*IJMMS, Vol. 13, No. 2, (July-December 2017): 403-415*

ISSN: 0973-3329

# NUMERICAL SOLUTION OF NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS USING NON-POLYNOMIAL SPLINES

***Shahna***

**Abstract**

*A new approach based on off step-points is discussed in this paper to solve second order singular boundary value problems. The developed method is second as well as fourth order accurate. Convergence analysis of the fourth order method is discussed. Two nonlinear second order singular boundary value problems have been solved by the presented method. Our results are more accurate than the results of other existing methods of order four. Graphs between exact and approximate solutions are also proved the accuracy and efficiency of the developed method.*

*2010 Mathematics Subject Classification. 65L10, 65D07.*

*Keywords: Second order, Non-linear, Singular, Non-polynomial, Spline, Boundary value problems, Convergence.*

## 1. INTRODUCTION

We consider the following second order non-linear singular boundary value problem(BVP):

$$U''(x) = A(x)U'(x) + B(x)U(x) + C(x), \qquad (1)$$

$$U(a) = \kappa_0, \quad U(b) = \kappa_1, \qquad (2)$$

where are smooth and bounded real functions, and are given constants. These problems arise in various fields of science and engineering such as nuclear engineering, control theory, fluid dynamics, fluid mechanics, elasticity, quantum mechanics, optimal control, hydrodynamics, convection diffusion processes etc. In previous years, many authors have solved BVPs using splines. For example, cubic spline method for solving second-order singularly perturbed BVPs is given by [8],

* Department of Mathematics, Jamia Millia Islamia, New-Delhi 110025, INDIA, Email - khan.shahana5@gmail.com

quadratic non-polynomial spline approach for solving system of second-order BVPs is given by [7]. In [4] iterative algorithm is given for the solution of non-linear singular second order ordinary differential equations. Methods for solving singular BVPs using splines are given in [2]. Existence and uniqueness of the BVPs of higher order are given in [5].

In this paper, we have derived a uniformly convergent tridiagonal scheme based on off-step points using non-polynomial mixed spline of lower degree for the solution of (1)-(2). For solving singular boundary value problems, many authors used L'Hospital rule to remove the singularity. But in this paper we use off-step points in place of nodal points to avoid singularity. Our method is second as well as fourth order accurate. The advantages of our second and fourth order methods are higher accuracy with the same computational effort. Error analysis is also discussed which prove the theoretical behavior of the scheme.

In Section 2, we give a brief derivation of the spline method. In Section 3 we present the application of the method and derivative approximations are also derived. Section 4 includes convergence analysis of the method and Section 5 contains the numerical examples and discussions that demonstrate the theoretical behaviour of our method. Graphs between exact and approximate solutions are also given in Section 5. Conclusion as given in Section 6.

## NON-POLYNOMIAL SPLINE FUNCTION

To develop the new method based on off-step points, we divide the interval *[a,b]* into $n$+1 subintervals, such that

$$a = x_0 < x_{1/2} < x_{3/2} < ..... < x_{n-1/2} < x_n = b.$$

We introduce a finite set of grid points $x_i$ as

$$x_i = a + (i - 1/2)h, \quad i = 0,1,\ldots,n \text{ and } h = \frac{(b-a)}{n}.$$

Let *S(x)* be a function of class $C^2[a,b]$ which interpolates *U(x)* at the mesh point $x_i$ depends on a parameter $\rho$, reduces to an ordinary quadratic spline in [*a*, *b*] as $\rho \to 0$ is termed as a non-polynomial quadratic spline function. Since the parameter $\rho$ can occur in $S(x)$ in many ways, such a spline is not unique.

