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ON MARTINGALE REPRESENTATION AND LOGARITHMIC-SOBOLEV INEQUALITY FOR FRACTIONAL BROWNIAN BRIDGE MEASURES

XIAOXIA SUN AND FENG GUO*

Abstract. In this paper, we consider the stochastic analysis for fractional Brownian bridge measures. We first give an integration by parts formula for such measures by Bismut's method and a pull back formula. Using this integration by parts formula, we then obtain a generalized Clark-Ocone martingale representation theorem for fractional Brownian bridge measures. Consequently, a Logarithmic-Sobolev inequality is derived by the martingale representation theorem for such measures.

1. Introduction

Fractional Brownian bridges are Gaussian bridges (see [7]). Measures determined by fractional Brownian bridges are called fractional Brownian bridge measures. In this paper, we consider the integration by parts formula, the martingale representation and the Logarithmic-Sobolev inequality for such measures.

Much work has been done on the integration by parts formula for bridge measures. Driver [5] gave an integration by parts formula for Brownian bridge measures on loop group with the vector field being C^1. For Cameron-Martin vector field, Enchev and Stroock [6] established an integration by parts formula for Brownian bridge measures on the loop space over Riemannian manifold with Levi-Civita connection. Similar results were also obtained in [10] by considering the path space and the estimates of derivatives of the heat kernel.

Through integration by parts formulas for bridge measures, the martingale representation and Logarithmic-Sobolev inequalities for bridge measures can be derived. A Logarithmic-Sobolev inequality for Brownian bridge measures on the loop group was obtained in [9]. For Brownian bridge measures on the loop space over Riemannian manifold, Gong and Ma [8] obtained a Logarithmic-Sobolev inequality by establishing a martingale representation theorem. For such measures, Aida [1] also gave a Logarithmic -Sobolev inequality with unbounded diffusion coefficients. A Logarithmic-Sobolev inequality for Gaussian measures was established in [3].

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The paper is organized as follows. In Section 2, we give some preliminaries about fractional Brownian bridge. We present in Section 3 a pull back formula and an integration by parts formula. In Section 4, we obtain a martingale representation theorem and a Logarithmic-Sobolev inequality for fractional Brownian bridge measures.

2. Preliminaries

It is known from [7] that the anticipative representation of fractional Brownian bridge $(X_t)_{0\leq t\leq 1}$ satisfies the following integral equation

$$X_t = B_t^H - \int_0^t \left(X_s + \int_0^s \Psi(s,u)dX_u \right) \frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2 du} ds, \tag{2.1}$$

where B^H is a fractional Brownian motion,

$$\begin{aligned} k(t,s) &= c_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}} du, \\ \Psi(t,s) &= \frac{\sin(\pi(H+\frac{1}{2}))}{\pi} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} \int_t^1 \frac{u^{H+\frac{1}{2}}(u-t)^{H+\frac{1}{2}}}{u-s} du, \end{aligned} \tag{2.2}$$

in which $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H,H-\frac{1}{2})}}$. By [7, Proposition 18], $(X_t)_{0\leq t\leq 1}$ admits the non-anticipative representation

$$X_t = B_t^H - \int_0^t \varphi(t,s) dB_s^H, \tag{2.3}$$

where

$$\varphi(t,s) = \int_s^t \left\{ \int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)dw)^2} dv - \frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2 dv} \right\} k(1,u)k(t,u)du. \tag{2.4}$$

We set $\Omega = \{\omega \in C([0,1];\mathbb{R}^n) \mid \omega_0 = \omega_1 = 0\}$ with the topology of local uniform convergence. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \nu)$ be a filtered probability space, where ν is the fractional Brownian bridge measure such that coordinate process $(X_t(\omega))_{0\leq t\leq 1} = (\omega_t)_{0\leq t\leq 1}$ satisfies integral equation (2.1), $\mathcal{F}$ is the ν- completion of the Borel σ-algebra of Ω and $\mathcal{F}_t$ is the ν-completed natural filtration of ω.

