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MAXIMAL ELEMENT THEOREMS IN FINITE CONTINUOUS SPACES

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Abstract: This paper deals with some maximal element theorems in finite continuous space. A well known result of Mehta is extended and we have applied it to achieving some maximal element theorems for condensing multimaps defined on a noncompact subset of a finite continuous space.

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1. INTRODUCTION

The existence of maximal elements for multimaps in topological vector spaces and its important applications in abstract economies, nonlinear analysis and other branches of mathematics have been studied by many authors in both mathematics and economies, for example see [4, 5, 6, 7, 21, 25, 31, 32, 35] and [36]. In the last two decades, the theory of fixed points and maximal elements for a family of multimaps defined on a product space have been investigated by many authors, see [1, 2, 4, 5, 6, 24, 25, 26] and [27].

The concept of generalized convex space introduced by Park and Kim in 1993 [33], which it generalizes topological vector space and several types of abstract convexity. Recently, as an extended version of generalized convex space, Ding introduced the notion of finite continuous space in [13, 17] and then many attempts have been applied this new space to various directions [9, 11, 12, 15, 19, 20, 22, 37] and especially to some fixed point and maximal element theorems by Ding in [13] and [14].

Kuratowski [23] introduced the set-measure of noncompactness, and later Gokhberg, Goldenstein and Markus (see Lloyd [29], Ch. 6) introduced the ball-measure of noncompactness. Petryshyn and Fitzpatrick [34], extended the notion of measure of noncompactness for subsets of a Hausdorff locally convex topological vector space.

At the present paper, we define FC-measure of noncompactness φ and φ_F-condensing multimaps in finite continuous spaces. Also, a well known Lemma of Mehta [31, 32], will be extend for an FC-measure of noncompactness φ and a φ_F-condensing multimap T. Then by using this result some maximal element theorems in finite continuous spaces are given.

2. PRELIMINARIES

A *multimap* $T : X \multimap Y$ is a function from a set X into $\mathcal{P}(Y)$, the power set of Y; that is, a function with the values $T(x) \subseteq Y$ for all $x \in X$ and

$$T^{-}(y) = \{x \in X : y \in T(x)\}$$

is a *fiber* for any $y \in Y$. Given $A \subseteq X$, set $T(A) = \bigcup_{x \in A} T(x)$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D and let Δ_n be the n-simplex with vertices $e_0, e_1, \ldots, e_n$, Δ_J be the face of Δ_n corresponding to $J \in \langle A \rangle$ where $A \in \langle D \rangle$; for example, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i0}, a_{i1}, \ldots, a_{ik}\} \subseteq A$, then $\Delta_J = co\{e_{i0}, e_{i1}, \ldots, e_{ik}\}$.

Suppose X and Y are topological spaces. A subset A of X is said to be compactly open (resp. closed) if for each nonempty compact subset K of X, $A \cap K$ is open (resp. closed) in K [8, 16]. The compact interior and the compact closure of A are defined as following

$$\text{cint}(A) = \bigcup \{B \subseteq X, B \subseteq A \text{ and } B \text{ is compactly open in } X\}$$

and

$$\text{ccl}(A) = \bigcap \{B \subseteq X, A \subseteq B \text{ and } B \text{ is compactly closed in } X\}$$

respectively.

Definition 2.1: [13, 17, 18, 37] Suppose X is a topological space. $(X, \{\varphi_N : N \in \langle X \rangle\})$ is said to be finite continuous space (briefly, FC-space) if for each $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to X$. Also, suppose A and B are two subsets of X. B is said to be a finite continuous subspace of X relative to A (briefly, an FC-subspace of X relative to A) if for each $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$ and for each $\{x_{i0}, x_{i1}, \ldots, x_{ik}\} \subseteq A \cap N$ we have $\varphi_N(\Delta_k) \subseteq B$. If $A = B$, then B is called a finite continuous subspace of X (briefly, an FC-subspace of X). It is easy to check that any FC-subspace of $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space too. For a subset A of an FC-space $(X, \{\varphi_N : N \subseteq \langle X \rangle\})$ the FC-hull of A is defined as following