Let $S_i(x) = a_i e^{\rho(x - x_{i-1/2})} + b_i[\cos\rho(x - x_{i-1/2}) + \sin\rho(x - x_{i-1/2})] + c_i$ (3)

To calculate the coefficient $a_i, b_i$, and $c_i$, we define the following interpolatory conditions as

$$S_i(x_{i-1/2}) = U_{i-1/2}, S_i(x_{i+1/2}) = U_{i+1/2} \text{ and } S_i''(x_{i-1/2}) = \frac{1}{2}(M_{i-1/2} + M_{i+1/2}), \quad (4)$$

Using condition (4), we calculated the coefficients as

$$a_i = \frac{U_{i+1/2} - U_{i-1/2}}{(e^\theta + \cos\theta + \sin\theta - 2)} + \frac{1}{2\rho^2}\left(\frac{(1-e^\theta)}{(e^\theta + \cos\theta + \sin\theta - 2)} + 1\right)(M_{i-1/2} + M_{i+1/2}),$$

$$b_i = \frac{U_{i+1/2} - U_{i-1/2}}{(e^\theta + \cos\theta + \sin\theta - 2)} + \frac{1}{2\rho^2}\left(\frac{(1-e^\theta)}{(e^\theta + \cos\theta + \sin\theta - 2)}\right)(M_{i-1/2} + M_{i+1/2}),$$

$$c_i = \frac{U_{i+1/2} + (e^\theta + \cos\theta + \sin\theta - 2)U_{i-1/2}}{(e^\theta + \cos\theta + \sin\theta - 2)} + \frac{1}{2\rho^2}\left(\frac{(1-2e^\theta)}{(e^\theta + \cos\theta + \sin\theta - 2)} + 1\right)(M_{i-1/2} + M_{i+1/2})$$

where $\theta = \rho h$.

Now by using the continuity of first derivative, $S_i^m(x_{i-1/2}) = S_{i-1}^m(x_{i-1/2})$, $m$=0,1 the following consistency relation is derived

$$A_1 U_{i-3/2} + A_2 U_{i-1/2} + A_3 U_{i+1/2} = h^2 (B_1 M_{i-3/2} + B_2 M_{i-1/2} + B_3 M_{i+1/2}), \quad i = 2,3......n-1 \qquad (5)$$

where $A_1 = \dfrac{e^\theta + \cos\theta + \sin\theta}{2}$, $A_2 = \dfrac{\sin\theta - \cos\theta - e^{\theta+2}}{2}$

$$A_3 = 1, \quad B_1 = \frac{e^\theta(2\sin\theta - 1) + \cos\theta - \sin\theta}{2},$$

$$B_2 = \frac{e^\theta \sin\theta - \sin\theta}{2\theta^2}, \; B_3 = \frac{e^\theta - \sin\theta - \cos\theta}{4\theta^2}.$$

**Remark**: Our method reduces to [3] based on quadratic spline when

$$(A_1, A_2, A_{3,}, B_1, B_2, B_3) = (1, -2, 1, 1/8, 6/8, 1/8)$$

**Truncation Error:** Truncation error is obtained by expanding equation (5) using Taylor's series which is as follows:

$$T_i=(A_1+A_2+A_3)U_i+(\frac{-3A_1-A_2+A_3}{2})hU_i^{'}+h^2[\frac{(9A_1+A_2+A_3)}{2^2 2!}-(B_1+B_2+B_3)]U_i^{(2)}$$
$$+h^3[\frac{(-27A_1-A_2+A_3)}{2^3 3!}-\frac{1}{2}(-3B_1-B_2+B_3)]U_i^{(3)}+h^4[\frac{(81A_1+A_2+A_3)}{2^4 4!}$$
$$-\frac{1}{2^2 2!}(9B_1+B_2+B_3)]U_i^{(4)}+h^5[\frac{(-243A_1-A_2+A_3)}{2^5 5!}-\frac{(-27B_1-B_2+B_3)}{2^3 3!}]U_i^{(5)}$$
$$+h^6[\frac{(729A_1+A_2+A_3)}{2^6 6!}-\frac{(81B_1+B_2+B_3)}{2^4 4!}]U_i^{(6)}+O(h^7),i=2,3,...,n-1. \tag{6}$$

For different values of parameters, we get the method of different orders. Here, we get the second as well as fourth order method. For $B_1+B_2+B_3=1$ and $B_1=B_3$,we get the second order method . For the choice of parameters $(A_1,A_2,A_3,B_1,B_2,B_3)=(1,-2,1,\frac{1}{12},\frac{10}{12},\frac{1}{12})$ we get the fourth order method.

