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DECOMPOSITION THEOREM IN FUZZY CONE NORMED LINEAR SPACE

Abstract: *In this paper, idea of generating space of quasi cone norm family is introduced. Decomposition theorem of a fuzzy cone norm into a family of cone norms is established and study some properties related to this theorem.*

keywords: *Fuzzy cone norm, strongly minihedral cone, normal cone, regular cone.*

1. INTRODUCTION

L. A. Zadeh [16] introduced the concept of fuzzy sets in 1965. After that, several authors develop many results of analysis, topology, algebra etc. in fuzzy setting. The concept of fuzzy norm was first introduced by A.K.Katsaras [9] in 1984. Based on the concept of fuzzy metric space given by Kaleva and Seikkala[8], in 1992, Felbin [7] introduced the notion of fuzzy normed linear space and discussed the completeness of a finite dimensional fuzzy normed linear space. After that, several authors try to develop these concept of fuzzy norm in different ways (please see [2], [5], [15]). With these notions of fuzzy norm, fuzzy analogous of several classical concepts of normed linear spaces have been established.

On the other hand, Huang and Zhang [11] introduced the notion of cone metric space as a generalization of metric space. Following the concept of cone, many authors have generalized the concept of fuzzy metric and fuzzy norm in cone setting (please see [3], [4], [13], [14]). The aim of the present paper is to established the decomposition theorem for fuzzy cone normed linear space and to study some important results in connection with the decomposition of fuzzy cone norm into a family of crisp cone norm.

The organisation of the paper is as follows:

In section 2, some preliminaries and essential concepts for the study are stated.

In section 3, generating space of quasi cone norm family is introduced and decomposition theorem for fuzzy cone normed linear space and some important results are established.

2. PRELIMINARIES

Definition 2.1. [11] Let E be a real Banach space and P be a subset of $E.P$ is called a cone if and only if:

(i) P is closed, non-empty and $P \neq \{\theta_E\}$;

(ii) $a, b \in R$; $a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta_E$.

Given a cone $P \subset E$, we define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ iff $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in IntP$, where $IntP$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, with $\theta_E \preccurlyeq x \preccurlyeq y$ implies $\| x \| \leq K \| y \|$.

The least positive number satisfying above is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that

$$x_1 \preccurlyeq x_2 \preccurlyeq \ldots \preccurlyeq x_n \preccurlyeq \cdots \preccurlyeq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if every decreasing sequence is bounded below is convergent. It is clear that a regular cone is a normal cone.

Definition 2.2. [6] The cone P is called strongly minihedral if every subset of E which is bounded above via the partial ordering obtained by P, must have a least upper bound. Hence, every subset which is bounded below must have greatest lower bound.

Lemma 2.3. [1] Every strongly minihedral normal cone is regular.

Definition 2.4. [10] A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is a t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative;

(2) $a * 1 = a \; \forall a \in [0, 1]$;

(3) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

If $*$ is continuous then it is called continuous t-norm. The following are examples of some t-norms that are frequently used and defined for all $a, b \in [0, 1]$.

(i) Standard intersection: $a * b = min(a, b)$.

(ii) Algebraic product: $a * b = ab$.

(iii) Bounded difference: $a * b = \max(0, a + b - 1)$.

(iv) Drastic intersection:

$$a * b = \begin{cases} a \; for \; b = 1 \\ b \; for \; a = 1 \\ 0 \; otherwise. \end{cases}$$

Definition 2.5. [12] Let V be a vector space over the field $\mathbb{R}$. The mapping $\| \; \|_c : V \to E$ is said to be a cone norm if it satisfies the following conditions:

(i) $\| x \|_c \geq \theta_E \; \forall x \in V$;

(ii) $\| x \|_c = \theta_E$ iff $x = \theta_V$;

(iii) $\| x \|_c = \alpha \| x \|_c \; \forall x \in V, \alpha \in \mathbb{R}$;

(iv) $\| x + y \|_c \| x \|_c + \| y \|_c \; \forall \, x, y \in V$.

Then $\| \, \|_c$ is called a cone norm on V and $(V, \| \, \|_c)$ is called a cone normed linear space.

Definition 2.6. [14] Let X be a linear space over the field K and E be a real Banach space with cone P, $*$ is a t-norm. A fuzzy subset $N_c : X \times E \to [0,1]$ is said to be a fuzzy cone norm if

(FCN1) $\forall t \in E$ with $t \preccurlyeq \theta_E$, $N_c (x, t) = 0$;
(FCN2) $(\forall \, \theta_E \prec t, N_c (x, t) = 1)$ iff $x = \theta_X$;

(FCN3) $\forall \, \theta_E \prec t$, and $0 \neq c \in K$, $N_c (cx, t) = N_c(cx, t) = N_c\left(x, \dfrac{t}{|c|}\right)$;

(FCN4) $\forall x, y \in X$ and $s, t \in E$, $N_c (x + y, s + t) \geq N_c (x, s) * N_c (y, t)$;

(FCN5) $\lim_{\|t\|\to\infty} N_c(x,t) = 1$.

Then $(X, N_c, *)$ is said to be a fuzzy cone normed linear space w.r.t E.

3. DECOMPOSITION THEOREM IN FUZZY CONE NORMED LINEAR SPACE

In this section we introduce generating space of quasi cone norm family and establish a decomposition theorem for fuzzy cone norm into crisp cone norm family by taking t-norm * as Λ (*min*) and studied some results.