For any $p \in [1,\infty)$, let $L^p(\Omega;\nu) = \{F \mid F : \Omega \to \mathbb{R}, \|F\|_p := (\mathbb{E}_\nu |F|^p)^{\frac{1}{p}} < \infty\}$. We denote $(H+\frac{1}{2})$-Hölder left fractional Riemann-Liouville integral operator by $I_{0+}^{H+\frac{1}{2}}(L^2(\Omega;\nu))$. In [4], the isomorphism operator $K : L^2(\Omega;\nu) \to I_{0+}^{H+\frac{1}{2}}(L^2(\Omega;\nu))$ is defined as $(Kh)_t = \int_0^t k(t,s)h_s ds$, where $h \in L^2(\Omega;\nu)$ and k satisfies (2.2). We denote K^{-1} as the inverse operator of K. The Cameron-Martin vector field on Ω is defined as

$$\mathcal{H}_0 = \{Kh \mid h \text{ is adapted process}, h \in L^2(\Omega;\nu) \text{ and } (Kh)_1 = 0\},$$

with scalar product $\langle Kh, Kg\rangle_{\mathcal{H}_0} = \langle h, g\rangle_{L^2(\Omega;\nu)} = \mathbb{E}_\nu\left[\int_0^1 \langle h_t, g_t\rangle dt\right]$. For $Kh \in \mathcal{H}_0$, the directional derivative of F along Kh is

$$D_h F(\omega) = \lim_{\delta\to 0} \frac{1}{\delta}\left(F(\omega + \delta(Kh)) - F(\omega)\right).$$

The set of all the smooth cylindrical functions on Ω is denoted by

$$\mathcal{F}C^\infty(\Omega) = \{F \mid F(\omega) = f(\omega_{t_1}, ..., \omega_{t_n}), 0 < t_1 \le \cdots \le t_n \le 1, f \in C^\infty(\mathbb{R}^n)\}.$$

For $F \in \mathcal{F}C^\infty(\Omega)$, the directional derivative of F is

$$D_h F(\omega) = \sum_{i=1}^n \langle \nabla^i F, (Kh)_{t_i}\rangle_{\mathbb{R}^n},$$

where $\nabla^i F = \nabla^i f(\omega_{t_1}, \cdots, \omega_{t_n})$ is the gradient with respect to the i-th variable of f. The gradient $DF : \Omega \to \mathcal{H}_0$ is determined by $\langle DF, Kh\rangle_{\mathcal{H}_0} = D_h F$. We denote the domain of D by $Dom(D)$.

3. Integration by Parts Formula for ν

To obtain an integration by parts formula for fractional Brownian bridge measures, as in [2], we first give a pull back formula for such measures. We need construct the stochastic integral equation for the flow of $(X_t)_{0\le t\le 1}$ as follows. For any $r \in (-\epsilon, \epsilon)$,

$$X_t(r) = B_t^H(r) - \int_0^t \left(X_s(r) + \int_0^s \Psi(s,u) dX_u(r)\right) \frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2 du} ds, \tag{3.1}$$

where $B_t^H(r)$ is defined as $B_t^H(r) = B_t^H + r\beta_t$, in which β is a $\mathbb{R}^n$-valued adapted process.We give the form of β in the following pull back formula.

Proposition 3.1. *If the solution of (3.1) satisfies*

(1) $(X_t(r))_{0\le t\le 1} \in \Omega$ *for any* r,

(2) $\frac{d}{dr}X_t(r)\Big|_{r=0}$ *exists and* $(Kh)_t = \frac{d}{dr}X_t(r)\Big|_{r=0}$ *for* $(h_t)_{0\le t\le 1} \in L^2(\Omega;\nu)$,

then

$$\beta_t = (Kh)_t + \int_0^t \left((Kh)_s + \int_0^s \Psi(s,u) d(Kh)_u\right) \frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2 du} ds. \tag{3.2}$$

Proof. Differentiating (3.1) with respect to r at $r = 0$, we obtain

$$\begin{aligned}\frac{d}{dr}X_t(r)\Big|_{r=0} =& \frac{d}{dr}B_t^H(r)\Big|_{r=0} \\ &- \int_0^t \left(\frac{d}{dr}X_s(r)\Big|_{r=0} + \int_0^s \Psi(s,u) d\frac{d}{dr}X_u(r)\Big|_{r=0}\right) \frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2 du} ds.\end{aligned}$$

By Condition 2, we have $\frac{d}{dr}X_t(r)\Big|_{r=0} = (Kh)_t$. Then,

$$(Kh)_t = \beta_t - \int_0^t \left((Kh)_s + \int_0^s \Psi(s,u) d(Kh)_u\right) \frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2 du} ds,$$

which yields (3.2). □

Now we can obtain an integration by parts formula for the fractional Brownian bridge measure ν.

