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A STUDY ON EFFECTS OF TEMPERATURE GRADIENTS ON FLOW OF CHOLESTERIC LIQUID CRYSTALS BETWEEN TWO PARALLEL PLATES USING COUPLE STRESS BOUNDARY CONDITIONS FOR DIRECTOR

Renu Virk, Rajesh Joshi* & A.K. Sharma

ABSTRACT

In the present paper, a study on flow of cholesteric liquid crystals between two parallel plates using couple stress boundary conditions for the director has been considered.

It is observed that the increase in thermo-gradient parameter corresponds to the increase in orientation and the horizontal velocity component while a decrease in vertical velocity component, thereby affecting the direction of molecules throughout the flow. The effects of temperature gradients in all the three perpendicular directions on the flow of the cholesteric liquid crystals between two parallel plates when the lower plate is at rest and the upper plate is in the motion have been investigated. For couple stress boundary conditions with the thermo-gradient parameter in all the three directions, a solution has been obtained.

Keywords: *Cholesteric liquid crystal; couple stress boundary conditions; director; thermo-gradients parameters*

INTRODUCTION

Liquid crystals are highly anisotropic fluids that exist between the boundaries of solid and liquid phase. Attention has been given to liquid crystal phenomenon because of their practical applications. Oseen [2] proposed a general theory of liquid, Frank [3] has provided a simpler derivation of the static part of the Oseen's theory and Ericksen [5] reformulated this theory as the theory of anisotropic liquids. Using the theory of Leslie [4] and Sharma [1], here we have discussed the effect of temperature gradient on flow of cholesteric liquid crystals between two parallel plates using couple stress boundary conditions for the director.

* Department of Mathematics, Statistics and Computer Science, Govind Ballabh Pant University of Agriculture and Technology, Pantnagar, Udham Singh Nagar, Uttaranchal, India

BASIC EQUATIONS

The conservation laws for incompressible cholesteric liquid crystals with director of unit magnitude takes the form

$$v_{i,i} = 0 \tag{2.1}$$

$$\rho\left(Dv_i / Dt\right) = \rho F_i + \sigma_{ji,j} \tag{2.2}$$

$$\rho_1\left(D^2 d_i / Dt^2\right) = \rho_1 G_i + g_i + \pi_{ji,j} \tag{2.3}$$

$$\rho\left(DU/Dt\right) = \sigma_{ji} v_{i,j} + \pi_{ji}\left(Dd_{i,j}/Dt\right) - g_i\left(Dd_i/Dt\right) - q_{i,i} \tag{2.4}$$

where, $\vec{v}, \rho, F_i, \sigma_{ji}, \rho_1, G_i, g_i, \pi_{ji}, U, \vec{q}$ *and* D/Dt represents the velocity vector, the uniform density, the body force per unit mass, the stress tensor, an inertial constant, the extrinsic director body force per unit mass, the intrinsic director body force per unit volume, the director stress tensor, the internal energy per unit mass, the heat flux vector and the material time derivative respectively.

The stress tensor σ_{ji}, the director stress tensor σ_{ji} and the intrinsic director body force g_i are given by the constitutive equations

$$\sigma_{ji} = -P\delta_{ij} - \rho\left(\partial F/\partial d_{k,j}\right)d_{k,i} + \alpha e_{jkp}\left(d_p d_i\right)_{,k} + \tilde{\sigma}_{ji} \tag{2.5}$$

$$\pi_{ji} = \beta_j d_i + \rho\left(\partial F/\partial d_{i,j}\right) + \alpha e_{ijk} d_k \tag{2.6}$$

$$g_i = \gamma d_i - \beta_j d_{i,j} - \rho\left(\partial F/\partial d_i\right) - \alpha e_{ijk} d_{k,j} + \tilde{g}_i \tag{2.7}$$

here, $P, F, \alpha, \vec{\beta}, \gamma, \tilde{\sigma}_{ji}$ *and* $\tilde{g}_i$ denotes the pressure, Helmholtz free energy per unit mass, a material coefficient, an arbitrary vector, the director tension, the non-equilibrium parts of extra stress tensor and extra intrinsic director body force respectively.