Equation (5) forms a system of $(n-2)$ linear equation in $n$ unknowns. $U_{i-1/2},i=1,2,3...n.$ Thus, we need two more equations, one at each end of range of integration. These boundary conditions are obtained by using method of undetermined coefficients.

The boundary equation for second order method are given in [7] as

$$2U_0-3U_{1/2}+U_{3/2}=\frac{h^2}{24}[15M_{1/2}+3M_{3/2}]+O(h^4),i=1 \tag{7}$$

$$2U_n-3U_{n-1/2}+U_{n-3/2}=\frac{h^2}{24}[15M_{n-1/2}+3M_{n-3/2}]+O(h^4),i=n \tag{8}$$

The truncation error of the second order method for

$(A_1,A_2,A_3,B_1,B_2,B_3)=(1,-2,1,\frac{1}{8},\frac{6}{8},\frac{1}{8})$ is given as follows:

$$T_i=\begin{cases}-\frac{1}{64}h^4U_0^{(4)}+O(h^5),i=1\\ -\frac{1}{24}h^{(4)}U_i^{(4)}+O(h^5),i=2,3,....n-1\\ -\frac{1}{64}h^{(4)}U_i^{(4)}+O(h^5),i=n\end{cases}$$

For fourth order method the boundary equations are obtained as follows

$$2U_0 - 3U_{1/2} + U_{3/2} = \frac{h^2}{384}[233M_{1/2} + 63M_{3/2} - 9M_{5/2} + M_{7/2}] + O(h^6), i = 1 \quad (9)$$

$$2U_n - 3U_{n-1/2} + U_{n-3/2} = \frac{h^2}{384}[233M_{n-1/2} + 63M_{n-3/2} - 9M_{n-5/2} + M_{n-7/2}] + O(h^6), i = n$$
(10)

The truncation error of the fourth order method for $(A_1, A_2, A_3, B_1, B_2, B_3) = (1,-2,1,\frac{1}{12},\frac{10}{12},\frac{1}{12})$ is given as follows:

$$T_i = \begin{cases} \frac{11}{7680} h^{(6)} U_0^{(6)} + O(h^7), i = 1 \\ -\frac{1}{120} h^6 U_i^{(6)} + O(h^7), i = 1,2,3.....n-1 \\ \frac{11}{7680} h^{(6)} U_n^{(6)} + O(h^7), i = n. \end{cases}$$

## APPLICATION OF THE ALGORITHM

We consider a second order boundary value problem of the form

$$U''(x) = A(x)U'(x) + B(x)U(x) + C(x) \quad (11)$$

Subject to boundary conditions

$$U(a) = \kappa_1, \text{ and } U(b) = \kappa_2$$

where $\kappa_1$ and $\kappa_2$ are finite real constants. Therefore by implementing the scheme (5)
on the boundary value problem (11), we get the following system of equations

$$A_1 U_{i-3/2} + A_2 U_{i-1/2} + A_3 U_{i+1/2} = h^2(B_1 M_{i-3/2} + B_2 M_{i-1/2} + B_3 M_{i+1/2}), \ i = 2,3,..,n-1 \quad (12)$$

The fourth order method is obtained by using the parameter $(B_1, B_2, B_3) = (\frac{1}{12},\frac{10}{12},\frac{1}{12})$ in the scheme (12)

$$A_1 U_{i-3/2} + A_2 U_{i-1/2} + A_3 U_{i+1/2} = h^2(\frac{1}{12} M_{i-3/2} + \frac{10}{12} M_{i-1/2} + \frac{1}{12} M_{i+1/2}), i = 2,3,..,n-1. \quad (13)$$

Where,

$$M_{i-3/2} = F(x, U_{i-3/2}, U^{(1)}{}_{i-3/2}).$$
$$M_{i-1/2} = F(x, U_{i-1/2}, U^{(1)}{}_{i-1/2}).$$
$$M_{i+1/2} = F(x, U_{i+1/2}, U^{(1)}{}_{i+1/2}).$$