Theorem 3.2. *For $T \in (0,1)$, $F \in Dom(D) \cap \mathcal{F}_T$ and $Kh \in \mathcal{H}_0$, an integration by parts formula for fractional Brownian bridge measure ν is*

$$\mathbb{E}_\nu \left[F \int_0^T \left\langle \left(K^{-1}\beta.\right)_t, dB_t \right\rangle \right] = \mathbb{E}_\nu [D_h F], \tag{3.3}$$

where

$$(K^{-1}\beta.)_t = h_t + \left((Kh)_t + \int_0^t \Psi(t,u) d(Kh)_u \right) \frac{k(1,t)}{\int_t^1 k(1,u)^2 du}.$$

Proof. It is proved in [4] that there is a Brownian motion $(B_t)_{0 \le t \le 1}$ such that $B_t^H = \int_0^t K(t,s) dB_s$. Thus, by Proposition 3.1, we obtain

$$B_t^H(r) = \int_0^t k(t,s) d \left(B_s + r \int_0^s \left(K^{-1}\beta.\right)_u du \right).$$

We set

$$\rho_t = \exp\left\{ -r \int_0^t \left\langle \left(K^{-1}\beta.\right)_s, dB_s \right\rangle - \frac{r^2}{2} \int_0^t \left(K^{-1}\beta.\right)_s^2 ds \right\}.$$

For $H > \frac{1}{2}$, by Proposition 3.1, we have

$$\left(K^{-1}\beta.\right)_t = h_t + \left((Kh)_t + \int_0^t \Psi(t,u) d(Kh)_u \right) \frac{k(1,t)}{\int_t^1 k(1,u)^2 du}. \tag{3.4}$$

It follows that

$$\begin{aligned} \int_0^1 \left(K^{-1}\beta.\right)_t^2 dt \le & 2 \int_0^1 h_t^2 dt + 4 \int_0^1 (Kh)_t^2 d \frac{1}{\int_t^1 k(1,u)^2 du} \\ & + 4 \int_0^1 \frac{\left(\int_0^t \Psi(t,u) d(Kh)_u \right)^2 k^2(1,t)}{(\int_t^1 k(1,u)^2 du)^2} dt. \end{aligned} \tag{3.5}$$

By the definition of k in (2.2), we have

$$\frac{c_H}{H - \frac{1}{2}} (1-t)^{H-\frac{1}{2}} \le k(1,t) \le \frac{c_H}{H-\frac{1}{2}} t^{\frac{1}{2}-H} (1-t)^{H-\frac{1}{2}}. \tag{3.6}$$

Since Kh is H-Hölder continuous and $(Kh)_1 = 0$, there is a constant C_K such that

$$|(Kh)_t| \le C_K (1-t)^H \left(\int_0^1 h_t^2 dt \right)^{\frac{1}{2}}. \tag{3.7}$$

By the expression of K,

$$\begin{aligned} (Kh)_t = & c_H \int_0^t s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{1}{2}} du h_s ds \\ = & c_H \int_0^t \int_0^u s^{\frac{1}{2}-H} u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} h_s ds du, \end{aligned}$$

which implies that

$$(Kh)'_t = c_H \int_0^t s^{\frac{1}{2}-H} t^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}} h_s ds. \tag{3.8}$$

Suppose that there is a constant C_h such that $|h| \le C_h$. By (3.6), (3.7) and (3.8), we get

$$\begin{aligned}
&\int_0^1 (Kh)_t^2 d\frac{1}{\int_t^1 k(1,u)^2 du} \\
\le& \left| \lim_{t\to 1} \frac{(Kh)_t^2}{\int_t^1 k(1,u)^2 du} \right| + \left| \int_0^1 \frac{2(Kh)_t (Kh)'_t}{\int_t^1 k(1,u)^2 du} dt \right| \\
\le& \frac{2H(H-\frac{1}{2})^2 C_K^2 C_h^2}{c_H^2} + \frac{4H(H-\frac{1}{2})^2 C_K C_h^2}{c_H} \int_0^1 t^{H-\frac{1}{2}} \frac{\int_0^t s^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}} ds}{(1-t)^H} dt \\
=& \frac{2H(H-\frac{1}{2})^2 C_K^2 C_h^2}{c_H^2} + \frac{4H(H-\frac{1}{2})^2 C_K C_h^2 B(H-\frac{1}{2}, \frac{3}{2}-H)}{(1-H)c_H}.
\end{aligned} \tag{3.9}$$