$$\begin{aligned}\tilde{\sigma}_{ji} = {} & \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 N_i d_j + \mu_3 N_j d_j + \mu_4 A_{ij} + \mu_5 A_{ik} d_k d_j + \\ & \mu_6 A_{jk} d_k d_i + \mu_7 e_{ipq} d_p T,_q d_j + \mu_8 e_{jpq} d_p T,_q d_i\end{aligned} \tag{2.8}$$

$$\tilde{g}_i = \lambda_1 N_i + \lambda_2 A_{ik} d_k + \lambda_3 e_{ipq} d_p T,_q d_i \tag{2.9}$$

$$q_i = K_1 T,_i + K_2 d_k T,_k d_i + K_3 e_{ipq} d_p N_q + K_4 e_{ipq} d_p A_{qk} d_k \tag{2.10}$$

where μ_i, λ_i *and* K_i are the material coefficients and

$$\left.\begin{aligned} 2A_{ij} &= v_{i,j} + v_{j,i} \\ 2w_{ij} &= v_{i,j} - v_{j,i} \\ N_i &= \left(Dd_i/Dt\right) - w_{ij}d_j \end{aligned}\right\rangle \tag{2.11}$$

and the Helmholtz free energy F for cholesteric liquid crystal s

$$2\rho F = \alpha_1\left(d_{i,i}\right)^2 + \alpha_2\left(\tau + d_i e_{ijk} d_{k,j}\right)^2 + \alpha_3 d_i d_j d_{k,i} d_{k,j} + (\alpha_2 + \alpha_4)\left\{d_{i,j}d_{j,i} - \left(d_{i,i}\right)^2\right\} \tag{2.12}$$

where α_i and τ are the material coefficients. In the present problem we have assumed all the material coefficients to be constant.

STATEMENT OF PROBLEM

We consider the lower plate is at rest while the upper plate is moving with a uniform velocity along a straight line in its own plane with velocity

$$W = \sqrt{U^2 + V^2} \tag{3.1}$$

making an angle with the positive direction of x-axis as $\delta = \tan^{-1}\left(V/U\right)$ (3.2) where U and V are the components of velocity of upper plate in x and y directions respectively. We choose a system of right handed Cartesian system (x, y, z) such that the lower and upper plate occupy the plane z=o and z=h respectively.

We examine solutions of the equations (2.1) to (2.4) in the form

$$\begin{aligned} &d_x = \cos\theta(z)\cos\phi(z), d_y = \cos\theta(z)\sin\phi(z), d_z = \sin\theta(z) \\ &v_x = u(z), v_y = v(z), v_z = 0, \\ &T = ax + by + f(z) \end{aligned} \tag{3.3}$$

where a and b are two distinct constant temperature gradients in x and y direction respectively, and unknowns are function of z only.

We consider the boundary conditions for orientation, velocity and temperature in the form

$$\begin{bmatrix} \theta = 0; \phi = \tau_0\left(z - z_0\right); at\ z = 0 \\ \theta = 0; \phi = \tau_0\left(z - z_0\right); at\ z = h \end{bmatrix} \tag{3.4}$$

$$\begin{bmatrix} u = 0; v = 0; at\ z = 0 \\ u = U; v = V; at\ z = h \end{bmatrix} \tag{3.5}$$

$$\begin{bmatrix} f = T_0 ; & at & z = 0 \\ f = T_1 ; & at & z = h \end{bmatrix} \tag{3.6}$$

FORMATION OF DIFFERENTIAL EQUATIONS

Equation (2.1) is clearly satisfied by the velocity components given by (3.3). In the absence of external body forces F_i and G_i, (2.2) to (2.4) takes the form

$$\sigma_{zx}' = 0, \sigma_{zy}' = 0, \sigma_{zz}' = 0 \tag{4.1}$$

$$\pi_{zx}' + g_x = 0 \tag{4.2}$$

$$\pi_{zy}' + g_y = 0 \tag{4.3}$$

$$\pi_{zz}' + g_z = 0 \tag{4.4}$$

$$2\left(\tilde{\sigma}_{zx}\xi + \tilde{\sigma}_{zy}\eta\right) - q_z' = 0 \tag{4.5}$$

where $$2\xi = u', \quad 2\eta = v', \quad \zeta = f' \tag{4.6}$$

and the prime represents the derivative with respect to z.