Definition 3.1. Let X be a real linear space and E be a real Banach space. Let $Q = \{|\,|_{c,\alpha} : \alpha \in (0,1)\}$ be a family of mappings from V into E. Then (X, Q) is called a generating space of quasi cone norm family if following conditions are satisfied,

(QN_c 1) $|x|_{c,\alpha}) = \theta_E \; \forall \alpha \in (0,1)$ iff $x = \theta_V$;

(QN_c 2) $|kx|_{c,\alpha} = |k|\,|x|_{c,\alpha} \; \forall x \in X, \forall \alpha \in (0, 1)$ and $k \in \mathbb{R}$;

(QN_c 3) for any $\alpha \in (0, 1)$, $\exists\, \beta \in (0, \alpha]$ such that

$$|x + y|_{c,\alpha} \preccurlyeq |x|_{c,\beta} + |y|_{c,\beta} \; \forall x, y \in X;$$

(QN_c 4) for any $x \in X$, $|x|_{c,\alpha}$ is non-increasing w.r.t. $\alpha \in (0,1)$.

Theorem 3.2. Let $(X, N_c, *)$ be a fuzzy cone normed linear space with strongly minihedral cone P and assume that $*$ is lower semi continuous. For $\alpha \in (0,1)$, define $|x|_{c,\alpha} = \wedge\{t \succ \theta_E : N_c(x, t) \geq 1 - \alpha\}$ and $Q_c = \{\,|\,|_{c,\alpha} : \alpha \in (0, 1)\}$. Then (X, Q_c) is a generating space of quasi cone norm family.

Proof: (i) $|x|_{c,\alpha} = \Lambda\, \{t \succ \theta_E : N_c(x, t) \geq 1 - \alpha\}$

If $x = \theta_X$. Then $N_c(x, t) = 1 \; \forall\, t \succ \theta_E$ by (FCN2)

$\Rightarrow N_c(x, t) \geq 1 - \alpha \; \forall\, t \succ \theta_E$ and $\forall\, \alpha \in (0,1)$.

$\Rightarrow |x|_{c,\alpha} = \theta_E \;\; \forall \alpha \in (0, 1)$.

Conversely, suppose $|x|_{c,\alpha} = \theta_E \; \forall\, \alpha \in (0,1)$.

$\Rightarrow \wedge\, \{t \succ \theta_E : N_c(x, t) \geq 1 - \alpha\} = \theta_E \; \forall \alpha \in (0,1)$.

$\Rightarrow$ for any $\epsilon \succ \theta_E$, $\Lambda\{t \succ \theta_E : N_c(x, t) \geq 1 - \alpha\} \prec \epsilon \; \forall \alpha \in (0,1)$.

$\Rightarrow N_c(x, \epsilon) \geq 1 - \alpha \; \forall \alpha \in (0,1)$.

$\Rightarrow N_c\,(x, \epsilon) = 1\ \forall\ \epsilon \succ \theta_E.$

$\Rightarrow x = \theta_X.$

(ii) Let $x \in X$ and $k \in \mathbb{R}$, $k \neq 0$;

Then $|kx|_{c,\alpha} = \wedge\{t \succ \theta_E : N_c\,(kx, t) \geq 1 - \alpha\}$

$$= \wedge\{|\,k\,|\cdot\frac{t}{|\,k\,|} \succ \theta_E : N_c\left(x, \frac{t}{|\,k\,|}\right) \geq 1 - \alpha\}$$

$= |k| \wedge \{s \succ \theta_E : N_c\,(x, s) \geq 1 - \alpha\} = |k|\,|x|_{c,\alpha}\ \ \forall \alpha \in (0,1).$

(iii) Since $*$ is lower semi continuous, for any $\alpha \in (0,1)$, $\exists\ \beta \in (0, \alpha]$ such that $(1 - \beta) * (1 - \beta) \geq 1 - \alpha$.

$$\begin{aligned}
\text{Now, } |x|_{c,\beta} + |y|_{c,\beta} &= \wedge\{s \succ \theta_E : N_c\,(x, s) \geq 1 - \beta\} + \Lambda\{t \succ \theta_E\, N_c\,(y, t) \geq 1 - \beta\} \\
&\succcurlyeq \wedge\{s + t \succ \theta_E : N_c\,(x, s) \geq 1 - \beta, N_c\,(y, t) \geq 1 - \beta\} \\
&\succcurlyeq \wedge\{s + t \succ \theta_E : N_c\,(x + y, s + t) \geq (1 - \beta) * (1 - \beta)\} \\
&\succcurlyeq \wedge\{s + t \succ \theta_E : N_c\,(x + y, s + t) \geq 1 - \alpha\} \\
&= |x + y|_{c,\alpha}
\end{aligned}$$

Thus for each $\alpha \in (0,1)$, $\exists\ \beta \in (0, \alpha]$ such that

$$|x + y|_{c,\alpha} \preccurlyeq |x|_{c,\beta} + |y|_{c,\beta}\ \forall x, y \in X.$$

(iv) Let $\alpha_1, \alpha_2 \in (0,1)$ and $\alpha_1 > \alpha_2$. So, $1 - \alpha_1 < 1 - \alpha_2$.

Now for $x \in X$ we have,

$\{s \succ \theta_E : N_c\,(x, s) \geq 1 - \alpha_2\} \subset \{s \succ \theta_E : N_c\,(x, s) \geq 1 - \alpha_1\}$

$\Rightarrow \wedge\ \{s \succ \theta_E : N_c\,(x, s) \geq 1 - \alpha_2\} \succcurlyeq \wedge\{s \succ \theta_E : N_c\,(x, s) \geq 1 - \alpha_1\}$

$\Rightarrow |x|_{c,\alpha 2} \succcurlyeq |x|_{c,\alpha 1}$

Thus $Q_c = \{|\,|_{c,\alpha} : \alpha \in (0,1)\}$ is non-increasing.

Hence (X, Q_c) is a generating space of quasi cone norm family.

Theorem 3.3. Let $(X, N_c, *)$ be a FCNLS with strongly minihedral cone P where $* = \min$.