By (2.2), there exists a constant C_Ψ such that

$$\Psi(t,s) \le C_\Psi s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} (1-t)^{H+\frac{1}{2}}. \tag{3.10}$$

By (3.8) and (3.10), we obtain

$$\begin{aligned}
&\left(\int_0^t \Psi(t,u)(Kh)'_u du \right)^2 \\
\le& \left(\int_0^t C_\Psi u^{\frac{1}{2}-H} (t-u)^{\frac{1}{2}-H} (1-t)^{H+\frac{1}{2}} c_H \int_0^u s^{\frac{1}{2}-H} u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} |h_s| ds du \right)^2 \\
=& \left(\int_0^t \left(\int_s^t C_\Psi u^{\frac{1}{2}-H} (t-u)^{\frac{1}{2}-H} (1-t)^{H+\frac{1}{2}} c_H s^{\frac{1}{2}-H} u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du \right) |h_s| ds \right)^2 \\
\le& c_H^2 C_\Psi^2 (1-t)^{2H+1} \int_0^t s^{1-2H} \left(\int_s^t (t-u)^{\frac{1}{2}-H} (u-s)^{H-\frac{3}{2}} du \right)^2 ds \int_0^t h_s^2 ds \\
\le& \frac{c_H^2 C_\Psi^2 B(\frac{3}{2}-H, H-\frac{1}{2})}{2-2H} (1-t)^{2H+1} \int_0^1 h_s^2 ds.
\end{aligned}$$

It follows that

$$\int_0^1 \frac{\left(\int_0^t \Psi(t,u) d(Kh)_u \right)^2 k^2(1,t)}{(\int_t^1 k(1,u)^2 du)^2} dt \le \frac{4H^2(H-\frac{1}{2})^2 C_\Psi^2 B(\frac{3}{2}-H, H-\frac{1}{2}) C_h^2}{(2-2H)^2}. \tag{3.11}$$

By (3.5), (3.9) and (3.11), we have that $\mathbb{E}_\nu[\rho_1] = 1$. It is easy to check that $\beta \in I_{0^+}^{H+\frac{1}{2}}(L^2(\Omega;\nu))$. Hence, by [12, Theorem 2],

$$B_t^H(r) = \left(\int_0^t K(t,s) d\left(B_s + r \int_0^s \left(K^{-1}\beta \right)_u du \right) \right)_{0\le t\le 1}$$

is a fractional Brownian motion under $\rho_1 \nu$. Then $(X_t(r))_{0\le t\le 1}$ and $(X_t)_{0\le t\le 1}$ have the same distribution under $\rho_1\nu$ and ν respectively. Therefore, for $F =$

$f(X_{t_1},...,X_{t_n}) \in \mathcal{FC}^\infty(\Omega)$,

$$\mathbb{E}_{\rho_1\nu}[f(X_{t_1}(r),\cdots,X_{t_n}(r))] = \mathbb{E}_\nu[f(X_{t_1},\cdots,X_{t_n})].$$

Differentiating above equation with respect to r we have

$$\begin{aligned}
&\frac{d}{dr}\mathbb{E}_\nu[\rho_1 f(X_{t_1}(r),\cdots,X_{t_n}(r))]\Big|_{r=0}\\
&=\mathbb{E}_\nu\left[\frac{d}{dr}\rho_1\Big|_{r=0} f(X_{t_1},\cdots,X_{t_n})\right] + \mathbb{E}_\nu\left[\frac{d}{dr}f(X_{t_1}(r),\cdots,X_{t_n}(r))\Big|_{r=0}\right]\\
&=-\mathbb{E}_\nu\left[F\int_0^1 \left\langle \left(K^{-1}\beta.\right)_t, dB_t\right\rangle\right] + \mathbb{E}_\nu[D_hF] = 0.
\end{aligned}$$

Thus for adapted bounded process h, we get

$$\mathbb{E}_\nu\left[F\int_0^1 \left\langle \left(K^{-1}\beta.\right)_t, dB_t\right\rangle\right] = \mathbb{E}_\nu[D_hF].$$