$$FC(A) = \bigcap \{B \subseteq X : A \subseteq B \text{ and } B \text{ is } FC\text{-subspace of } X\},$$

also the closed FC-hull of A is defined as following

$$FC(A) = \bigcap \{B \subseteq X : A \subseteq B \text{ and } B \text{ is a closed } FC\text{-subspace of } X\}.$$

Example 2.2: Let $X = (0, 1) \cup (2, 3)$ with the usual topology. For each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$, define the mapping $\varphi_N : \Delta_n \to X$ by $\varphi_N(\alpha) = \frac{1}{3} \sum_{i=o}^{n} \alpha_i x_i$ for all $\alpha = (\alpha_0, \ldots, \alpha_n) \in \Delta_n$, then, it is easy to check that φ_N is continuous and hence $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space without convexity structure.

Proposition 2.3: [13] Suppose $(X_i, \{\varphi_{N_i} : N_i \in \langle X_i \rangle\})_{i \in I}$ is a family of FC-spaces. Then $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space, where $X = \prod_{i \in I} X_i$ and $\varphi_N = \prod_{i \in I} \varphi_{N_i}$.

Proposition 2.4: [37] Suppose $(X_i, \{\varphi_{N_i} : N_i \in \langle X_i \rangle\})_{i \in I}$ is a family of FC-spaces, $X = \prod_{i \in I} X_i$ and $\varphi_N = \prod_{i \in I} \varphi_{N_i}$. Also, suppose that A_i is an F_C-subspace of X_i relative to B_i for each $i \in I$. Then $A = \prod_{i \in I} A_i$ is an FC-subspace of X relative to $B = \prod_{i \in I} Bi$.

Proposition 2.5: [37] Suppose $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space. Then $\bigcap_{i \in I} A_i$ is an FC-subspace of X relative to $\bigcap_{i \in I} B_i$, if A_i is an FC-subspace of X relative to B_i for each $i \in I$.

Corollary 2.6: Suppose $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space. If A_i is an FC-subspace of X for each $i \in I$, then $\bigcap_{i \in I} A_i$ is an FC-subspace of X too.

Proof: It is an immediate consequence of Proposition 2.5.

3. MAXIMAL ELEMENT THEOREMS

Kuratowski [23] introduced the set-measure of noncompactness ∞ defined in a Banach space by $\alpha(A) = +\infty$, if A is unbounded and $\alpha(A) = \inf\{\epsilon > 0 : A$ can be covered by a finite number of sets with diameter less than $\epsilon\}$, if A is bounded. Analogously, Gokhberg, Goldenstein and Markus (see Lloyd [29], Ch. 6) introduced the ball-measure of noncompactness β defined in a Banach space by $\beta(A) = +\infty$, if A is unbounded and $\beta(A) = \inf\{r > 0 : A$ can be covered by a finite number of balls with radius less than $r\}$, if A is bounded. Clearly, in both cases, the range is $\mathbb{R}^+ \cup \{+\infty\}$ with the usual ordering. For $\varphi = \alpha$ or $\varphi = \beta$ the following statements satisfy,

(a) $\varphi(A) = \varphi(\overline{co}(A))$,

(b) $\varphi(A) = 0$ if and only if A is relatively compact,

(c) $\varphi(A \cup B) = \max\{\varphi(A), \varphi(B)\}$.

For another definitions and more details see [4] and [29] and references therein. Petryshyn and Fitzpatrick [34], extended the notion of measure of noncompactness for subsets of a Hausdorff locally convex topological vector space. In this section, measure of noncompactness will be defined in a finite continuous space similar to Petryshyn and Fitzpatrick by a suitable modifications. Also, an extended version of Mehta's result is given. Finally, by using these we obtain existence of maximal elements for condensing multimaps defined on a noncompact subset of a finite continuous space.

