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SANDWICH COMPOSITION OPERATOR ON WEIGHTED HARDY SPACES

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Abstract: Let $C_{\varphi,}$ D be the composition and differentiation operators defined by $C_\varphi f = f o\varphi$ and Df=f respectively. In this paper, the boundedness and compactness of the Sandwich composition operator DC_φD on the weighted Hardy spaces have been characterized.

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1. INTRODUCTION

Let φ be an analytic self-map of the open unit disc $\mathbb{D}$ in the finite complex plane $\mathbb{C}$ and H($\mathbb{D}$) be the set of all complex valued analytic functions on $\mathbb{D}$. By $\partial\mathbb{D}$ we denote the boundary of $\mathbb{D}$; and H^p $(1 \leq p < \infty)$ the classical Hardy space.

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ = be the sequence of positive numbers such that $\beta_0 = 1$ and $\underset{n\to\infty}{Lt} \frac{\beta_{n+1}}{\beta_n} = 1$. Then the weighted Hardy spaces $H^2(\beta)$ is the Banach space of all analytic functions f on the open unit disk $\mathbb{D}$ defined by

$$H^2(\beta) = \{f : z \to \sum_{n=0}^{\infty} a_n z^n \ s.t \ \|f\|^2_{H^2(\beta)} = \sum_{n=0}^{\infty} |a_n|^2.\beta_n^2 < \infty\}$$

where $\|.\|_{H^2}$ is a norm on $H^2(\beta)$.

If $\beta \equiv 1$, then $H^2(\beta)$ becomes the classical Hardy space $H^2(\mathbb{D})$.

Also, $H^2(\beta)$ is a Hilbert space w.r.t the inner product

$$< f, g > = \sum_{n=0}^{\infty} a_n \bar{b}_n . \beta_n^2$$

where f, $g \in H^2(\beta)$. For a detailed discussion on $H^2(\beta)$ one can see [14].

Associated with j, the classical linear operator C_j H($\mathbb{D}$) → H($\mathbb{D}$) is defined by $f \to f\, o\, \varphi$ and this operator is called the composition operator induced by self-map.

j. Let D be the differentiation operator defined by D$f = f'$. It has been known that the composition operator C_φ is bounded on almost all spaces of analytic functions for example see [1, 2, 3] and D is usually unbounded on spaces of analytic functions. Recently, the above defined operators have received the attention of many researchers see, for example [5, 7, 11, 12] and [17].

The product of composition operator C_φ and differentiation operator D is written as $C_\varphi D$ and DC_φ which are defined as $C_\varphi D f = f' o \varphi$ and $DC_\varphi f = (f o \varphi)'$ respectively, for function f analytic in the disc $\mathbb{D}$. In [5], Hibschweiles and Portony defined the product $C_\varphi D$ and DC_φ and studied the boundedness and compactness of these operators between Bergman and Hardy spaces by using the Carleson-type measure, where as in [12] the author studied the boundedness and compactness of $C_\varphi D$ and DC_φ between Hardy type spaces.

This paper is organised as follows. In the second section, we shall discuss the boundedness of the operator $DC_\varphi D$ on weighted Hardy spaces $H^2(\beta)$. In the third section, we shall study the compactness of the operator $DC_\varphi D$ on weighted Hardy spaces $H^2(\beta)$ and in the final section, we shall give necessary and sufficient condition for the operator $DC_\varphi D$ to be the Hilbert-Schmidt operator on weighted Hardy spaces $H^2(\beta)$.

2. Boundedness of the operator $DC_\varphi D$.

In this section, we shall characterize the boundedness of the Sandwich composition operator $DC_\varphi D$ on the weighted Hardy spaces.

Theorem 2.1. Let $\varphi: \mathbb{D} \to \mathbb{D}$ is an analytic self map of $\mathbb{D}$ and $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then the Sandwich composition operator $DC_\varphi D : H^2(\beta)$’! H^2(â)is bounded iff

$$\left\| \varphi^{n-2} \,.\, \varphi' \right\|_{H^2(\beta)} \leq \frac{M.\ \beta_n}{n(n-1)} \quad \text{for all } n \in \mathbb{N}\cup\{0\}..$$

Proof: First, suppose that an operator $DC_\varphi D: H^2(\beta) \to H^2(\beta)$ is bounded.

Then there exist a +ve number M such that

$$\left\| DC\varphi D f \right\|_{H^2(\beta)} \leq M \left\| f \right\|_{H^2(\beta)} \ \forall \ f \in H^2(\beta). \tag{2}$$

Let. $f(z) = Z^n$. Then $f \in H^2_{(\beta)}$ and so from (2.1), we have

$$\left\| n(n-1)\varphi^{n-2}.\varphi' \right\|_{H^2(\beta)} \leq M \left\| z^n \right\| = M.\beta_n$$

That is

$$\left\| \varphi^{n-2}.\varphi' \right\|_{H^2(\beta)} \leq \frac{M.\ \beta_n}{n(n-1)} \text{ for all } n \in \mathbb{N}\cup\{0\}.$$

Conversely, assume that

$$\| \varphi^{n-2} . \varphi' \|_{H^2(\beta)} \leq \frac{M.\ \beta_n}{n(n-1)} \tag{2.2}$$

Then, we have to prove that $DC_\varphi D$ is bounded.