The general solution of (4.1) is

$$\sigma_{zx} = k, \sigma_{zy} = l, \sigma_{zz} = m \tag{4.7}$$

where k,l,m are constants. Putting the values of $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$ from (2.5), (2.8) calculated with the help of (2.11), (2.12) and (3.3) in (4.7), we get

$$\begin{aligned} &\{H_1(\theta) + H_2(\theta)\cos^2\phi\}\xi + H_2(\theta)\eta\sin\phi\cos\phi + H_3(\theta)\zeta\sin\phi - H_5 a(\theta)\sin\phi\cos\phi - \\ &b\{H_4(\theta) - H_5(\theta)\cos^2\phi\} = k \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\{H_2(\theta)\xi\sin\phi\cos\phi\} + \{H_1(\theta) + H_2(\theta)\sin^2\phi\}\eta + a\{H_4(\theta) - H_5(\theta)\sin^2\phi\} \\ &- H_3(\theta)\zeta\cos\phi + bH_5 a(\theta)\sin\phi\cos\phi = l \end{aligned} \tag{4.9}$$

$$\begin{aligned} p = {} & \alpha_2\tau\cos^2\theta(\phi') - m - F_1(\theta)(\theta')^2 - F_2(\theta)(\phi')^2 + \\ & H_6(\theta)(\xi\cos\phi + \eta\sin\phi)\{H_3(\theta) + H_7(\theta)\}\{b\cos\phi - a\sin\phi\} \end{aligned} \tag{4.10}$$

$$\left.\begin{aligned}
H_1(\theta) &= \mu_4 + (\mu_5 - \mu_2)\sin^2\theta \\
H_2(\theta) &= (2\mu_1 \sin^2\theta + \mu_3 + \mu_6)\cos^2\theta \\
H_3(\theta) &= \mu_7 \sin\theta\cos\theta \\
H_4(\theta) &= \mu_7 \sin^2\theta \\
H_5(\theta) &= \mu_8 \cos^2\theta \\
H_6(\theta) &= (2\mu_1 \sin^2\theta + \mu_2 + \mu_3 + \mu_5 + \mu_6)\sin\theta\cos\theta \\
H_7(\theta) &= \mu_8 \sin\theta\cos\theta \\
H_5(\theta) &= \mu_8 \cos^2\theta \\
H_6(\theta) &= (2\mu_1 \sin^2\theta + \mu_2 + \mu_3 + \mu_5 + \mu_6)\sin\theta\cos\theta \\
H_7(\theta) &= \mu_8 \sin\theta\cos\theta
\end{aligned}\right\rangle \tag{4.11}$$

$$\left.\begin{aligned}
F_1(\theta) &= \alpha_1 \cos^2\theta + \alpha_3 \sin^2\theta \\
F_2(\theta) &= (\alpha_2 \cos^2\theta + \alpha_3 \sin^2\theta)\cos^2\theta
\end{aligned}\right\rangle \tag{4.12}$$

In the equation of energy (4.5) the first two terms represent viscous heating and are quadratic in ξ, ζ, η, a, b whereas the last term is linear in these quantities, hence following Leslie [4], we neglect these to get $q_z{}' = 0$. Solution of this is $q_z = r$ where r is the constant. Thus,

$$K_1(\theta)\zeta + K_2(\theta)(\xi\sin\phi - \eta\cos\phi) + K_3(\theta)(a\cos\phi + b\sin\phi) = r \tag{4.13}$$

where,

$$\left.\begin{aligned}
K_1(\theta) &= k_1 + k_2 \sin^2\theta \\
K_2(\theta) &= (k_3 - k_4)\sin\theta\cos\theta \\
K_3(\theta) &= k_2 \sin\theta\cos\theta
\end{aligned}\right\} \tag{4.14}$$

