NOTE ON THE STABILITY OF MAGNETOGASDYNAMIC SHEAR FLOWS

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ABSTRACT: *We consider the stability of magnetogasdynamic shear flows studied earlier in Dandapat and Gupta (1977) under the approximation $M << 1$ and $c_i << 1$ so that their product can be neglected in comparison to unity. For this problem it is shown that a sufficiently strong magnetic field stabilizes all subsonic disturbances. Consequently the same problem is considered under a weak magnetic field and it is shown that short waves are stable.*

Keywords: *Stability, shear flows, magnetogasdynamics.*

AMS subject classification: *76 E.*

1. INTRODUCTION

Consider the linear stability of the basic plane parallel flow $U(y)$ of an inviscid perfectly conducting gas in the x-direction with transverse variation along the y-axis. A uniform magnetic field H_0 acts along the x-axis, the flow being unaffected by this magnetic field. We assume that the basic thermodynamic state is constant. Thus if the velocity, pressure and magnetic field in the perturbed state are denoted by $(U(y) + u, v)$, $p_0 + \phi$, $(H_0 + h_x, h_y)$ respectively, where p_0, ρ_0 denote the constant pressure and density in the basic state and in the perturbed state, any physical variable is taken as a (function of y). $\exp[i\alpha(x - ct)$ then the stability equation is (*cf.* Dandapat and Gupta (1977))

$$\frac{d}{dy}\left[\left\{\frac{a_0^2(U-c)^2}{a_0^2-(U-c)^2}-V_A^2\right\}\frac{dF}{dy}\right]-\alpha^2\left[(U-c)^2-V_A^2\right]F=0, \tag{1}$$

and the boundary conditions are

$$F = 0 \text{ at } y = y_1 \text{ and } y = y_2. \tag{2}$$

Here $a_0\left(=\left(\frac{\gamma p_0}{\rho_0}\right)^{\frac{1}{2}}\right)$ is the acoustic speed in the unperturbed state, γ being the ratio of the specific heats of gas and $V_A\left(=\left(\frac{\mu_e H_0^2}{4\pi\rho_0}\right)^{\frac{1}{2}}\right)$ represents the Alfven velocity along the magnetic lines of force. If $v(y)e^{i\alpha(x-ct)}$ is the perturbed velocity in the y-direction then $F(y)=\frac{v(y)}{U-c}$ and so $F(y)$ is a non-dimensional variable. Let L be characteristic length and V be characteristic velocity of the flow. Introduce the nondimensional variables $x'=\frac{x}{L},\ y'=\frac{y}{L}, U'=\frac{U}{V},\ c'=\frac{c}{V}$ and note that F is already nondimensional. Rewriting (1) in terms of the nondimensional variables we have, upon dropping $'$, the equation

$$\frac{d}{dy}\left[\left\{\frac{(U-c)^2}{1-M^2(U-c)^2}-S\right\}\frac{dF}{dy}\right]-k^2\left[(U-c)^2-S\right]F=0, \tag{3}$$

with boundary conditions

$$F=0 \text{ at } y=y_1,\ y_2. \tag{4}$$

Here $M=\frac{V}{a_0}$ is the Mach number, $S=\frac{V_A^2}{V^2}$ is the magnetic field parameter and $k=\alpha L$ is the nondimensional wave number. Using the standard procedure to obtain the semicircular instability region (*cf.* Dandapat and Gupta (1977)) one can show that

$$\left[\left(c_r-\frac{a+b}{2}\right)^2+c_i^2-\left(\frac{b-a}{2}\right)^2\right]\int Q dy+M^2\int\frac{|U-c|^4\left|\frac{dF}{dy}\right|^2 dy}{\left|1-M^2(U-c)^2\right|^2}+S\int\left(\left|\frac{dF}{dy}\right|^2+k^2|F|^2\right)dy\le 0 \tag{5}$$

where $a=U_{\min}$, $b=U_{\max}$ and $Q=\frac{\left|\frac{dF}{dy}\right|^2}{\left|1-M^2(U-c)^2\right|^2}+k^2|F|^2$.

Since $S=\frac{V_A^2}{V^2}>0,$ one gets from this that the instability region is

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 \le \left(\frac{b-a}{2}\right)^2,$$

which has been obtained in Dandapat and Gupta (1977) in terms of dimensional variables. In the case of incompressible flows $M = 0$ and (5) reduces to

$$\left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right]\int\left(\left|\frac{dF}{dy}\right|^2 + k^2|F|^2\right)dy + S\int\left(\left|\frac{dF}{dy}\right|^2 + k^2|F|^2\right)dy \le 0.$$

From this one gets the instability region for incompressible flow as

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 \le \left(\frac{b-a}{2}\right)^2 - S, \tag{6}$$

and this result has already been obtained (see Agrawal and Agrawal (1969), Kochar and Jain (1979), Kailash Chandra (1973) and Shukhman (1998)). An interesting conclusion from this result is that a sufficiently strong magnetic field, that is, a magnetic field with $S \ge \left(\frac{b-a}{2}\right)^2$ stabilizes the shear flow. Consequently, the stability of an incompressible shear flow in the presence of a magnetic field has been studied under the weak magnetic field approximation (see, for example Shukhman (1998) where one can find a brief survey of all the known results on MHD shear flows).