Assume that (FCN6) $N_c\,(x, t) > 0\ \forall\ t \succ \theta_E \Rightarrow x = \theta_X$.

Define $\| x \|_{c,\alpha} = \wedge\{t \succ \theta_E : N_c(x, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

Then $\{\| \|_{c,\alpha}, \alpha \in (0,1)\}$ is an ascending family of cone norms on X. We call these α-cone norms corresponding to the fuzzy cone norm N_c.

Proof: (i) $\forall\, x \in X,\ N_c(x,t) = 0$ for $t \preccurlyeq \theta_E$

$\Rightarrow \wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha\} \succcurlyeq \theta_E$
$\Rightarrow \| x \|_{c,\alpha} \succcurlyeq \theta_E\ \forall\, \alpha \in (0,1)$.

(ii) $\| x \|_{c,\alpha} = \theta_E$

$\Rightarrow \wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha\} = \theta_E$

$\Rightarrow \forall\, t \succ \theta_E,\ N_c(x,t) \geq \alpha > 0$

$\Rightarrow x = \theta_X$ by (FCN6)

Conversely, $x = \theta_X$

$\Rightarrow N_c(x,t) = 1,\ \forall\, t \succ \theta_E$

$\Rightarrow \forall\, \alpha \in (0,1),\ \wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha\} = \theta_E$

$\Rightarrow \| x \|_{c,\alpha} = \theta_E\ \ \forall\, \alpha \in (0,1)$

(iii) If $k \neq 0$.

Then $\| kx \|_{c,\alpha} = \wedge\{ s \succ \theta_E : N_c(kx,s) \geq \alpha\}$

$= \wedge\{s \succ \theta_E : N_c(x, \frac{s}{|k|}) \geq \alpha\}$

$= \wedge\{ |k|t \succ \theta_E : N_c(x,t) \geq \alpha\}$

$= |k| \wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha\}$

$= |k|\ \| x \|_{c,\alpha}\ \ \forall\, \alpha \in (0,1)$.

If $k = 0$. Then $\| kx \|_{c,\alpha} = \| \theta_X \|_{c,\alpha} = \theta_E = 0\ \| x \|_{c,\alpha} = |k|\ \| x \|_{c,\alpha}\ \ \forall\, \alpha \in (0,1)$.

(iv) $\| x \|_{c,\alpha} + \| y \|_{c,\alpha}$

$= \wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha\} + \wedge\{ s \succ \theta_E : N_c(y,s) \geq \alpha\}$

$= \wedge\{ t + s \succ \theta_E : N_c(x,t) \geq \alpha,\ N_c(y,s) \geq \alpha\}$

$\succcurlyeq \wedge\{ r \succ \theta_E : N_c(x+y,r) \geq \min\{ N_c(x,t), N_c(y,s)\} \geq \alpha\}$

$= \| x+y \|_{c,\alpha}$

$\Rightarrow \| x+y \|_{c,\alpha} \preccurlyeq \| x \|_{c,\alpha} + \| y \|_{c,\alpha}$

(v) Now taking $0 < \alpha_1 < \alpha_2$

$\| x \|_{c,\alpha_1} = \wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha_1\}$

$\| x \|_{c,\alpha_2} = \wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha_2\}$

Since $0 < \alpha_1 < \alpha_2$

$\{ t \succ \theta_E : N_c(x,t) \geq \alpha_2\} \subset \{ t \succ \theta_E : N_c(x,t) \geq \alpha_1\}$

$\wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha_2\} \succcurlyeq \wedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha_1\}$

$\Rightarrow \| x \|_{c,\alpha_1} \preccurlyeq \| x \|_{c,\alpha_2}$

Thus $\{\| \|_{c,\alpha},\ \alpha \in (0,1)\}$ is an ascending family of cone norms on *X*.

Theorem 3.4. Let $\{\| \|_{c,\alpha},\ \alpha \in (0,1)\}$ be an ascending family of cone norms on *X*, and *P* be strongly minihedral normal cone with normal constant *K*. Now define a function,

$N_c' : X \times E \longrightarrow [0,1]$ as

$$N_c'(x,t) = \vee\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t\} \text{ when } (x,t) \neq (\theta_X, \theta_E)$$
$$= 0 \qquad \text{when } (x,t) = (\theta_X, \theta_E)$$

Then N_c' is a fuzzy cone norm on *X* and $(X, N_c', *)$ is a fuzzy cone normed linear space where * = min.

Proof:

(FCN1) $\forall\, t \in E$ with $t \prec \theta_E$ we have

$N_c'(x,t) = \vee\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t\} = 0\ \forall\, x \in X.$

(since $\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t\} = \emptyset$ when $t \prec \theta_E$)

For $t = \theta_E$ and $\neq \theta_X$. $\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t\} = \emptyset$

$\Rightarrow N_c'(x,t) = 0.$

When $t = \theta_E$ and $= \theta_X$. $N_c'(x, t) = 0$. (by definition)

(FCN2) Let $N_c'(x, t) = 1 \;\forall\, t \succ \theta_E$

Choose any $\epsilon \in (0,1)$.

Then for any $t \succ \theta_E$, there exists $\alpha_t \in (\epsilon, 1)$ such that $\| x \|_{c,\alpha_t} \preccurlyeq t$ and hence $\| x \|_{c,\epsilon} \preccurlyeq t$.

Since $t \succ \theta_E$ is arbitrary, $\| x \|_{c,\epsilon} = \theta_E \Rightarrow x = \theta_X$.