Hence, for $F \in \mathcal{F}_T$,

$$\mathbb{E}_\nu\left[F\int_0^T \left\langle \left(K^{-1}\beta.\right)_t, dB_t\right\rangle\right] = \mathbb{E}_\nu[D_hF]. \tag{3.12}$$

By (3.5), (3.9) and (3.11), we can easily obtain that $(K^{-1}\beta.) \in L^2(\Omega;\nu)$ for any adapted process $h \in L^2(\Omega;\nu)$. Therefore, (3.12) holds for any adapted process $h \in L^2(\Omega;\nu)$. Moreover, since D is a closable operator, the integration by parts formula (3.12) holds for any $F \in Dom(D) \cap \mathcal{F}_T$. □

4. Martingale Representation Theorem and Logarithmic-Sobolev Inequality for ν

Inspired by [8] and [11], we first established a martingale representation theorem for ν through its integration by parts formula, then we prove a Logarithmic-Sobolev inequality for ν by the martingale representation theorem.

Theorem 4.1. *Suppose that $F \in Dom(D) \cap \mathcal{F}_T$, there exists a $\mathcal{F}_t$-predictable process $(\eta_t)_{0\le t\le 1}$ such that*

$$F = \mathbb{E}_\nu[F] + \int_0^T \langle \eta_t, dB_t\rangle,$$

where

$$\begin{aligned}
\eta_t = \mathbb{E}_\nu\Bigg[&(K^{-1}DF)_t\\
&-\int_t^T\left(c_H t^{\frac{1}{2}-H}s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}}\int_s^T \delta(u,s)(K^{-1}DF)_u du\right)ds\Bigg|\mathcal{F}_t\Bigg],
\end{aligned} \tag{4.1}$$

in which

$$\delta(u,s) = \left(\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)dw)^2}dv - \frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2 dv}\right)k(1,u).$$

Proof. By the definition of D_hF, we have

$$\mathbb{E}_\nu[D_hF] = \mathbb{E}_\nu[\langle DF, Kh\rangle_{\mathcal{H}^H}] = \mathbb{E}_\nu\left[\int_0^T \langle (K^{-1}DF)_t, h_t\rangle dt\right]. \tag{4.2}$$

By (3.3), we obtain

$$\begin{aligned}\mathbb{E}_\nu[D_hF] =&\mathbb{E}_\nu\left[\int_0^T \langle \eta_t, dB_t\rangle \int_0^T \left\langle \left(K^{-1}\beta.\right)_t, dB_t\right\rangle\right]\\ =&\mathbb{E}_\nu\left[\int_0^T \left\langle \eta_t, \left(K^{-1}\beta.\right)_t\right\rangle dt\right].\end{aligned} \tag{4.3}$$

For any $j \in L^2(\Omega;\nu)$, let $j_t = \left(K^{-1}\beta.\right)_t$. Then

$$(Kh)_t + \int_0^t \left((Kh)_s + \int_0^s \Psi(s,u)d(Kh)_u\right)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2du}ds = (Kj)_t,$$

by (2.3) and (2.4), we have

$$(Kh)_t = (Kj)_t - \int_0^t \varphi(t,s)d(Kj)_s,$$

and $Kh \in \mathcal{H}_0$. Thus

$$h_t = j_t - \left(K^{-1}\left(\int_0^\cdot \varphi(\cdot,s)d(Kj)_s\right)\right)_t. \tag{4.4}$$

By (4.2), (4.3) and (4.4), we get

$$\begin{aligned}&\mathbb{E}_\nu\left[\int_0^T \left\langle (K^{-1}DF)_t, j_t - \left(K^{-1}\left(\int_0^\cdot \varphi(\cdot,s)d(Kj)_s\right)\right)_t\right\rangle dt\right]\\ &= \mathbb{E}_\nu\left[\int_0^T \langle \eta_t, j_t\rangle\, dt\right].\end{aligned} \tag{4.5}$$

It is obvious that

$$\begin{aligned}&\int_0^t \varphi(t,s)(Kj)'_s ds\\ &= \int_0^t\left\{\int_s^t\left(\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)^2dw)^2}dv - \frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2dv}\right)k(1,u)k(t,u)du\right\}(Kj)'_s ds\\ &= \int_0^t k(t,u)\left\{\int_0^u\left(\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)^2dw)^2}dv - \frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2dv}\right)k(1,u)(Kj)'_s ds\right\}du\\ &= \left(K\int_0^\cdot \delta(\cdot,s)(Kj)'_s ds\right)_t,\end{aligned}$$

where

$$\delta(u,s) = \left(\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)^2dw)^2}dv - \frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2dv}\right)k(1,u). \tag{4.6}$$