Now returning to the stability of magnetogasdynamic shear flows we ask the question whether a strong magnetic field can stabilize the shear flow? This question has not been answered so far. In this paper we answer this question for the stability of a special class of basic flows with respect to a special class of disturbances.

In the absence of the magnetic field the stability of compressible shear flows has been studied in detail in Shivamoggi ((1977) under the conditions $M^2 << 1$ and $c_i << 1$ so that their product can be neglected in comparison to unity. Under Shivamoggi's approximation the stability equation (3) becomes

$$\frac{d}{dy}\left[\left\{\frac{(U-c)^2}{1-M^2(U-c_r)^2} - S\right\}\frac{dF}{dy}\right] - k^2\left[(U-c)^2 - S\right]F = 0, \tag{7}$$

and the boundary conditions (4) remain the same. Here it may be noted that the condition $M^2 << 1$ means that the basic flow is subsonic while the condition $c_i << 1$ means that the unstable modes under consideration are very close to neutral modes. The disturbances are classified to be subsonic, sonic or supersonic according as $1 - M^2(U - c_r)^2 >, =, < 0$ (see for example Shivamoggi (1977)).

In this paper we consider only subsonic disturbances for which

$$1 - M^2(U - c_r)^2 > 0. \tag{8}$$

2. SUFFICIENT CONDITION FOR STABILITY

Following the standard procedure to obtain the instability region one can easily show that

$$\left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right]\int Q dy + S\int\left(\left|\frac{dF}{dy}\right|^2 + k^2|F|^2\right)dy \le 0. \tag{9}$$

where

$$Q = \frac{\left|\frac{dF}{dy}\right|^2}{\left|1 - M^2(U - c_r)^2\right|^2} + k^2|F|^2 \ge 0, \tag{10}$$

$a = U_{\min}$ and $b = U_{\max}$. From this it follows that the instability region is given by

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2 \le 0. \tag{11}$$

From this semicircular instability region it follows that $(U - c_r) \le (b - a)$ and that

$$\frac{1}{1 - M^2(U - c_r)^2} \le \frac{1}{1 - M^2(b - a)^2}.$$

Substituting this in (9), we get

$$\left[\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 - \left(\frac{b-a}{2}\right)^2\right]\int\left(\frac{\left|\frac{dF}{dy}\right|^2}{\left|1 - M^2(b-a)^2\right|^2} + k^2|F|^2\right)dy + S\int\left(\left|\frac{dF}{dy}\right|^2 + k^2|F|^2\right)dy \le 0. \tag{12}$$

Since for the subsonic modes under consideration $0 < 1 - M^2(b - a)^2 < 1$ we get from (12) the result

$$\left[\left(c_r-\frac{a+b}{2}\right)^2+c_i^2-\left(\frac{b-a}{2}\right)^2\right]\int\left(\frac{\left|\frac{dF}{dy}\right|^2+k^2|F|^2}{\left|1-M^2(b-a)^2\right|^2}\right)dy+S\int\left(\left|\frac{dF}{dy}\right|^2+k^2|F|^2\right)dy\le 0. \quad (13)$$

From this it follows that

$$\left[\left(c_r-\frac{a+b}{2}\right)^2+c_i^2-\left(\frac{b-a}{2}\right)^2\right]\frac{1}{1-M^2(b-a)^2}+S\le 0, \quad (14)$$

and as a consequence we get the new instability region

$$\left(c_r-\frac{a+b}{2}\right)^2+c_i^2\le\left(\frac{b-a}{2}\right)^2-S\left[1-M^2(b-a)^2\right]. \quad (15)$$

This instability region depends on the magnetic field parameter S and furthermore it lies within the instability region given by (11).

For incompressible flows $M = 0$ and this instability region reduces to the region given by (6).

More importantly we can conclude that a sufficient condition for stability is

$$S\ge\frac{\left(\frac{b-a}{2}\right)^2}{1-M^2(b-a)^2}; \quad (16)$$

i.e., a sufficiently strong magnetic field stabilizes the shear flow.

3. WEAK MAGNETIC FIELD

Since we have seen that a strong magnetic field stabilizes a shear flow one can study the instability of shear flows under the weak magnetic field approximation as has been done for incompressible shear flows in Agrawal and Agrawal (1969). If we consider a weak magnetic field with S<<1 then equation (7) is replaced by

$$\frac{d}{dy}\left[\frac{(U-c)^2\frac{dF}{dy}}{1-M^2(U-c_r)^2}\right]-k^2[(U-c)^2-S]\,F=0. \quad (17)$$

where the boundary conditions (4) are the same as before. Here it should be noted that the two terms $S\frac{d^2F}{dy^2}$ and SF the first one is neglected since S is very small but the last term in (17) is retained as it multiplies k^2.