The finite difference approximations to derivatives are given as

$$U^{(1)}_{i-1/2} = \frac{U_{i+1/2} - U_{i-3/2}}{2h},$$
$$U^{(1)}_{i-3/2} = \frac{-3U_{i-3/2} + 4U_{i-1/2} - U_{i+1/2}}{2h},$$
$$U^{(1)}_{i+1/2} = \frac{U_{i-3/2} - 4U_{i-1/2} + 3U_{i+1/2}}{2h},$$
$$\overline{U}^{(1)}_{i-1/2} = \frac{U_{i+1/2} - U_{i-3/2}}{2h} - \frac{h}{20}(M_{i+1/2} - M_{i-3/2}).$$

The fourth order approximations of derivatives involved in the end equation (9)-(10) are as follows:

$$hU^{(1)}_{1/2} = -\frac{32}{35}U_0 + \frac{1}{6}U_{1/2} + U_{3/2} - \frac{3}{10}U_{5/2} + \frac{1}{21}U_{7/2},$$
$$hU^{(1)}_{3/2} = -\frac{32}{105}U_0 - U_{1/2} + \frac{1}{6}U_{3/2} + \frac{3}{5}U_{5/2} - \frac{1}{14}U_{7/2},$$
$$hU^{(1)}_{5/2} = -\frac{32}{105}U_0 + \frac{5}{6}U_{1/2} - \frac{5}{3}U_{3/2} + \frac{9}{10}U_{5/2} + \frac{5}{21}U_{7/2},$$
$$hU^{(1)}_{1/2} = -\frac{32}{35}U_0 - \frac{7}{3}U_{1/2} + \frac{7}{2}U_{3/2} - \frac{21}{5}U_{5/2} + \frac{89}{42}U_{7/2}.$$

Here we derive only the second order method .By using the second derivative approximation we obtain the following final scheme as

$$J_i U_{i-3/2} + K_i U_{i-1/2} + L_i U_{i+1/2} = H_i \tag{14}$$

where, $J_i = -A_1 - \frac{3h}{2}B_1 a_{i-3/2} + h^2 B_1 b_{i-3/2} - \frac{h}{2}B_2 a_{i-1/2} + \frac{h}{2}B_3 a_{i+3/2},$

$$K_i = -A_2 + 2hB_1 a_{i-3/2} + h^2 B_2 b_{i-3/2} - 2hB_3 a_{i+1/2}$$
$$L_i = -A_3 - \frac{h}{2}B_1 a_{i-3/2} + \frac{1}{2}hB_2 a_{i-1/2} + \frac{3}{2}hB_3 a_{i+1/2} + h^2 B_3 b_{i+1/2},$$
$$H_i = -h^2(B_1 f_{i-3/2} + B_2 f_{i-1/2} + B_3 f_{i+1/2}), i = 2,3,..,n-1.$$

Now for $i$=1, we have

$$K_1U_{1/2}+L_1U_{3/2}=H_1 \tag{15}$$

where,
$$K_1=3-\frac{15}{24}ha_{1/2}+\frac{15}{24}h^2b_{1/2}-\frac{3}{24}ha_{3/2},$$
$$L_1=-1+\frac{15}{24}ha_{1/2}+\frac{3}{24}h^2b_{1/2}+\frac{3}{24}ha_{3/2},$$
$$H_1=2U_0-\frac{h^2}{24}(15f_{1/2}+3f_{3/2}).$$

Now for $i$=$n$, we have

$$K_nU_{n-3/2}+L_nU_{n-1/2}=H_n \tag{16}$$

where,
$$K_n=3-\frac{15}{24}ha_{n-1/2}+\frac{15}{24}h^2b_{n-1/2}-\frac{3}{24}ha_{n-3/2},$$
$$L_n=-1+\frac{15}{24}ha_{n-1/2}+\frac{3}{24}h^2a_{n-3/2}+\frac{3}{24}h^2b_{n-3/2},$$
$$H_n=2U_n-\frac{h^2}{24}(15f_{n-1/2}+3f_{n-3/2})$$