If $x = \theta_X$. Then $\forall\, t \succ \theta_E$

$N_c'(x,t) = \bigvee\{\alpha \in (0,1) : \| \theta_X \|_{c,\alpha} \preccurlyeq t\} = \bigvee\{\alpha \in (0,1) : \alpha \in (0,1)\} = 1.$

(FCN3) $\forall\, t \succ \theta_E$

$$N_c'(kx,t) = \bigvee\{\alpha \in (0,1) : \| kx \|_{c,\alpha} \preccurlyeq t\}$$
$$= \bigvee\{\alpha \in (0,1) : |k| \, \| x \|_{c,\alpha} \preccurlyeq t\}$$
$$= \bigvee\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq \frac{t}{|k|}\}$$
$$= N_c'\left(x, \frac{t}{|k|}\right) \quad \forall x \in X.$$

(FCN4) We can consider four cases.

(i) $s \preccurlyeq \theta_E\,,\; t \prec \theta_E$

(ii) $s \preccurlyeq \theta_E\,,\; \theta_E \prec t$

(iii) $s \preccurlyeq \theta_E\,, t = \theta_E$

In the above cases the (FCN4) holds. Now we consider the case

(iv) $\theta_E \prec s,\; \theta_E \prec t$

Let $p = N_c'(x,s), q = N_c'(y,t)$ and $p \leq q$.

If $p = 0$, $q = 0$ then obviously (FCN4) holds.

Let $0 < r < p \leq q$.

Then there exists $\alpha > r$ such that $\| x \|_{c,\alpha} \preccurlyeq s$ and there exists $\beta > r$ such that $\| y \|_{c,\beta} \preccurlyeq t$.

Let $\gamma = \alpha \wedge \beta > r$

Therefore, $\| x \|_{c,\gamma} \preccurlyeq \| x \|_{c,\alpha} \preccurlyeq s$ and $\| y \|_{c,\gamma} \preccurlyeq \| y \|_{c,\beta} \preccurlyeq t$.

Now, $\| x+y \|_{c,\gamma} \preccurlyeq \| x \|_{c,\gamma} + \| y \|_{c,\gamma} \preccurlyeq s+t$

$\Rightarrow N_c'(x+y, s+t) \geq \gamma > r$.

Since $0 < r < \gamma$ is arbitrary, thus

$N_c'(x+y, s+t) \geq p = \min\{N_c'(x,s), N_c'(y,t)\}$

Similarly if $p \geq q$, then the relation holds.

(FCN5) Let $x \in X$, $\alpha \in (0,1)$.

$$N_c'(x,t) = \vee\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t\}$$
$$\leq \vee\{\alpha \in (0,1) : \| \| x \|_{c,\alpha} \| \leq K \| t \|\}$$

(P is a normal cone with normal constant K)

($\| \|$ is a norm on E)

Then as $\| t \| \rightarrow \infty$,

$N_c'(x,t) = \vee\{\alpha \in (0,1) : \alpha \in (0,1)\} = 1$ (since it holds for all $\alpha \in (0,1)$)

Therefore, $\lim_{\|t\| \rightarrow \infty} N_c'(x,t) = 1$.

Note 3.5. $N_c'(x,t)$ (x, t) is non-decreasing function of E.

(i) If $t_1 \prec t_2 \preccurlyeq \theta_E$, $N_c'(x,t_1) = N_c'(x,t_2) = 0 \ \forall x \in X$.

(ii) $\theta_E \prec t_1 \prec t_2$, then

$\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t_1\} \subset \{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t_2\}$

$\Rightarrow \vee\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t_1\} \leq \vee\{\alpha \in (0,1) : \| x \|_{c,\alpha} \preccurlyeq t_2\}$

$\Rightarrow N_c'(x,t_1) \leq N_c'(x,t_2)$.

Proposition 3.6. Between any two elements u, v of E satisfying $u \prec v$ there exists $w \in E$ such that $u \prec w \prec v$.

Proof: Suppose $u, v \in E$ such that $u \prec v$.

Let $w = \frac{u+v}{2}$. Then $v - w = v - \frac{u+v}{2} = \frac{v-u}{2} \in P$. (since $v - u \in P$ and $\frac{1}{2} > 0$)

Therefore $v - w \in P$.
i.e, $w \prec v$.
In a similar way it can be proved $u \prec w$.
Therefore $u \prec w \prec v$.

Definition 3.7. $N_c(x,.)$ is said to be upper semi-continuous at $c \in E$ if for any $\epsilon > 0, \exists\, \delta \succ \theta_E$ such that for all t in E, with $c - \delta \prec t \prec c + \delta$
$N_c(x,t) \le N_c(x,c) + \epsilon$.

Definition 3.8. Let X be a linear space and N_c be a fuzzy cone norm on X.
We define
$\lim_{s \downarrow t} \mathrm{N_c(x,s)} = \mathrm{l}$ if for every $\epsilon > 0$, $\exists\, c \succ \theta_E$ such that $|N_c(x,s) - l| < \epsilon$ whenever $t \prec s \prec t + c$.
and $\lim_{s \uparrow t} \mathrm{N_c(x,s)} = \mathrm{q}$ if for every $\epsilon > 0$, $\exists\, d \succ \theta_E$ such that $|N_c(x,s) - q| < \epsilon$ whenever $t - d \prec s \prec t$.
We denote $\lim_{s \downarrow t} \mathrm{N_c(x,s)}$ by $\mathrm{N_c(x,t+)}$ or $\mathrm{N_{c_+}(x,t)}$ and $\lim_{s \uparrow t} \mathrm{N_c(x,s)}$ by $\mathrm{N_c(x,t-)}$ or $\mathrm{N_{c\_}(x,t)}$.

Definition 3.9. Let X be a linear space and N_{c_1}, N_{c_2} be two fuzzy cone norms on X. Then $N_{c_1}$and N_{c_2} are said to be equipotent if $N_{c_1}(x,t+) = N_{c_2}(x,t+)$ and $N_{c_1}(x,t-) = N_{c_2}(x,t-)\ \forall\, x \in X,\ \forall\, t \in E$.

