \;\; as \;\; m \to \infty$$

$$\sum_{n=1}^{m} n \; X_n \left|\Delta^2 \lambda_n\right| = 0(1)$$

and

$$\sum_{n=1}^{m} \frac{1}{P_n} |d_n|^k = 0(X_m) \ as \ m \to \infty,$$

where d_n is the n-th (C, 1) transform of the sequence (na_n) then the series $\sum_{n=1}^{\infty} a_n \lambda_n$

is summable $|N, p_n|_k$, $k \geq 1$.

Putting $q_n = 1$ for all n, we get the corollary.

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