## CONVERGENCE OF THE METHOD

In this section, we study the convergence analysis of the second order method developed in Section 2 where $(A_1,A_2,A_3,B_1,B_2,B_3)=(1,-2,1,\frac{1}{8},\frac{6}{8},\frac{1}{8})$. The method has the following form:

$$\text{MW=V} \tag{17}$$

$$M=\begin{bmatrix} K_1 & L_1 & 0 & & & & & & \\ J_2 & K_2 & L_2 & & & & & & \\ 0 & J_3 & K_3 & L_3 & 0 & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & J_{n-1} & K_{n-1} & L_{n-1} \\ & & & & & & 0 & K_n & L_n \end{bmatrix}$$

where, $M$ is a tridiagonal matrix of order $n$, $W = [\mathrm{U}_{1/2}, \mathrm{U}_{3/2}, \mathrm{U}_{5/2}, .....\mathrm{U}_{n\text{-}1/2}]^{\mathrm{T}}$ and the right hand side vector $V = [\mathrm{v}_{1/2}, \mathrm{v}_{3/2}, \mathrm{v}_{5/2}, .....\mathrm{v}_{n\text{-}1/2}]^{\mathrm{T}}$.

We also have,

$$M\overline{W} = V + T \tag{18}$$

where $\overline{W} = [\overline{U}_{1/2}, \overline{U}_{3/2}, ..., \overline{U}_{n-1/2}]^T$ be the exact solution and $T = [t_1, t_2, t_3, ..., t_n]^T$ be the local truncation error. From equation (17) and (18) we have,

$$M(\overline{W} - W) = T,$$
$$AE = T$$
$$E = \overline{W} - W = [\overline{e}_1, \overline{e}_2, \overline{e}_3, ......\overline{e}_n]^T.$$

Let $0 < R \in Z^+$ is the minimum of $|A_i|, |B_i|$ and $|C_i|$.

Then

$$\|L_i\| \le \max_{1\le i\le n-1} (1 + A_3 + \frac{1}{2}h(B_1 + B_2 + 3B_3)R + h^2 B_3 R + \frac{18}{24}hR + \frac{3}{24}h^2 R),$$

$$\|K_i\| \le \max_{1\le i\le n-1} (3 + A_1 + \frac{1}{2}h(3B_1 + B_2 + B_3)R + h^2 B_1 R + \frac{18}{24}hR + \frac{15}{24}h^2 R).$$

Further,

$$\|L_i\|_\infty \le \max_{1\le i\le n-1} (2 + \frac{11}{8}hR + \frac{1}{4}h^2 R),$$
$$\|K_i\|_\infty \le \max_{1\le i\le n-1} (4 + \frac{11}{8}hR + \frac{1}{2}h^2 R).$$

This shows that $\|L_i\|_\infty \le 2$ and $\|K_i\|_\infty \le 4$ for very small $h$. Hence $M$ is irreducible.

Now we have to show that $M$ is monotone. To show matrix $M$ is monotone first we calculate the sum of each row of the matrix $M$.

$$Sum_1 = K_1 + L_1,$$

$$Sum_i = J_i + K_i + L_i, i = 2,3,...,n-1$$

$$Sum_n = J_n + K_n$$

Then we have

$$Sum_1 = 2 + \frac{3}{4}h^2 b_{1/2},$$

$$Sum_i = -(A_1 + A_2 + A_3) + h^2(B_1 b_{i-3/2} + B_2 b_{i-1/2} + B_3 b_{i+1/2}), i = 2,3,...,n-1$$

$$Sum_n = 2 + \frac{3}{4}h^2 b_{n-1/2}.$$

For sufficiently small *h*, we easily show that the matrix *M* is irreducible and monotone .Therefore, $M^{-1}$ exist and $M^{-1} \geq 0.$

Hence,

$$\|E\| = \|M^{-1}\|\|T\|.$$

Now for sufficiently small *h*, we have

$$Sum_1 \geq \frac{3}{4}h^2 R,$$

$$Sum_i \geq h^2 R, i = 2,3,...,n-1$$

$$Sum_n \geq \frac{3}{4}h^2 R$$

Therefore, we get

$$\frac{1}{Sum_i} \leq \begin{cases} \frac{4}{3Rh^2}, i = 1 \\ \frac{1}{Rh^2}, i = 2,3....n-1 \\ \frac{4}{3Rh^2}, i = n \end{cases}$$