Theorem 3.10. Let X be a linear space and N_{c_1}, N_{c_2}be two fuzzy cone norm on X with a strongly minihedral cone P satisfying (FCN6) and any two elements of P are comparable..
Then $\forall\, x \in X,\ \forall\, t \in E\ \ N_{c_1}(x,t+) = N_{c_2}(x,t+)$ and $N_{c_1}(x,t-) = N_{c_2}(x,t-)$ iff $\| x \|_{c_1,\alpha} = \| x \|_{c_2,\alpha}\ \ \forall\, \alpha \in (0,1)$, where $\| \|_{c_1,\alpha}$ and $\| \|_{c_2,\alpha}$ denote the corresponding α-cone norms of N_{c_1} and N_{c_2} respectively.

Proof: First suppose $\| x \|_{c_1,\alpha} = \| x \|_{c_2,\alpha}\ \ \forall\, \alpha \in (0,1)$.
If possible for some $t_0 \in E$, $\ N_{c_1}(x,t_0+) \neq N_{c_2}(x,t_0+)$
Without loss of generality, we assume $N_{c_1}(x,t_0+) < N_{c_2}(x,t_0+)$.
Then for $t_0 \prec t \prec t_0 + \epsilon\ (\epsilon \succ \theta_E)$, $\ N_{c_1}(x,t) < N_{c_2}(x,t)$.
Choose β such that $N_{c_1}(x,t) < \beta < N_{c_2}(x,t)$. (3.10.1)
Note that,
$\| x \|_{c_1,\alpha} = \bigwedge\{ t \succ \theta_E : N_{c_1}(x,t) \ge \alpha \},\ \ \alpha \in (0,1)$. (3.10.2)

$\| x \|_{c_2,\alpha} = \bigwedge\{ t \succ \theta_E : N_{c_2}(x,t) \ge \alpha \},\ \ \alpha \in (0,1)$. (3.10.3)

Using (3.10.1), (3.10.2) and (3.10.3) We have,
$\parallel x \parallel_{c_2,\beta} \preccurlyeq t_0$, $\parallel x \parallel_{c_1,\beta} \succcurlyeq t_0 + \epsilon \succ t_0$ which is a contradiction to the hypothesis.
Therefore, $N_{c_1}(x, t+) = N_{c_2}(x, t+)\ \ \forall\, t \in E$.
Similarly, $N_{c_1}(x, t-) = N_{c_2}(x, t-)\ \ \forall\, t \in E$.

Conversely suppose that $N_{c_1}(x, t+) = N_{c_2}(x, t+)$, $N_{c_1}(x, t-) = N_{c_2}(x, t-)$ hold $\forall\, t \in E$.
We have to show that $\parallel x \parallel_{c_1,\alpha} = \parallel x \parallel_{c_2,\alpha}\ \ \forall\, \alpha \in (0,1)$.
If possible $\exists\, \alpha_0 \in (0,1)$ such that $\parallel x \parallel_{c_1,\alpha_0} \neq \parallel x \parallel_{c_2,\alpha_0}$.
Without loss of generality, we assume $\parallel x \parallel_{c_2,\alpha_0} \prec \parallel x \parallel_{c_1,\alpha_0}$ (since every element of P are comparable.)
Choose t_1, t_2, t_3 such that $\parallel x \parallel_{c_2,\alpha_0} \prec t_3 \prec t_2 \prec t_1 \prec \parallel x \parallel_{c_1,\alpha_0}$ (3.10.4)
Using (3.10.2) and (3.10.3), we have
$N_{c_1}(x, t_1) < \alpha_0$, $N_{c_2}(x, t_3) \geq \alpha_0$ (3.10.5)
Now from (3.10.4) and (3.10.5),
$N_{c_1}(x, t_2 +) \leq N_{c_1}(x, t_1) < \alpha_0$, $N_{c_2}(x, t_2 -) \geq N_{c_2}(x, t_3) \geq \alpha_0$.
Combining the above two results, we have
$N_{c_1}(x, t_2 +) < \alpha_0 \leq N_{c_2}(x, t_2 -) \leq N_{c_2}(x, t_2 +)$.
$\Rightarrow N_{c_1}(x, t_2 +) < N_{c_2}(x, t_2 +)$ a contradiction to the assumption.
Thus $\parallel x \parallel_{c_1,\alpha} = \parallel x \parallel_{c_2,\alpha}\ \ \forall\, \alpha \in (0,1)$.

Theorem 3.11. Let $(X, N_c, *)$ be a fuzzy cone normed linear space where $* = min$ with a strongly minihedral normal cone P with normal constant K satisfying (FCN6) and $\parallel\ \parallel_{c,\alpha}$ denotes the α-cone norm of N_c, $0 < \alpha < 1$.
Let

$$N'_c(x, t) = \vee\{\alpha \in (0,1) : \parallel x \parallel_{c,\alpha} \preccurlyeq t\} \text{ when } (x, t) \neq (\theta_X, \theta_E)$$
$$= 0 \qquad \text{ when } (x, t) = (\theta_X, \theta_E)$$

Then N'_c is a fuzzy cone norm on X and N_c and N'_c are equipotent.