Definition 3.1: Suppose $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space and C is a lattice with a minimal element 0. A mapping $\varphi : \mathcal{P}(X) \to C$ is said to be FC-measure of noncompactness if for all $A, B \in \mathcal{P}(X)$,

(a) $\varphi(A) = \varphi(\overline{FC}(A))$,

(b) $\varphi(A) = 0$ if and only if A is relatively compact,

(c) $\varphi(A \cup B) = \max\{\varphi(A), \varphi(B)\}$.

Definition 3.2: Suppose $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space, C is a lattice with a minimal element 0 and $\varphi : \mathcal{P}(X) \to C$ is an FC-measure of noncompactness. A multimap $T : Y \subseteq X \multimap X$ is said to be φ_F-condensing if $\varphi(M) \leq \varphi(T(M))$ implies that M is relatively compact for each $M \in \mathcal{P}(Y)$.

Theorem 3.3: Suppose $(X_i, \{\varphi_{N_i} : N_i \in \langle X_i \rangle\})_{i \in I}$ is a family of closed FC-spaces and φ is an FC-measure of noncompactness on $X = \prod_{i \in I} X_i$. Also, suppose that $T_i : X \multimap X_i$ are multimaps such that $T = \prod_{i \in I} T_i$ is a φ_F-condensing multimap. Then there exists a nonempty compact FC-subspace K of X in which $T(K) \subseteq K$.

Proof: Fix an arbitrary $x \in X$. Set

$$\mathcal{F} = \{C \subseteq X : C \text{ is closed and } FC\text{-subspace of } X, x \in C, T(C) \subseteq C\}.$$

$\mathcal{F}$ is nonempty, since $X \in \mathcal{F}$. Put $K = \bigcap_{C \in F} C$, so $x \in K$, therefore K is nonempty. For each $k \in K$, $T(k) \subseteq C$ for all $C \in \mathcal{F}$, thus $T(K) \subseteq C$ for all $C \in \mathcal{F}$ which implies that $T(K) \subseteq K$. Clearly K is closed and also Corollary 2.6 implies K is FC-subspace of X. Now, if K is not compact, then it is not relatively compact. Since T is φ_F-condensing, so $\varphi(T(K)) < \varphi(K)$. Put $H = \overline{FC}(\{x\} \cup T(K))$, since $\{x\} \cup T(K) \subseteq K$ and since K is closed, so one can deduce that $H \subseteq K$. By the structure of H, $T(H) \subseteq T(K) \subseteq H$, so $H \in \mathcal{F}$. According to the definition of

F, K H, consequently $K = H$. On the other hand

$$\begin{aligned}\varphi(H) &= \varphi(\overline{FC}(\{x\} \cup T(K))) \\ &= \varphi(\{x\} \cup T(K)) \\ &= \max\{\varphi(\{x\}), \varphi(T(K))\} \\ &= \max\{0, \varphi(T(K))\} \\ &= \varphi(T(K)) \\ &< \varphi(K) = \varphi(H),\end{aligned}$$

which is a contradiction.

Remark 3.4: The originate version of Theorem 3.3 goes back to Martin [30]. Also, an adaptation of the argument of Martin for multimaps when $I = \{1\}$ is due to Mehta [31] and when I is an arbitrary index set, the result is due to Chebbi and Florenzano [4]. Finally, we note that some applications of Mehta's Lemma can be found in [3, 4, 26, 27] and [28].

Definition 3.5: [10] Suppose X is a nonempty set and Y is a topological space. A multimap $T : X \multimap Y$ is said to be transfer compactly open valued if for any $x \in X$ and for each compact subset K of Y, $y \in T(x) \cap K$ implies that there is $x_0 \in X$ for which $y \in \operatorname{int}_K(T(x_0) \cap K)$. In other words, if for every $x \in X$, $y \in T(x)$, there exists a point $\tilde{x} \in X$ for which $y \in c\operatorname{int} T(\tilde{x})$.