Solving the equations (4.8), (4.9) and (4.13) we obtain

$$\begin{aligned}
\xi = a[\sin\phi\cos\phi\{G_1(\theta) + G_2(\theta)\}] + b\{G_2(\theta)\sin^2\phi - G_1(\theta)\cos^2\phi\} + \\
k\{G_4(\theta) - G_5(\theta)\sin^2\phi\} - l\sin\phi\cos\phi G_5(\theta) - r\sin\phi G_6(\theta)
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\eta = a\{G_1(\theta)\sin^2\phi - G_2(\theta)\cos^2\phi\} - b\sin\phi\cos\phi\{G_1(\theta) + G_2(\theta)\} + \\
l\{G_7(\theta) - G_5(\theta)\sin^2\phi\} - k\sin\phi\cos\phi G_5(\theta) + r\cos\phi G_6(\theta)
\end{aligned} \tag{4.16}$$

$$\zeta = -\{a\cos\phi + b\sin\phi\}G_3(\theta) - \{k\sin\phi - l\cos\phi\}G_8(\theta) + rG_9(\theta) \tag{4.17}$$

Here,

$$\left.\begin{aligned}
G_1(\theta) &= \{H_5(\theta)-H_4(\theta)\}/\{H_1(\theta)+H_2(\theta)\}\\
G_2(\theta) &= \{H_4(\theta)K_1(\theta)+H_3(\theta)K_3(\theta)\}/\{H_1(\theta)K_1(\theta)-H_3(\theta)K_2(\theta)\}\\
G_3(\theta) &= \{H_1(\theta)K_3(\theta)+H_4(\theta)K_2(\theta)\}/\{H_1(\theta)K_1(\theta)-H_3(\theta)K_2(\theta)\}\\
G_4(\theta) &= 1/\left(H_1(\theta)+H_2(\theta)\right)\\
G_5(\theta) &= \{H_2(\theta)K_1(\theta)+H_3(\theta)K_2(\theta)\}/\{H_1(\theta)+H_2(\theta)\}\{H_1(\theta)K_1(\theta)-H_3(\theta)K_2(\theta)\}\\
G_6(\theta) &= H_3(\theta)/\{H_1(\theta)K_1(\theta)-H_3(\theta)K_2(\theta)\}\\
G_7(\theta) &= K_1(\theta)/\{H_1(\theta)K_1(\theta)-H_3(\theta)K_2(\theta)\}\\
G_8(\theta) &= K_3(\theta)/\{H_1(\theta)K_1(\theta)-H_3(\theta)K_2(\theta)\}\\
G_9(\theta) &= H_1(\theta)/\{H_1(\theta)K_1(\theta)-H_3(\theta)K_2(\theta)\}
\end{aligned}\right\rangle \tag{4.18}$$

Now, putting the values of $\pi_{zx}, \pi_{zy}, \pi_{zz}, g_x, g_y, g_z$ in (4.2), (4.3), (4.4) and then eliminating γ between them, we get