Hence, the left side of (4.5) can be written as

$$\begin{aligned}
&\mathbb{E}_\nu\left[\int_0^T \left\langle (K^{-1}DF)_t, j_t\right\rangle dt\right] - \mathbb{E}_\nu\left[\int_0^T \left\langle (K^{-1}DF)_t, \int_0^t \delta(t,s)(Kj)'_s ds\right\rangle dt\right]\\
=&\mathbb{E}_\nu\left[\int_0^T \left\langle (K^{-1}DF)_t, j_t\right\rangle dt\right] - \mathbb{E}_\nu\left[\int_0^T \left\langle \int_s^T \delta(t,s)(K^{-1}DF)_t dt, (Kj)'_s\right\rangle ds\right].
\end{aligned} \tag{4.7}$$

By (3.8), the second term for above equation is

$$\begin{aligned}
&\mathbb{E}_\nu\left[\int_0^T \left\langle \int_s^T \delta(t,s)(K^{-1}DF)_t dt, (Kj)'_s\right\rangle ds\right]\\
=&\mathbb{E}_\nu\left[\int_0^T \left\langle \int_s^T \delta(u,s)(K^{-1}DF)_u du, c_H\int_0^s t^{\frac{1}{2}-H}s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}}j_t dt\right\rangle ds\right]\\
=&\mathbb{E}_\nu\left[\int_0^T \left\langle \int_t^T \left(c_H t^{\frac{1}{2}-H}s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}}\int_s^T \delta(u,s)(K^{-1}DF)_u du\right) ds, j_t\right\rangle dt\right].
\end{aligned}$$

Then by (4.5) and (4.7), we have

$$\begin{aligned}
&\mathbb{E}_\nu\left[\int_0^T \left\langle (K^{-1}DF)_t \right.\right.\\
&\qquad \left.\left. - \int_t^T \left(c_H t^{\frac{1}{2}-H}s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}}\int_s^T \delta(u,s)(K^{-1}DF)_u du\right) ds, j_t\right\rangle dt\right]\\
=&\mathbb{E}_\nu\left[\int_0^T \langle \eta_t, j_t\rangle dt\right],
\end{aligned}$$

which yields

$$\begin{aligned}
\eta_t =&\mathbb{E}_\nu\left[(K^{-1}DF)_t \right.\\
&\left. - \int_t^T \left(c_H t^{\frac{1}{2}-H}s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}}\int_s^T \delta(u,s)(K^{-1}DF)_u du\right) ds \middle| \mathcal{F}_t\right].
\end{aligned}$$

□

Now we can prove a Logarithmic-Sobolev inequality for ν by Theorem 4.1.

Theorem 4.2. *For $F \in Dom(D) \cap \mathcal{F}_T$, we have*

$$\mathbb{E}_\nu[F^2 \ln F^2] \le 4\left(1 + \frac{4C}{2-2H}\right)\mathbb{E}_\nu\left[\int_0^T |(K^{-1}DF)_s|^2 ds\right] + \mathbb{E}_\nu[F^2]\ln \mathbb{E}_\nu[F^2],$$

where

$$\begin{aligned}
C =&\frac{c_H^2 C_1^2}{(2-2H)^2(H-\frac{1}{2})^2} + \left(\frac{c_H C_1 C_\Psi B(H-\frac{1}{2}, \frac{3}{2}-H)}{(2-2H)\sqrt{2-2H}}\right)^2 + \frac{c_H^2 C_2^2}{(H-\frac{1}{2})^2}\\
&+ \left(\frac{c_H C_2 C_\Psi B(H-\frac{1}{2}, \frac{3}{2}-H)}{\sqrt{2-2H}}\right)^2,
\end{aligned}$$

in which $C_1 = \frac{(H-\frac{1}{2})(2H)^2}{c_H(1-T)^{2H+1}}$, $C_2 = \frac{2H(H-\frac{1}{2})}{c_H(1-T)^{H+\frac{1}{2}}}$ *and* C_Ψ *satisfies (3.10).*