Theorem 3.6: [13] Suppose $(X_i, \{\varphi_{N_i} : N_i \in \langle X_i \rangle\})_{i \in I}$ is a family of FC-spaces and $X = \prod_{i \in I} X_i$. Let $S_i : X \multimap X_i$ satisfying

(a) for each $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$ and for each $\{x_{l_0}, x_{l_1}, \ldots, x_{l_k}\} \subseteq N$,

$$\varphi_N(\Delta_k) \cap \left(\bigcap_{j=0}^{k} c\operatorname{int} S_i^-(\pi_i(x_{l_j}) \right) = \emptyset,$$

(b) S_i^- is transfer compactly open valued,

(c) $I(x) = \{i \in I : S_i(x) \neq \emptyset\}$ is finite, for each $x \in X$,

(d) there exists a compact subset K of X such that for each $N_i \in \langle X_i \rangle$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i such that for each $x \in X \setminus K$, there exists $y \in L_N = \prod_{i \in I} L_{N_i}$ such that for each $i \in I(x)$, $x \in c\operatorname{int} S_i^-(y_i)$.

Then there exists $\bar{x} \in K$ for which $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Theorem 3.7: Suppose $(X_i, \{\varphi_{N_i} : N_i \in \langle X_i \rangle\})_{i \in I}$ is a family of FC-spaces and $X = \prod_{i \in I} X_i$. Let $S_i, T_i : X \multimap X_i$ satisfying

(a) $T_i(x)$ is FC-subspace of X_i relative to $S_i(x)$ for each $x \in X$,

(b) for each $x \in X$, $xi \notin T_i(x)$ and S_i^- is transfer compactly open valued,

(c) $I(x) = \{i \in I : S_i(x) \neq \emptyset\}$ is finite for each $x \in X$,

(d) there exists a compact subset K of X and $N_i \in \langle X_i \rangle$ and there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i such that for each $x \in X \setminus K$, there exists $y \in L_N = \prod_{i \in I} L_{N_i}$ such that for each $i \in I(x)$, $x \in c\operatorname{int} S_i^-(y_i)$.

Then there exists $\bar{x} \in K$ for which $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Proof: We claim that, condition (a) and $x_i \notin T_i(x)$ imply that the condition (a) of Theorem 3.6 holds. On the contrary, suppose there exists $N = \{x_0, ..., x_n\} \in \langle X \rangle$ and $\{x_{l_0}, ..., x_{l_k}\} \subseteq N$ such that

$$\varphi_N(\Delta_k) \cap \left(\bigcap_{j=0}^{k} c \operatorname{int} S_i^-(\pi_i(x_{l_j})) \right) \neq \emptyset .$$

Then there exists $\hat{x} \in \varphi_N(\Delta_k)$ in which $\hat{x} \in c \operatorname{int} S_i^-(\pi_i(x_{l_j})) \subseteq S_i^-(\pi_i(x_{l_j}))$ for all $j = 0, ..., k$. Therefore, $\{\pi_i(x_{l_j}) : j = 0, ..., k\} \subseteq S_i(\hat{x})$ for each $i \in I$. Hence, for $N_i = \pi_i(N)$ we have $\{\pi_i(x_{l_j}) : j = 0, ..., k\} \subseteq N_i \cap S_i(\hat{x})$. Now, since $T_i(\hat{x})$ is an FC-subspace of X_i relative to $S_i(\hat{x})$,

$$\hat{x}_i = \pi_i(\hat{x}) \in \pi_i(\varphi_N(\Delta_k)) = \varphi_{N_i}(\Delta_k) \subseteq T_i(\hat{x}),$$

a contradiction. That there exists $\bar{x} \in K$ for which $S_i(\bar{x}) = \emptyset$ for each $i \in I$ follows from Theorem 3.6.