For an unstable mode $c_i > 0$ and so the function $G = (U-c)^{\frac{1}{2}} F$ is well defined.

From (17) one can obtain the equation

$$\frac{d}{dy}\left[\frac{(U-c)\frac{dG}{dy}}{1-M^2(U-c_r)^2}\right]-\frac{1}{2}\frac{d}{dy}\left[\frac{\frac{dU}{dy}}{1-M^2(U-c_r)^2}\right]G-\frac{\left(\frac{dU}{dy}\right)^2 G}{4(U-c)\left(1-M^2(U-c_r)^2\right)}$$

$$-k^2(U-c)G+\frac{k^2SG}{(U-c)}=0 \tag{18}$$

and the boundary conditions are

$$G(y_1) = 0 = G(y_2). \tag{19}$$

One can multiply (18) by G^* (the complex conjugate of G), integrate the resultant equation by using (19) to get

$$\int (U-c)\left[\frac{\left|\frac{dG}{dy}\right|^2}{1-M^2(U-c_r)^2}+k^2|G|^2\right]dy+\int\frac{1}{2}\frac{d}{dy}\left(\frac{\frac{dU}{dy}}{1-M^2(U-c_r)^2}\right)|G|^2\,dy$$

$$+\int\left[\frac{\left(\frac{dU}{dy}\right)^2}{4\left(1-M^2(U-c_r)^2\right)}-k^2\right]\frac{|G|^2}{(U-c)}dy=0. \tag{20}$$

Taking the imaginary part of (20) and noting that $c_i > 0$ for an unstable mode one gets the equation

$$\int\left[\frac{\left|\frac{dG}{dy}^2\right|}{1-M^2\left(U-c_r\right)^2}+k^2\left|G\right|^2\right]dy=\int\left[\frac{\left(\frac{dU}{dy}\right)^2}{4\left(1-M^2\left(U-c_r\right)^2\right)}-k^2\right]\frac{\left|G\right|^2}{\left(U-c\right)}dy\ . \quad (21)$$

This is impossible if

$k^2 > \dfrac{\left(\frac{dU}{dy}\right)^2}{4\left(1-M^2\left(U-c_r\right)^2\right)}$ throughout the flow domain.

Hence it follows that for an unstable mode it is necessary that

$$k^2 \le \frac{\left(\frac{dU}{dy}\right)^2_{\max}}{4\left(1-M^2\left(b-a\right)^2\right)}; \quad (22)$$

i.e., only long waves can be unstable. Also it follows that an estimate for the growth rate of an unstable mode is given by

$$k^2c_i^2 \le \left[\frac{\left(\frac{dU}{dy}\right)^2_{\max}}{4\left(1-M^2\left(b-a\right)^2\right)}-k^2\right]. \quad (23)$$

4. CONCLUDING REMARKS

In this paper we have considered the stability of magnetogasdynamic shear flows studied earlier in Dandapat and Gupta (1977) under the approximation $M << 1$ and $c_i << 1$ so that their product can be ignored in comparison to unity and for the special class of subsonic disturbances as has been done earlier in Shivamoggi (1977) for the non-conducting case. For the problem under consideration it is proved that a strong magnetic field stabilizes the shear flow completely and as a consequence, the instability of magnetogasdynamic shear flow is considered under the weak magnetic field approximation and it is shown that short waves are stable.

It is an open problem to prove or disprove that a sufficiently strong magnetic field stabilizes a compressible shear flow without the approximations of Shivamoggi (1977) and without restriction to subsonic modes.

REFERENCES

[1] Agrawal, S. C. and Agrawal, G. S., "Hydromagnetic Stability of Hetrogeneous Shear Flow", *J. Phy. Soc. Japan*, **27**(1), (1969), 218–223.

[2] Dandapat, B. S. and Gupta, A. S. "Stability of Magnetogasdynamic Shear Flow", *Acta Mechanica*, **28**, (1977), 77–83.

[3] Kailash Chandra, "Hydromagnetic Stability of Plane Hetrogeneous Shear Flow", *J. Phy. Soc. Japan*, **34(2)** (1973), 539–542.

[4] Kochar, G. T. and Jain, R. K., "On Howard's Semi Circle Theorem in Hydromagnetics", *J. Phy. Soc. Japan*, **47**(2), (1979), 654–658.

[5] Shivamoggi. B. K., "Inviscid Theory of Stability of Parallel Compressible Flows", *Journal de Mécanique*, **16**(2), (1977), 227–255.

[6] Shukhman. I. G., "Nonlinear Evolution of a Weakly Unstable Wave in a Free Shear Flow with a Weak Parallel Magnetic Field", *J. Fluid Mech.*, 369, (1998), 217–252.

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