Proof: Similar to Theorem 3.4, N'_c is a fuzzy cone norm on X.
We have, $\parallel x \parallel_{c,\alpha} = \wedge\{\, t \succ \theta_E : N_c(x, t) \geq \alpha\}$, $\alpha \in (0,1)$. (3.11.1)
Now we have to show that,
$N'_c(x, t -) = N_c(x, t -)$ and $N'_c(x, t +) = N_c(x, t +)\ \forall\, x \in X,\ \forall\, t \in E$.
If possible for some $t_0 \in E$, $N'_c(x, t_0 +) \neq N_c(x, t_0 +)$.
Without loss of generality, we assume $N_c(x, t_0 +) < N'_c(x, t_0 +)$.
Then for $t_0 \prec t \prec t_0 + \epsilon\ (\epsilon \succ \theta_E)$, $N_c(x, t) < N'_c(x, t)$.
Choose β such that $N_c(x, t) < \beta < N'_c(x, t)$.

Now for $t_0 \prec t \prec t_0 + \epsilon$ $(\epsilon \succ \theta_E)$, $N_c(x,t) < \beta \Rightarrow \| x \|_{c,\beta} \succ t_0$ (using (3.11.1)) and for $t_0 \prec t \prec t_0 + \epsilon$ $(\epsilon \succ \theta_E)$, $N'_c(x,t) > \beta \Rightarrow \| x \|_{c,\beta} \preccurlyeq t_0$ (using definition of N'_c).
Thus we arrive at a contradiction.
Therefore $N'_c(x, t+) = N_c(x, t+)\ \forall\, x \in X,\ \forall\, t \in E$.
Similarly, we can verify that $N'_c(x, t-) = N_c(x, t-)\ \forall\, x \in X,\ \forall\, t \in E$.
Hence N_c and N'_c are equipotent.

Lemma 3.12. Let $(X, N_c, *)$ be a fuzzy cone normed linear space where $* = min$ with a strongly minihedral normal cone P with normal constant K satisfying (FCN6), $x_0 (\neq \theta_X)$ and $\| \ \|_{c,\alpha}$ denotes the α-cone norm of N_c, $0 < \alpha < 1$.
Then
(1) If $N_c(x_0, .)$ is upper semicontinuous and if for $\theta_E \prec t_0$, $N_c(x_0, t_0) = \alpha_0 \in (0,1)$ then $N_c(x_0, \| x_0 \|_{c,\alpha_0}) = \alpha_0$.
(2) If $N_c(x_0, .)$ is continuous, then for any $\alpha \in (0,1)$, $N_c(x_0, \| x_0 \|_{c,\alpha}) = \alpha$.
(3) If $N_c(x_0, .)$ is continuous and strictly increasing for $\theta_E \prec t$, then $N_c(x_0, t) = \alpha \Leftrightarrow \| x_0 \|_{c,\alpha} = t$.

Proof:
(1) $\| x_0 \|_{c,\alpha_0} = \wedge\{ t \succ \theta_E : N_c(x_0, t) \geq \alpha_0\}$, (3.12.1)
Since $N_c(x_0, t_0) = \alpha_0$, we get from (i), $\| x_0 \|_{c,\alpha_0} \preccurlyeq t_o$ (3.12.2)
Since $N_c(x_0, .)$ is non-decreasing, we have from (3.12.2),
$\alpha_0 = N_c(x_0, t_0) \geq N_c(x_0, \| x_0 \|_{c,\alpha_0})$
i.e, $N_c(x_0, \| x_0 \|_{c,\alpha_0}) \leq \alpha_0$. (3.12.3)
If possible, $N_c(x_0, \| x_0 \|_{c,\alpha_0}) < \alpha_0$.
Then by the upper semi continuity of $N_c(x_0, .)$, $\exists\, t' \succ \| x_0 \|_{c,\alpha_0}$ such that $N_c(x_0, t') < \alpha_0$.
Then $\| x_0 \|_{c,\alpha_0} = \wedge\{ t \succ \theta_E : N_c(x_0, t) \geq \alpha_0\} \succcurlyeq t' \succ \| x_0 \|_{c,\alpha_0}$. a contradiction.
Thus $N_c(x_0, \| x_0 \|_{c,\alpha_0}) = \alpha_0$.
(2) Since $N_c(x_0, .)$ is continuous and non-decreasing i.e, $\theta_E \prec t_1 \prec t_2 \Rightarrow N_c(x, t_1) \leq N_c(x, t_2)$, for each $\alpha \in (0,1)$, $\exists\, t \succ \theta_E$ such that $N_c(x_0, t) = \alpha$.
Then by (1) , the proof follows.
(3) Let $N_c(x_0, t) = \alpha$, then by (1) $N_c(x_0, \| x_0 \|_{c,\alpha}) = \alpha$.
Since $N_c(x_0, t) = \alpha$, $\| x_0 \|_{c,\alpha} \preccurlyeq t$.
If $\| x_0 \|_{c,\alpha} \prec t$, then $N_c(x_0, \| x_0 \|_{c,\alpha}) = \alpha < N_c(x_0, t) = \alpha$.
(since $N_c(x_0, .)$ is strictly increasing)
which is a contradiction.
Thus $\| x_0 \|_{c,\alpha} = t$.
Conversely, Let $\| x_0 \|_{c,\alpha} = t$.

Since $N_c(x_0,.)$ is continuous, by (2) for any $\alpha \in (0,1)$, $N_c\big(x_0, \parallel x_0 \parallel_{c,\alpha}\big) = \alpha$.
$\Rightarrow N_c(x_0,t) = \alpha$.

(FCN7). for $x \neq \theta_X$, $N_c(x,.)$ is a continuous function of E and strictly increasing on subset $\{t \in E : 0 < N_c(x,t) < 1\}$.