Remark 3.8: It should be noticed that

(a) Theorem 3.7 for $S_i = T_i$ reduces to Corollary 2.5 of Ding in [13] and hence it generalizes and improves Theorem 4.1 in [27].

(b) When X_i is compact, then Theorem 3.7 holds without condition (d). Because, it is sufficient to put $L_{N_i} = X_i$ or $K = \prod_{i \in I} X_i$, in condition (d) of this theorem.

Lemma 3.9: Suppose X is a topological space, $T_i : X \multimap X_i$ are multimaps and T_i^- is a transfer compactly open valued multimap on X_i for each $i \in I$. Also, suppose that there is a subset A of X for which $T(A) = \prod_{i \in I} T_i(A) \subseteq A$. Then $(T_i|_A)^- : A_i \multimap A$ is transfer compactly open valued on A_i, where $A_i = \pi_i(A)$.

Proof: From $T(A) \subseteq A$, we have $T_i(A) \subseteq A_i$, so $T_i|_A : A \multimap A_i$ is a multimap. Now, we prove that $(T_i|_A)^-$ is a transfer compactly open valued multimap on A_i. Consider $a_i \in A_i$ with $(T_i|_A)^-(a_i) \neq \emptyset$. For each $a \in A$ with $a \in (T_i|_A)^-(a_i)$, since T_i^- is a transfer compactly open valued multimap on X_i, there exists $\tilde{a}_i \in X_i$ in which $a \in c \operatorname{int} T_i^-(\tilde{a}_i) \subseteq T_i^-(\tilde{a}_i)$. So, $\tilde{a}_i \in T_i(a) = T_i|_A(a) \in A_i$. We are done.

Lemma 3.10: Suppose $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space and B is an FC-subspace of X relative to A. Then each subset C of X containing B is FC-subspace of X relative to each subset D of A.

Proof: It is straightforward.

Lemma 3.11: Suppose $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space, Y is an FC-subspace of X and $A, B \subseteq Y$. Then B is an FC-subspace of X relative to A if and only if it is an FC-subspace of Y relative to A.

Proof: Let B is an FC-subspace of X relative to A. Suppose $N = \{x_0, ..., x_n\} \in \langle Y \rangle$, so $N \in \langle X \rangle$. If $\{x_{i0}, ..., x_{ik}\} \subseteq N \cap A$ then $\varphi_N(\Delta_k) \subseteq B$. It implies that B is an FC-subspace of Y relative to A. Conversely, suppose B is an FC-subspace of Y relative to A. For each $N \in \langle X \rangle$ we have $N \cap A \in \langle Y \rangle$. If $\{x_{i0}, ..., x_{ik}\} \subseteq N \cap A$ then $\varphi_N(\Delta_k) \subseteq B$; i.e., B is an FC-subspace of X relative to A.

Theorem 3.12: Suppose $(X_i, \{\varphi_{N_i} : N_i \in \langle X_i \rangle\})_{i \in I}$ is a family of closed FC-spaces and $X = \prod_{i \in I} X_i$. Assume that φ is an FC-measure of noncompactness on X and $S_i, T_i : X \multimap X_i$ are multimaps satisfying

(a) $FC(S_i(x)) \subseteq T_i(x)$ for each $x \in X$,

(b) for each $x \in X$, $x_i \notin T_i(X)$ and S_i^- is transfer compactly open valued on X_i,

(c) $I(x) = \{i \in I : S_i(x) \neq \emptyset\}$ is finite, for each $x \in X$,

(d) $T = \prod_{i \in I} T_i$ is φ_F-condensing.