Let $f \in H^2(\beta)$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, we have

$$\begin{aligned}
\| DC\varphi Df \|^2_{H^2(\beta)} &= \left\| \sum_{n=0}^{\infty} a_n (DC\varphi D)(z^n)) \right\|^2_{H^2(\beta)} \\
&= \left\| \sum_{n=0}^{\infty} a_n . n(n-1)\varphi^{n-2} . \varphi' \right\|^2_{H^2(\beta)} \\
&\leq \left\| \sum_{n=0}^{\infty} | n(n-1) |^2 | a_n |^2 . \| \varphi^{n-2} . \varphi' \right\|^2_{H^2(\beta)} \\
&\leq \sum_{n=0}^{\infty} | n(n-1) |^2 | a_n |^2 \frac{M^2 . \beta_n^2}{[n(n-1)]^2} \\
&= M \sum_{n=0}^{\infty} | a_n |^2 . \beta_n^2 \\
&= M \| f \|^2_{H^2(\beta)}
\end{aligned}$$

This implies that

$$\| DC\varphi Df \|_{H^2(\beta)} \leq M \| f \|_{H^2(\beta)} \ \forall f \in H^2(\beta)$$

That is $DC_{5\text{Øß}}D$ is bounded.

3. Compactness of the operator $DC_\varphi D$.

In this section, we study the boundedness of the operator $DC_\varphi D$ on the weighted Hardy spaces $H^2(\beta)$. For this we need the following Lemma.

Lemma 3.1 Let φ is an analytic self-map of $\mathbb{D}$. Then the Sandwich composition operator $DC_\varphi D$: $H^2(\beta)$'→ $H^2(\beta)$is compact iff for every bounded sequence converging uniformly on compact subsets of $\mathbb{D}$, we have

$$\| DC\varphi Df_n \|_{H^2(\beta)} \to 0$$

Proof: The proof of this Lemma can be written by using the similar arguments as in [2, P-128].

Theorem 3.2. Let $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of $\mathbb{D}$ and $\{\varphi^n : n \geq 1\}$ be an orthogonal family. Then $DC_\varphi D : H^2(\beta) \to H^2(\beta)$ is compact iff

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{n(n-1)}{\beta_n} \| \varphi^{n-2}.\varphi' \|_{H^2(\beta)} = 0$$

Proof: Let us suppose that $DC_\varphi D$: $H^2(\beta) \to H^2(\beta)$ is compact and $\left\{\frac{z^n}{\beta_n}\right\}_{n=1}^{\infty}$ converges uniformly to zero on compact subsets of E!, so by Lemma 3.1

$$\left\| DC\varphi D \left\{\frac{z^n}{\beta_n}\right\} \right\|_{H^2(\beta)} \to 0 \text{ as } n \to \infty.$$

Hence $$\underset{n\to\infty}{\mathrm{Lt}} \frac{n(n-1)}{\beta_n} \| \varphi^{n-2}.\varphi' \|_{H^2(\beta)} = 0$$

Conversely, suppose that

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{n(n-1)}{\beta_n} \| \varphi^{n-2}.\varphi' \|_{H^2(\beta)} = 0$$

Then for given any $\in > 0$, there exists +ve integer m, such$_n$ that

$$\| \varphi^{n-2}.\varphi' \|.\frac{n(n-1)}{\beta_n} < \epsilon \ \forall \ n \geq m.$$

Now, let $f \in H^2(\beta)$ such that $f(z) = \sum_{n=0}^{\infty} a_n.n$.

Define an operator T_k on $H^2(\beta)$as

$$T_k f = \sum_{n=0}^{\infty} a_n (DC\varphi D) z^n = \sum_{n=0}^{\infty} a_n.n(n-1)\varphi^{n-2}.\varphi'$$

Then T_k is a finite rank operator on $H^2(\beta)$.

Now for k $\geq m$

$$\| (DC\varphi D - T_k) f \|_{H^2(\beta)}^2 = \left\| \sum_{n=k+1}^{\infty} a_n.n(n-1)\varphi^{n-2}.\varphi' \right\|_{H^2(\beta)}^2$$

$$= \sum_{n=k+1}^{\infty} [n(n-1)]^2 |^2 \| \varphi^{n-2}.\varphi' \|_{H^2(\beta)}^2$$

$$\leq \sum_{n=0}^{\infty} | a_n |^2 . \in^2 . \beta_n^2$$

$$= \in^2 \sum_{n=0}^{\infty} | a_n |^2 . \beta_n^2$$

$$= \in^2 \| f \|_{H^2(\beta)}^2$$

This implies that

$$\| \mathrm{DC\varphi D} - T_k \|_{H^2(\beta)} < \epsilon \ \forall\, k \geq m$$

and so the operator $DC_\varphi D$ is compact.