Proof. Let $G = F^2$. We let G_t be a right continuous version of $\mathbb{E}_\nu[G|\mathcal{F}_t]$, then by Theorem 4.1, we have $dG_t = \langle \eta_t, dB_t\rangle$. By Itô formula, we obtain

$$d(G_t \ln(G_t)) = (1 + \ln(G_t))dG_t + \frac{1}{2}\frac{|\eta_t|^2}{G_t}dt = \langle (1 + \ln(G_t))\eta_t, dB_t\rangle + \frac{1}{2}\frac{|\eta_t|^2}{G_t}dt,$$

which implies

$$\mathbb{E}_\nu[G \ln G] - \mathbb{E}_\nu[G] \ln \mathbb{E}_\nu[G] = \frac{1}{2}\mathbb{E}_\nu\left[\int_0^T \frac{|\eta_t|^2}{G_t}dt\right]. \tag{4.8}$$

Since $DF^2 = 2FDF$,

$$\begin{aligned}\eta_t =& \mathbb{E}_\nu\Big[2F\Big((K^{-1}DF)_t \\ &- \int_t^T \int_t^u \Big(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}}\delta(u,s)(K^{-1}DF)_u\Big)\, dsdu\Big)\Big|\mathcal{F}_t\Big].\end{aligned}$$

It follows that

$$\begin{aligned}|\eta_t|^2 \le& 8\mathbb{E}_\nu\left[F^2|\mathcal{F}_t\right]\mathbb{E}_\nu\Big[|(K^{-1}DF)_t|^2 \\ &+ \left|\int_t^T \int_t^u \Big(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}}\delta(u,s)(K^{-1}DF)_u\Big)\, dsdu\right|^2 \Big|\mathcal{F}_t\Big] \\ \le& 8\mathbb{E}_\nu\left[F^2|\mathcal{F}_t\right]\mathbb{E}_\nu\Big[|(K^{-1}DF)_t|^2 \\ &+ \int_t^T \left(\int_t^u \Big(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}}\delta(u,s)\Big)\, ds\right)^2 du \\ &\times \int_t^T \left|(K^{-1}DF)_u\right|^2 du\Big|\mathcal{F}_t\Big].\end{aligned} \tag{4.9}$$

By (3.6), (3.10) and (4.6), for the constants

$$C_1 = \frac{(H-\frac{1}{2})(2H)^2}{c_H(1-T)^{2H+1}} \quad \text{and} \quad C_2 = \frac{2H(H-\frac{1}{2})}{c_H(1-T)^{H+\frac{1}{2}}},$$

we have

$$\begin{aligned}|\delta(u,s)| =& \left|\left(\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)^2 dw)^2}dv - \frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2 dv}\right)k(1,u)\right| \\ =& A_1 + A_2 + A_3 + A_4,\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}&A_1 = \frac{C_1 u^{\frac{1}{2}-H}}{2-2H}, \quad A_2 = C_1 C_\Psi \int_s^u s^{\frac{1}{2}-H}(v-s)^{\frac{1}{2}-H} v^{1-2H} dv u^{\frac{1}{2}-H} \\ &A_3 = C_2 u^{\frac{1}{2}-H}, \quad A_4 = C_2 C_\Psi s^{\frac{1}{2}-H}(u-s)^{\frac{1}{2}-H} u^{\frac{1}{2}-H}.\end{aligned}$$

Therefore, by (4.9),

$$\begin{aligned}
|\eta_t|^2 \leq & 8\mathbb{E}_\nu \left[F^2|\mathcal{F}_t\right] \mathbb{E}_\nu \Bigg[|(K^{-1}DF)_t|^2 \\
& + 4\int_t^T \left(\int_t^u \left(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} A_1\right) ds\right)^2 \\
& + \left(\int_t^u \left(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} A_2\right) ds\right)^2 \\
& + \left(\int_t^u \left(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} A_3\right) ds\right)^2 \\
& + \left(\int_t^u \left(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} A_4\right) ds\right)^2 du \int_t^T \left|(K^{-1}DF)_u\right|^2 du \Bigg|\mathcal{F}_t\Bigg].
\end{aligned} \tag{4.11}$$

It is obvious that

$$\left(\int_t^u \left(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} A_1\right) ds\right)^2 \leq \frac{c_H^2 C_1^2 t^{1-2H}}{(2-2H)^2(H-\frac{1}{2})^2}. \tag{4.12}$$