Theorem 3.13. Let $(X, N_c, *)$ be a fuzzy cone normed linear space where $* = min$ with a strongly minihedral normal cone P with normal constant K satisfying (FCN6) and (FCN7) and any two elements of P are comparable.
Define $\parallel x \parallel_{c,\alpha} = \bigwedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha\}$, $\alpha \in (0,1)$ and $N_c' : X \times E \longrightarrow [0,1]$ as

$$N_c'(x,t) = \bigvee\{\alpha \in (0,1) : \parallel x \parallel_{c,\alpha} \preccurlyeq t\} \text{ when } (x,t) \neq (\theta_X, \theta_E)$$
$$= 0 \qquad \text{when } (x,t) = (\theta_X, \theta_E).$$

Then
(1) $\{ \parallel \parallel_{c,\alpha},\ \alpha \in (0,1)\}$ is an ascending family of α -cone norms on X.
(2) N_c' is a fuzzy cone norm on X.
(3) $N_c' = N_c$.

Proof: Results (1) and (2) are followed from Theorem 3.3 and Theorem 3.4.
(3) Let $(x_0, t_0) \in X \times E$ and $N_c(x_0, t_0) = \alpha_0$.
Case I. $x_0 = \theta_X,\ t_0 \preccurlyeq \theta_E$.
$N_c(x_0,t_0) = N_c'(x_0,t_0) = 0$

Case II. $x_0 = \theta_X,\ t_0 \succ \theta_E$.
$N_c(x_0,t_0) = N_c'(x_0,t_0) = 1$.

Case III. $x_0 \neq \theta_X,\ t_0 \preccurlyeq \theta_E$.
$N_c(x_0,t_0) = N_c'(x_0,t_0) = 0$.

Case IV. $x_0 \neq \theta_X,\ t_0 \succ \theta_E$ such that $N_c(x_0,t_0) = 0$.
We have $\parallel x_0 \parallel_{c,\alpha} = \bigwedge\{ t \succ \theta_E : N_c(x_0,t) \geq \alpha\}$,
By Lemma 3.12(2) , We have $N_c\big(x_0, \parallel x_0 \parallel_{c,\alpha}\big) = \alpha\ \ \forall \alpha \in (0,1)$.
Since $N_c(x_0,t_0) = 0 < \alpha$, it follows that $t_0 \prec \parallel x_0 \parallel_{c,\alpha}, \forall \alpha \in (0,1)$
So, $N_c'(x_0,t_0) = \bigvee\{\alpha \in (0,1) : \parallel x_0 \parallel_{c,\alpha} \preccurlyeq t_0\}$
$= 0$
Thus, $N_c(x_0,t_0) = N_c'(x_0,t_0)$.

Case V. $x_0 \neq \theta_X,\ t_0 \succ \theta_E$ such that $0 < N_c(x_0,t_0) < 1$.
Let $N_c(x_0,t_0) = \alpha_0$. Then $0 < \alpha_0 < 1$.
Now $N_c'(x,t) = \bigvee\{\alpha \in (0,1) : \parallel x \parallel_{c,\alpha} \preccurlyeq t\}$ when $(x,t) \neq (\theta_X, \theta_E)$ (3.13.1)
$\parallel x \parallel_{c,\alpha} = \bigwedge\{ t \succ \theta_E : N_c(x,t) \geq \alpha\}$, $\alpha \in (0,1)$ (3.13.2)

Since $N_c(x_0, t_0) = \alpha_0$.
We have from (3.13.2)
$\parallel x_0 \parallel_{c,\alpha_0} \preccurlyeq t_0$. (3.13.3)
Using (3.13.3) we have
$N'_c(x_0, t_0) \geq \alpha_0$
$\Rightarrow N'_c(x_0, t_0) \geq N_c(x_0, t_0)$. (3.13.4)
From Lemma 3.12(3), we have $\parallel x_0 \parallel_{c,\alpha_0} = t_0$.
For $\alpha_0 < \alpha < 1$, Let $\parallel x_0 \parallel_{c,\alpha} = t'$.
Then $t_0 \preccurlyeq t'$.
By Lemma 3.12(3) $N_c(x_0, t') = \alpha$.
So, $N_c(x_0, t') = \alpha > \alpha_0 = N_c(x_0, t_0)$.
Since $N_c(x_0, .)$ is non-decreasing and any two elements of P are comparable, so $t_0 \prec t'$.
So for $\alpha_0 < \alpha < 1$, $\parallel x_0 \parallel_{c,\alpha} = t' \preccurlyeq t_0$ does not hold.
Hence $N'_c(x_0, t_0) \leq \alpha_0 = N_c(x_0, t_0)$ (3.13.5)
From (3.13.4) and (3.13.5), $N_c(x_0, t_0) = N_c(x_0, t_0)$.

Theorem 3.14. Let $(X, N_c, *)$ be a fuzzy cone normed linear space where $* = min$ with a strongly minihedral normal cone P with normal constant K satisfying (FCN6) and (FCN7) and $\{\parallel \parallel_{c,\alpha}, \ \alpha \in (0,1)\}$ be the family of corresponding α-cone norms of N_c on X defined by
$\parallel x \parallel_{c,\alpha} = \ \wedge\{ t \succ \theta_E : N_c(x, t) \geq \alpha\}, \ \alpha \in (0,1)$.
Then for any increasing (or decreasing) sequence $\{\alpha_n\}$ in $(0,1)$,
$\alpha_n \longrightarrow \alpha \Rightarrow \parallel x \parallel_{c,\alpha_n} \longrightarrow \parallel x \parallel_{c,\alpha} \ \ \forall x \in X$.