Then there is $\bar{x} \in X$ for which $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Proof: By (d), T is φ_F-condensing and according to Theorem 3.3 there is a nonempty compact FC-subspace K of X such that $T(K) \subseteq K$. Set $K_i := \pi_i(K)$, then each $T_i|_K : K \multimap K_i$ is multimap. Condition (a) and Lemma 3.10 imply that $T_i(x)$ is FC-subspace of X_i relative to $S_i(x)$, for each $x \in X$. Clearly, for each $k \in K$, using Lemma 3.11 each $T_i(k)$ is FC-subspace of K_i relative to $S_i(k)$. Therefore, condition (a) of Theorem 3.7 satisfies for $T_i|_K$ and $S_i|_K$ instead of T_i and S_i respectively. For each $k \in K$, $k \notin T_i|_K(k)$ and from condition (a), $S_i(k) \subseteq FC(S_i(k)) \subseteq T_i(k)$ for each $k \in K$, so $S_i(K) \subseteq K$. Lemma 3.9 implies that $(S_i|K)^-$ is transfer compactly open valued on K_i, hence condition (b) of Theorem 3.7 is satisfies. After all, since K is compact, part (b) of Remark 3.8 implies that the condition (d) of Theorem 3.7 is not necessary. Finally, all conditions of Theorem 3.7 valid for $T_i|_K$ and $S_i|_K$. Then there is $\bar{x} \in K$ for which $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

Theorem 3.13: Suppose $(X_i, \{\varphi_{N_i} : N_i \in \langle X_i \rangle\})_{i \in I}$ is a family of closed FC-spaces and $X = \prod_{i \in I} X_i$. Also suppose that φ is an FC-measure of noncompactness on X and each $S_i : X \multimap X_i$ is a multimap satisfying

(a) $x_i \notin FC(S_i(x))$ for each $x \in X$,

(b) S_i^- is transfer compactly open valued on X_i for each $i \in I$,

(c) $I(x) = \{i \in I : S_i(x) \neq \emptyset\}$ is finite for each $x \in X$,

(d) $S = \prod_{i \in I} S_i$ is φ_F-condensing.

Then there is $\bar{x} \in X$ for which $S_i(\bar{x}) =$; for each $i \in I$.

Proof: For each $i \in I$, consider the multimap T_i defined by $T_i(x) = FC(S_i(x))$ for each $x \in X$. Since S is φ_F-condensing, so there is a nonempty compact FC-subspace K of X in which $T(K) \subseteq K$. Let $K_i = \pi_i(K)$, clearly, $T_i|_K$ and $S_i|_K$ maps K to K_i; i.e., $T_i|_K, S_i|_K : K \multimap K_i$ are well defined multimaps. It follows from (a) that $k_i \notin T_i|_K(k)$ for all $k \in K$ and also Lemma 3.9 implies that $(S_i|_K)^-$ is transfer compactly open valued multimap on K.

Therefore, all conditions of Theorem 3.7 satisfy for $(S_i|_K, T_i|_K)$ instead of (S_i, T_i), so there exists $\bar{x} \in$ X for which $S_i(\bar{x}) = \emptyset$ for each $i \in I$.

For the case that I is singleton we have the following direct result of Theorem 3.13.

Corollary 3.14: Suppose $(X, \{\varphi_N : N \in \langle X \rangle\})$ is an FC-space, φ be an FC-measure of noncompactness on X and $S : X \multimap X$ is a multimap satisfying

(a) $x \notin FC(S(x))$, for each $x \in X$,

(b) S^- is transfer compactly open valued on X,

(d) S is φ_F-condensing.

Then there is $\bar{x} \in X$ for which $S(\bar{x}) = \emptyset$.

Remark 3.15: Theorem 3.12 and Theorem 3.13 are extensions of Theorem 4.2 and Theorem 4.3 in [27] respectively. Corollary 3.14 is a generalized version of Corollary 4.1 in [27] and since it generalizes Corollary 2 in [4], Theorem 2 in [25], Theorem 2.2 in [32] and Theorem 3.1. in [36].

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