It holds that

$$\begin{aligned}
&\left(\int_t^u \left(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} A_2\right) ds\right)^2 \\
&= (c_H C_1 C_\Psi)^2 \\
&\quad \left(\int_t^u \int_s^u \left(t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} s^{\frac{1}{2}-H}(v-s)^{\frac{1}{2}-H} v^{1-2H} u^{\frac{1}{2}-H}\right) dv ds\right)^2 \\
&= (c_H C_1 C_\Psi)^2 \left(\int_t^u t^{\frac{1}{2}-H} v^{1-2H} u^{\frac{1}{2}-H} \int_t^v \left((s-t)^{H-\frac{3}{2}}(v-s)^{\frac{1}{2}-H}\right) ds dv\right)^2 \\
&= \left(c_H C_1 C_\Psi B(H-\frac{1}{2}, \frac{3}{2}-H)\right)^2 t^{1-2H} u^{1-2H} \left(\int_t^u v^{1-2H} dv\right)^2 \\
&\leq \left(\frac{c_H C_1 C_\Psi B(H-\frac{1}{2}, \frac{3}{2}-H)}{2-2H}\right)^2 t^{1-2H} u^{1-2H}.
\end{aligned} \tag{4.13}$$

We can easily obtain that

$$\left(\int_t^u \left(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} A_3\right) ds\right)^2 \leq \frac{c_H^2 C_2^2}{(H-\frac{1}{2})^2} t^{1-2H}. \tag{4.14}$$

It is easy to check that

$$\begin{aligned}
&\left(\int_t^u \left(c_H t^{\frac{1}{2}-H} s^{H-\frac{1}{2}}(s-t)^{H-\frac{3}{2}} A_4\right) ds\right)^2 \\
&= (c_H C_2 C_\Psi)^2 t^{1-2H} u^{1-2H} \left(\int_t^u (s-t)^{H-\frac{3}{2}}(u-s)^{\frac{1}{2}-H} ds\right)^2 \\
&= \left(c_H C_2 C_\Psi B(H-\frac{1}{2}, \frac{3}{2}-H)\right)^2 t^{1-2H} u^{1-2H}.
\end{aligned} \tag{4.15}$$

By (4.11), (4.12), (4.13), (4.14) and (4.15), we have

$$|\eta_t|^2 \leq 8\mathbb{E}_\nu \left[F^2|\mathcal{F}_t\right] \mathbb{E}_\nu \left[|(K^{-1}DF)_t|^2 + 4Ct^{1-2H} \int_t^T \left|(K^{-1}DF)_u\right|^2 du \middle| \mathcal{F}_t \right]. \tag{4.16}$$

where

$$\begin{aligned} C = & \frac{c_H^2 C_1^2}{(2-2H)^2(H-\frac{1}{2})^2} + \left(\frac{c_H C_1 C_\Psi B(H-\frac{1}{2}, \frac{3}{2}-H)}{(2-2H)\sqrt{2-2H}} \right)^2 \\ & + \frac{c_H^2 C_2^2}{(H-\frac{1}{2})^2} + \left(\frac{c_H C_2 C_\Psi B(H-\frac{1}{2}, \frac{3}{2}-H)}{\sqrt{2-2H}} \right)^2. \end{aligned}$$

Then it holds that

$$\begin{aligned} \mathbb{E}_\nu \left[\int_0^T \frac{|\eta_t|^2}{G_t} dt \right] \leq & 8 \left(1 + 4C \int_0^T t^{1-2H} dt \right) \mathbb{E}_\nu \left[\int_0^T |(K^{-1}DF)_s|^2 ds \right] \\ \leq & 8 \left(1 + \frac{4C}{2-2H} \right) \mathbb{E}_\nu \left[\int_0^T |(K^{-1}DF)_s|^2 ds \right]. \end{aligned}$$

Hence, by (4.8), we obtain a Logarithmic-Sobolev inequality for ν as follows

$$\mathbb{E}_\nu[F^2 \ln F^2] \leq 4 \left(1 + \frac{4C}{2-2H} \right) \mathbb{E}_\nu \left[\int_0^T |(K^{-1}DF)_s|^2 ds \right] + \mathbb{E}_\nu[F^2] \ln \mathbb{E}_\nu[F^2].$$

□

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Xiaoxia Sun: School of Mathematics, Dongbei University of Finance and Economics, Dalian, 116025, China
E-mail address: xiaoxiasun@dufe.edu.cn

Feng Guo: School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China
E-mail address: fguo@dlut.edu.cn