Proof: For $x = \theta_X$, $\alpha_n \longrightarrow \alpha \Rightarrow \parallel x \parallel_{c,\alpha_n} \longrightarrow \parallel x \parallel_{c,\alpha}$
Suppose $x \neq \theta_X$.
From lemma 3.12(3), for $x \neq \theta_X$, $\alpha \in (0,1)$ and $t' \succ \theta_E$ we have
$\parallel x \parallel_{c,\alpha} = t' \Leftrightarrow N_c(x, t') = \alpha$.
Let $\{\alpha_n\}$ be an increasing sequence in $(0,1)$ such that $\alpha_n \longrightarrow \alpha \in (0,1)$.
Let $\parallel x \parallel_{c,\alpha_n} = t_n$ and $\parallel x \parallel_{c,\alpha} = t$. (3.14.1)
Then $N_c(x, t_n) = \alpha_n$ and $N_c(x, t) = \alpha$.
Since $\{\parallel \parallel_{c,\alpha}, \ \alpha \in (0,1)\}$ is an increasing family of α-cone norms. $\{t_n\}$ is an increasing sequence in E w.r.t. cone ordering which is bounded above by t. (since$\parallel x \parallel_{c,\alpha_n} \preccurlyeq \parallel x \parallel_{c,\alpha}$)
Since P is a strongly minihedral normal cone, so it is a regular cone. Thus $\{t_n\}$ is convergent.
Thus, $\lim_{n\to\infty} N_c(x, t_n) = \lim_{n\to\infty} \alpha_n$
$\Rightarrow N_c\left(x, \lim_{n\to\infty} t_n\right) = \alpha$. (3.14.2)

From (3.14.1) and (3.14.2), $\lim_{n\to\infty} t_n = t.$ by (FCN7)
Therefore, $\lim_{n\to\infty} \parallel x \parallel_{c,\alpha_n} = \parallel x \parallel_{c,\alpha}$.
Similarly, if $\{\alpha_n\}$ is a decreasing sequence in $(0,1)$, such that $\alpha_n \longrightarrow \alpha \in (0,1)$ then it can be shown $\parallel x \parallel_{c,\alpha_n} \longrightarrow \parallel x \parallel_{c,\alpha}$ $\forall x \in X$.

Theorem 3.15. Let $(X, N_c, *)$ be a fuzzy cone normed linear space where $* = min$ with a strongly minihedral normal cone P with normal constant K satisfying (FCN6) and (FCN7) and any two elements of P are comparable and $\{\parallel \parallel_{c,\alpha}, \ \alpha \in (0,1)\}$ be the ascending family of α-cone norms of N_c on X. Let N_c' be the fuzzy cone norm defined by

$$N_c'(x,t) = \vee\{\alpha \in (0,1) : \parallel x \parallel_{c,\alpha} \preccurlyeq t\} \text{ when } (x,t) \neq (\theta_X, \theta_E)$$
$$= 0 \qquad \text{when } (x,t) = (\theta_X, \theta_E). \qquad (3.15.1)$$

Let $\parallel x \parallel'_{c,\alpha} : X \longrightarrow E$ be a function defined by

$$\parallel x \parallel'_{c,\alpha} = \wedge\{ t \succ \theta_E : N_c'(x,t) \geq \alpha\}, \ \alpha \in (0,1) \qquad (3.15.2)$$

Then $\parallel x \parallel'_{c,\alpha} = \parallel x \parallel_{c,\alpha}$ $\forall \alpha \in (0,1)$.

Proof: For $x = \theta_X$, $\parallel x \parallel'_{c,\alpha} = \parallel x \parallel_{c,\alpha}$ $\forall \alpha \in (0,1)$.
Suppose $x \neq \theta_X$.
Let $\alpha_0 \in (0,1)$ and $\parallel x \parallel_{c,\alpha_0} = t_0$. Then $t_0 \succ \theta_E$.
From (3.15.1), we get $N_c'(x, t_0) \geq \alpha_0$.
Now from (3.15.2)

$$\parallel x \parallel'_{c,\alpha_0} \preccurlyeq t_0 = \parallel x \parallel_{c,\alpha_0}. \qquad (3.15.3)$$

Next let $\parallel x \parallel'_{c,\alpha_0} \prec s$.
Then $\exists\, t_1 \prec s$ such that $N_c'(x, t_1) \geq \alpha_0$.
$\Rightarrow \vee\{\alpha \in (0,1) : \parallel x \parallel_{c,\alpha} \preccurlyeq t_1\} \geq \alpha_0$.
If $\vee\{\alpha \in (0,1) : \parallel x \parallel_{c,\alpha} \preccurlyeq t_1\} = \alpha_0$, then $\exists$ an increasing sequence $\{\alpha_n\}$ in $(0,1)$ such that $\alpha_n \uparrow \alpha_0$ and $\parallel x \parallel_{c,\alpha_n} \preccurlyeq t_1$.
Since P is a strongly minihedral normal cone, so it is a regular cone. Thus from Theorem 3.14, $\parallel x \parallel_{c,\alpha_n} \longrightarrow \parallel x \parallel_{c,\alpha_0}$.
Thus, $\parallel x \parallel_{c,\alpha_0} \preccurlyeq t_1 \prec s$.
If $\vee\{\alpha \in (0,1) : \parallel x \parallel_{c,\alpha} \preccurlyeq t_1\} > \alpha_0$, then it follows, $\parallel x \parallel_{c,\alpha_0} \preccurlyeq t_1 \prec s$.
Thus in any cases, $\parallel x \parallel_{c,\alpha_0} \prec s$.
Hence $\parallel x \parallel_{c,\alpha_0} \preccurlyeq \parallel x \parallel'_{c,\alpha_0}$. (3.15.4)
From (3.15.3) and (3.15.4), we get $\parallel x \parallel'_{c,\alpha} = \parallel x \parallel_{c,\alpha}$ $\forall x \in X$.
Since $\alpha_0 \in (0,1)$ is arbitrary, $\parallel x \parallel'_{c,\alpha} = \parallel x \parallel_{c,\alpha}$ $\forall x \in X$, $\alpha \in (0,1)$.

$$\parallel x \parallel'_{c,\alpha_0} = \parallel x \parallel_{c,\alpha_0}.$$

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