Let $M^{-1} = (m^*_{i,j}),$ then by theory of matrices [6], we get

$$\sum_{i=1}^{n} m_{i,j}^{*} Sum_i = 1, \; j = 1,2,3......n.$$

Therefore ,

$$m_{i,j}^{*} \le \frac{1}{Sum_i}$$

$$\left\| M^{-1} \right\| = \max_{1 \le i \le n} \sum_{j=1}^{n} \left| m_{i,j}^{*} \right| \le \sum_{i=1}^{n} \frac{1}{Sum_i} = \frac{1}{h^2 R}\left( \frac{8}{3} + 1 \right), \; i=1,2,\ldots,n.$$

$$\left\| T_i \right\| = \max_{1 \le i \le n} \left| T_i \right|.$$

The error is given by

$$\left\| E \right\| = \left\| M^{-1} \right\| \left\| T \right\| \le \frac{1}{h^2 R}\left( \frac{11}{3} \right) \left\| T \right\|.$$

Therefore, using (6) we get $\left\| T \right\| = O(h^4)$ for second order method.

$$\left\| E \right\| \le \frac{1}{h^2 R}\left( \frac{11}{3} \right) O(h^4) = O(h^2).$$

Hence, our method is second order convergent. By repeating the same procedure we can find the bound of error for fourth order method which is as follows:

$$\left\| E \right\| = \left\| M^{-1} \right\| \left\| T \right\| \le \frac{1}{h^2 R}\left( \frac{11}{3} \right) \left\| T \right\|.$$

Therefore, using (6) we get $\left\| T \right\| = O(h^6)$ for fourth order method.

$$\left\| E \right\| \le \frac{1}{h^2 R}\left( \frac{11}{3} \right) O(h^6) = O(h^4).$$

Hence, our method is fourth order convergent.

**Theorem.** The method given by equation (5) for solving the boundary value problem (1)-(2) for sufficiently small *h* has a second as well as fourth order convergence depending upon the parameters.

## NUMERICAL ILLUSTRATIONS

We have implemented our method on two problems which supports the theoretical analysis for second and fourth order convergence. The maximum absolute errors at the off-step points, max $|U(x_{i-1/2})-U_{i-1/2}|$ are tabulated in Tables 1 and 2 for different values of $N$.

**Example 1**: Consider the following second order non-linear singular BVP

$$(xU'(x))' = xe^{U(x)},$$
$$U(0) = 2\ln(d+1), U(1) = 0.$$

The exact solution is given by $U(x) = 2\ln(\frac{d+1}{dx^2+1})$,

where d=-5+2√6.

Maximum absolute errors of second as well as fourth order method for example 1 are given in Table 1.

**Table 1: Maximum absolute errors of example 1**

| $N$ | Our Second Order Method ***(B1, B2, B3) = (1/8,6/8,1/8)*** | Our Fourth Order Method ***(B1, B2, B3) = (1/12,10/12,1/12)*** | [1] |
|---|---|---|---|
| 16 | 1.3000E-03 | 2.5251E-07 | 2.52E -03 |
| 32 | 3.2243E-04 | 2.0754E-08 | 1.83E -04 |
| 64 | 8.1470E-05 | 1.6614E-09 | 1.28E -05 |
| 128 | 2.0473E-05 | 1.2811E-10 | 8.33E -07 |
| 256 | 5.1312E-06 | 9.6749E-12 | - |

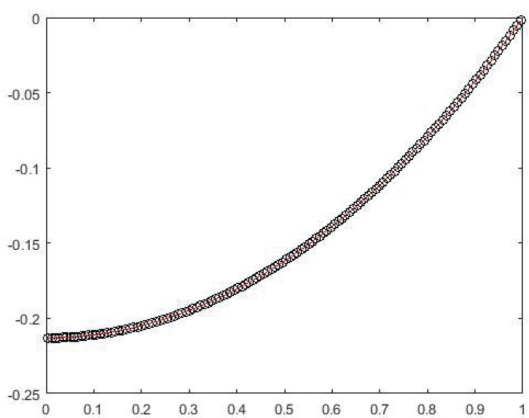


**Figure 1: Graph of the exact solution versus the approximate solution for *N*=256 for example 1**

**Example 2:** Consider the following second order non-linear singular BVP

$$(x^2 U'(x))' = -x^2 U^5(x),$$

$$U(0) = 1, U(1) = \sqrt{\frac{3}{4}}.$$

The exact solution is given by $U(x) = \sqrt{\frac{3}{3+x^2}}$.