4. NECESSARY AND SUFFICIENT CONDITION FOR THE OPERATOR $DC_\varphi D$ TO BE HILBERT SCHMIDT OPERATOR ON $H^2(\beta)$.

Recall that a linear operator T on Hilbert space H is said to be Hilbert Schmidt operator if $\sum_{n=0}^{\infty} \| Te_n \|^2 < \infty$ for some orthonormal basis $\{e_n\}$ of H.

Theorem 4.1. The operator $DC\varphi D$ is a Hilbert Schmidt operator on H^2(â) iff .

$$\sum_{n=0}^{\infty} \frac{n^2(n-1)^2}{\beta_n{}^2} \| \varphi^{n-2}.\varphi' \|_{H^2(\beta)}^2 < \infty.$$

Proof: Since $\left\{ \frac{z^n}{\beta_n} : n \geq 0 \right\}$ is an orthonormal basis for $H^2(\beta)$. The operator $DC\varphi D$ is Hilbert Schmidt operator

$$\text{iff } \sum_{n=0}^{\infty} \| DC_\varphi D\left(\frac{z^n}{\beta_n}\right) \|_{H^2(\beta)}^2 < \infty$$

$$\text{iff } \sum_{n=0}^{\infty} \| \frac{n(n-1)}{\beta_n} \varphi^{n-2}.\varphi' \|_{H^2(\beta)}^2 < \infty$$

$$\text{iff } \sum_{n=0}^{\infty} \| \frac{n^2(n-1)^2}{\beta_n{}^2} \| \varphi^{n-2}.\varphi' \|_{H^2(\beta)}^2 < \infty$$

This complete the proof.

REFERENCES

[1] A. K. Sharma and S. D. Sharma,: Weighted composition operators between Bergman type spaces, Comm. Korean Math. Soc. 21 No. 3(2006), 465-474.

[2] A. K. Sharma,: Product of composition, multiplication and differentiation operators between Bergman and Bloch type spaces, Turkish. J. Math. 34 (2010), 117.

[3] A. K. Sharma,: Volterra composition operators between Bergman Nevanlinna and Bloch-type spaces, Demonstratio Mathematica Vol XLII No. 3, 2009.

[4] A. L. Shield,: Weighted shift operator and analytic function theory, in topic in operator theory, Math surveys, No 13, Amer. Math. soc. Providence 1947.

[5] B. Garnett,: Bounded analytic functions , Revised first edition. Graduate Texts in Mathematics, 236. Springer, New York, 2007.

[6] C. C. Cowen and B. D. MacCluear,: Composition operators on spaces of analytic functions, Stud. Adv. Math., CRC Press. Boca Ration, 1995.

[7] D. M. Boyd,: Composition operators on Hp(A), Pacific J. Math. 62 (1976), 55-60.

[8] H. J. Schwartz,: Composition operators on ?H^p?^ , Ph.D thesis, University of Toledo,1969.

[9] J. H. Shapiro,: Composition operators and classical function theory, Springer-Verlag New York 1993.

[10] K. Zhu,: Operator theory in function spaces, Marcel-Dekker, New York 1990.

[11] K. Zhu,: Spaces of holomorphic functions in the unit ball. Graduate Text in Mathematics, vol. 226. Springer, New York (2005).

[12] P. Duren,: Theory of H^(p)spaces, Academic Press New York, 1973.

[13] Pawan Kumar and S. D. Sharma,: Generalized composition operators from weighted Bergman-Nevanlinna spaces to zygmund spaces, Int. J. Mod. Math. Sci.1(3),(2012),160-162.

[14] Pawan Kumar and S. D. Sharma,: Weighted composition operators from weighted Bergman-Nevanlinna spaces to zygmund spaces, Int. J. Mod. Math. Sci.3(1), (2012),31-54.

[15] Pawan Kumar and Zaheer Abbas,: Composition operator between weighted Hardy type spaces, International journal of pure and Applied mathematics Vol.106, No.3,2016.

[16] Pawan Kumar and Zaheer Abbas,: Product of multiplication and composition operators on weighted Hardy spaces , S.S International Journal of Pure and Applied Mathematics Vol.1, Issue 2,2015.

[17] Pawan Kumar and Zaheer Abbas,: Product of multiplication, composition and differentiation operators on weighted Hardy space , International Journal of computational and applied mathematics,Vol.12, No.3, 2017.

[18] R. A. Hibschweiles and N. Portnoy,: Composition operators followed by differentiation between Bergman and Hardy spaces, Rocky Mountain J.Math.35, 843-855 (2005).

[19] S. D. Sharma and R. Kumar,: Substitution operators on Hardy -Orlicz spaces, Proc. Nat. Acad. Sci. India Sect.A61 (1991),535-541.

[20] S. Li and S. Stevic,: Composition followed by differentiation between H^(?)and ?-Bloch spaces, Houston J. Math. 35(2009), 327-340.

[21] S. Ohno,: Product of composition and differentiation between Hardy spaces, Bull Austral Math. Soc. 73(2006), 235-243.

[22] X. Zhu,: Weighted composition operators from Area Nevanlinna spaces into Bloch spaces, Applied Mathematics and computation, 215 (2010), 4340-4346.

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