$$\begin{aligned}
&2F_1(\theta)\frac{d^2\theta}{dz^2}+\frac{d}{d\theta}F_1(\theta)\left(\frac{d\theta}{dz}\right)^2-\frac{d}{d\theta}F_2(\theta)\left(\frac{d\phi}{dz}\right)^2-4\alpha_2\tau\sin\theta\cos\theta\frac{d\theta}{dz}+\\
&2(\lambda_1+\lambda_2\cos 2\theta)(\xi\cos\theta+\eta\sin\phi)-2\lambda_3(a\sin\phi-b\cos\phi)=0
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&F_2(\theta)\frac{d^2\phi}{dz^2}+\frac{d}{d\theta}F_2(\theta)\frac{d\theta}{dz}\frac{d\phi}{dz}+2\alpha_2\tau\sin\theta\cos\theta\frac{d\theta}{dz}-\lambda_3\xi\cos^2\theta+\\
&(\lambda_1-\lambda_2)(\xi\sin\phi-\eta\cos\phi)\sin\theta\cos\theta+\lambda_3\sin\theta\cos\theta(a\cos\phi+b\sin\phi)=0
\end{aligned} \tag{4.20}$$

SOLUTION OF DIFFERENTIAL EQUATIONS FOR THE COUPLE STRESS BOUNDARY CONDITIONS AT THE PLATE

Following Sharma [1], the non-dimensional solutions of (4.15), (4.16), (4.17), (4.19), (4.20) in case of zero couple stress boundary conditions are obtained as

$$\begin{aligned}
u/U = {}& A[d - B\sin d\{\cos(d-2\tau_0 z_0)+C\sin(d-2\tau_0 z_0)\}]+\\
& E[d+\sin d\{\cos(d-2\tau_0 z_0)-F\sin(d-2\tau_0 z_0)\}
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
v/V = {}& A(U/V)[Cd + B\sin d\{\cos(d-2\tau_0 z_0)-\sin(d-2\tau_0 z_0)\}]+\\
& E(U/V)[Fd - \sin d\{\cos(d-2\tau_0 z_0)+\sin(d-2\tau_0 z_0)\}]
\end{aligned} \tag{5.2}$$

$$\phi = d-\tau_0 z_0 + Q.h.d\{(d/(\tau_0 h))-1\}\{(\tau_1/\tau_0)-1\} \quad ,Q=\lambda_3/(2\alpha_2 h) \tag{5.3}$$

$$\theta = D\begin{bmatrix}\{d/(\tau_0 h)\}\sin\{(\tau_0 h)/2\}\{\sin(\tau_0(h-2z_0)/2) - C\cos(\tau_0(h-2z_0)/2)\} - \\ \sin(d/2)\{\sin\{(d-2\tau_0 z_0)/2\} - C\cos\{(d-2\tau_0 z_0)/2\}\}\end{bmatrix} - G\begin{bmatrix}\{d/(\tau_0 h)\}\sin\{(\tau_0 h)/2\}\{\sin(\tau_0(h-2z_0)/2) + F\cos(\tau_0(h-2z_0)/2)\} - \\ \sin(d/2)\{\sin\{(d-2\tau_0 z_0)/2\} + F\cos\{(d-2\tau_0 z_0)/2\}\}\end{bmatrix} \quad (5.4)$$

where A, B, C, D, E, F, G are non-dimensional terms which remains constant throughout the investigation and are given by

$$A = (k\mu_a)/(\tau_0 U) = \big[\tau_0 h + B\sin(\tau_0 h)\{\cos\tau_0(h-2z_0) + (V/U)\sin\tau_0(h-2z_0)\} \{(b\mu_a)/(\tau_0 U)\}\sin(\tau_0 h)(\tau_0 h)(1+B)\{\cos\tau_0(h-2z_0) - F\sin\tau_0(h-2z_0)\} + \{(b\mu_a)/(\tau_0 U)\}\{B\sin^2(\tau_0 h) + (\tau_0 h)^2\}\big] \Big/ \{(\tau_0 h)^2 - B^2\sin^2(\tau_0 h)\},$$