Maximum absolute errors of second as well as fourth order method for example 2 are given in Table 2.

**Table 2: Maximum absolute errors of exmaple 2**

| *N* | Our Second Order Method ***(B₁, B₂, B₃)*** = **(1/8,6/8,1/8)** | Our Fourth Order Method ***(B₁, B₂, B₃)*** = **(1/12,10/12,1/12)** | [1] |
|---|---|---|---|
| 16 | 3.2000E-03 | 4.7046E-07 | 3.64E -04 |
| 32 | 7.9163E-04 | 3.0691E-08 | 2.49E -05 |
| 64 | 1.9626E-04 | 1.9372E-09 | 1.60E -06 |
| 128 | 4.8875E-05 | 1.2130E-10 | 1.01E -07 |
| 256 | 1.2196E-05 | 7.5364E-12 | - |

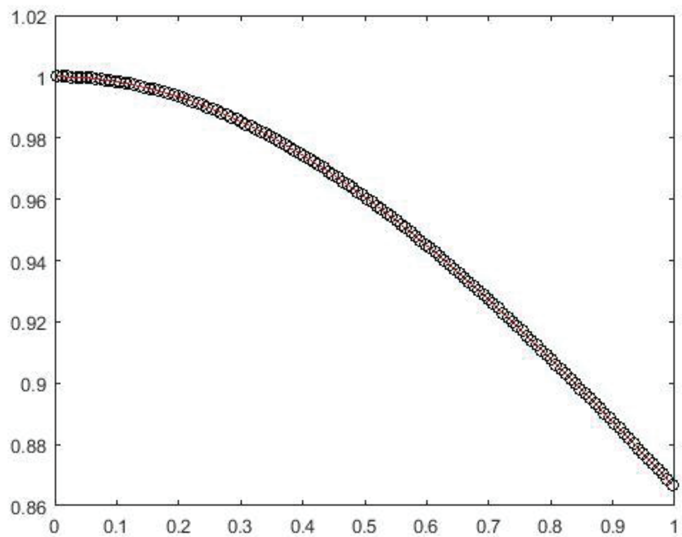


**Figure 2: Graph of the exact solution versus the approximate solution for *N*=256 for example 2**

## CONCLUSION

In this paper, a non-polynomial mixed spline of lower degree based on off-step points is used for solving second order non-linear singular boundary value problems. The presented method is second as well as fourth order accurate. Fourth order derivative approximations are derived. Convergence analysis is also discussed. Numerical examples are solved to show the applicability of the method. Comparison of our method with the existing methods of order four are shown in Tables 1 and 2. Graphs between exact and approximate solutions of the examples 1 and 2 are shown in figures 1 and 2 respectively also prove the accuracy of the method. Our approximated results are more accurate than the other existing methods.

## REFERENCES

[1] M. Chawla, R. Subramanian. and H.L. Sathi, A fourth order method for a singular two-point boundary value problem, BIT 28(1988) 88–97.

[2] M. Kumar and Y. Gupta, Methods for solving singular boundary value problems using splines: a review, Journal of Applied Mathematics and Computing volume 32(2010)265–278.

[3] M.A. Ramadan, I.F. Lashien and W.K. Zahra, Polynomial and non-polynomial spline approaches to the numerical solution of second order boundary value problems. Applied Mathematics and Computations, 184(2007)476–484.

[4] N. Jha. and R.K. Mohanty, TAGE iterative algorithm and non-polynomial spline basis for the solution of non-linear singular second order ordinary differential equations, Applied Mathematics and Computations, 218(2011)3289-3296.

[5] R.P. Agarwal, Boundary Value Problems for Higher-Order Differential Equations, World Scientific, Singapore, 1986.

[6] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[7] S. Islam, M.A. Noor, I.A. Tirmizi and M.A. Khan, Quadratic non-polynomial spline approach to the solution of a system of second-order boundary-value problems, Applied Mathematics and Computation 179(2006)153-160.

[8] T. Aziz and A. Khan, A spline method for second-order singularly perturbed boundary value problems, Journal of Computational and Applied Mathematics, 147(2002)445-452.