$$B = \mu_c/\mu_b$$

$$C = l/k = \big[(V/U)(\tau_0 h) + B\sin(\tau_0 h)\{\sin\tau_0(h-2z_0) - (V/U)\cos\tau_0(h-2z_0)\} - (a/b)\{b\mu_a/\tau_0 U\}\{(\tau_0 h)^2 + B\sin^2(\tau_0 h)\} - (a/b)\{b\mu_a/\tau_0 V\}(\tau_0 h)\{(1+B)\sin(\tau_0 h)\cos\tau_0(h-2z_0)\} + \{b\mu_a/\tau_0 U\}(\tau_0 h)\{(1+B)\sin(\tau_0 h)\sin\tau_0(h-2z_0)\}\big] \Big/ \big[(\tau_0 h) + B\sin(\tau_0 h)\{\cos\tau_0(h-2z_0) + (V/U)\sin\tau_0(h-2z_0)\} - (a/b)\{b\mu_a/\tau_0 U\}(\tau_0 h)\{(1+B)\sin(\tau_0 h)\sin\tau_0(h-2z_0)\} + \{b\mu_a/\tau_0 U\}\{(\tau_0 h)^2 + B\sin^2(\tau_0 h)\} + \{b\mu_a/\tau_0 V\}(\tau_0 h)\{(1+B)\sin(\tau_0 h)\cos\tau_0(h-2z_0)\}\big]$$

$$D = kl = (2\mu AU)/(\alpha_1\tau_0)\big[\{(\mu_2+\mu_4)/\mu_3 + \mu_4 + \mu_6\} - B\big];\ E = (b\mu_a)/(\tau_0 U)$$

$$F = a/b; \qquad G = (2b\mu_a)/(\alpha_1\tau_0^{\,2})\big[(\lambda_3/\mu_a) - (\lambda_1+\lambda_2)\big]$$

DISCUSSION

For couple stress boundary conditions in all three directions, equation (5.1), (5.2) determine the velocity distribution, (5.3) and (5.4) the orientation of molecules between two plates. The effects of thermo-gradient parameter on the orientation of molecule varies parabolically between two plates, being zero on both the plates and maximum in the middle of stream while velocity components maximize or minimize according as the constant coefficient takes negative or positive value.

Note: This paper is a part of Master's thesis research work.

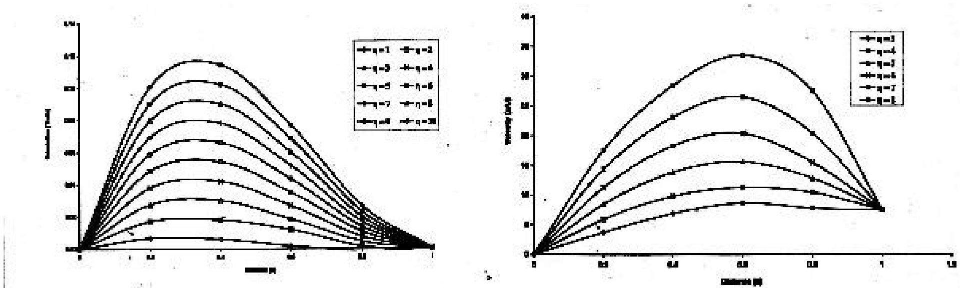

Fig. 1: Orientation profiles between the plates for various values of [illegible] parameter

Fig. 2: Velocity component [illegible] profiles

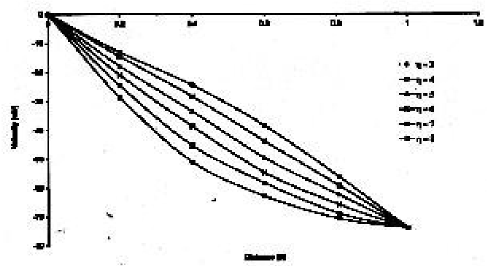

Fig. 3: Velocity component [illegible] profiles

REFERENCES

A. K. Sharma (1977), *Indian J. of Theo. Phy.* 2, 65-72.

C. W. Oseen (1933), Trans. Faraday Soc. 29, 883-889.

F. C. Frank (1958), Discussions Fartaday Soc. 25, 19-28.

F.M. Leslie, Proc. Roy. Soc. A. 307 (1968) 359-372.

J. L. Ericksen, Arch. Rational Mech. Anal. 9 (1962) 371-378.