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A Computational Method to H_∞ Control Problem of Second-Order Linear Systems Using Haar Wavelets

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ABSTRACT

This paper presents a computational methodology to solve the problem of H_∞ control design for second-order linear systems using Haar wavelets. The main focus is on development of an H_∞ controller for the system under consideration computationally that guarantees both the closed-loop stability and satisfies a desired H_∞ performance. The Haar wavelet properties are utilized to find the approximate solutions of trajectories and robust optimal control by solving only algebraic equations instead of solving the Riccati differential equation.

Index Terms: Second-order linear system, Haar wavelet; H_∞ control; computational method.

1. INTRODUCTION

In recent years, wavelet theory has obtained more attention from both mathematical and practical perspectives [1]. It has been applied in a wide range of engineering disciplines such as signal processing, pattern recognition and computational graphics. Recently, some of the attempts are made in solving surface integral equations, improving the finite difference time domain method, solving linear differential equations and nonlinear partial differential equations and modelling nonlinear semiconductor devices [2, 3, 4, 6, 8, 9, 10, 11, 12]

On the other hand, as a special class of wavelets, orthogonal functions like Haar wavelets (HWs) [6, 8], Walsh functions [4], block pulse functions [12], Laguerre polynomials [7], Legendre polynomials [2], Chebyshev functions [5] and Fourier series [13], often used to represent an arbitrary time functions, have received considerable attention in dealing with various problems of dynamic systems. The main feature of this method is to convert the problem described by differential equations to the problem of solving a system of algebraic equations for the solution of problems, such as analysis of linear time-invariant, time-varying systems, model reduction, optimal control and system identification. Thus, the solution, identification and optimisation procedure are either greatly reduced or much simplified accordingly. The available sets of orthogonal functions can be divided into three classes such as piecewise constant basis functions (PCBFs) like HWs, Walsh functions and block pulse functions; orthogonal polynomials like Laguerre, Legendre and Chebyshev as well as sine-cosine functions in Fourier series [11].

In the present paper, a computational method is presented to the finite-time robust optimal control problem of the second-order linear systems based on HWs. To this aim, the properties of HWs, Haar wavelet integral operational matrix and Haar wavelet product operational matrix are given and are utilized to provide a systematic computational framework to find the approximated robust optimal trajectory and finite-time H_∞ control of the system with respect to an H_∞ performance by solving only the linear algebraic equations instead of solving the differential equations.

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The notations used throughout the paper are fairly standard. The matrices I_r, 0_r and $0_{r\times s}$ are the identity matrix with dimension $r\times r$ and the zero matrices with dimensions $r\times r$ and $r\times s$, respectively. The symbol $\otimes$ and $tr(A)$ denote Kronecker product and trace of the matrix A, respectively. Also, operator $vec(X)$ denotes the vector obtained by putting matrix X into one column. Finally, given a signal $x(t)$, $\|x(t)\|_2$ denotes the L_2 norm of $x(t)$; i.e., $\|x(t)\|_2^2 = \int_0^\infty x(t)^T x(t)\ dt$.

2. PROPERTIES OF HAAR WAVELETS

The oldest and most basic of the wavelet systems is named Haar wavelet that is a group of square waves with magnitude of ±1 in the interval [0, 1) [3]. In other words, the HWs are defined on the interval [0, 1) as

$$\psi_0(t)=1\ ,\ t\in[0,\ 1)\ ,$$
$$\psi_1\ \ (t)=\begin{cases}1\ ,\ for\ \ t\in[0,\ \tfrac{1}{2})\ ,\\ -1\ ,\ for\ \ t\in[\tfrac{1}{2},\ 1)\ ,\end{cases} \tag{1}$$

and $\psi_i(t)=\psi_1(2^j t-k)$ for $i \geq 1$ and we write $i = 2^j + k$ for $j \geq 0$ and $0 \leq k < 2^j$. We can easily see that the $\psi_0(t)$ and $\psi_1(t)$ are compactly supported, they give a local description, at different scales j, of the considered function.

The finite series representation of any square integrable function $y(t)$ in terms of an orthogonal basis in the interval $[0,1)$, namely $\hat{y}(t)$, is given by $\hat{y}(t)=\sum_{i=0}^{m-1} a_i\Psi_i(t) := a^T\Psi_m(t)$ where $a := [a_0 a_1\ \cdots\ a_{m-1}]^T$ and $\Psi_m(t) := [\psi_0(t)\ \psi_1(t)\cdots\psi_{m-1}(t)]^T$ for $m = 2j$ and the Haar coefficients a_i are determined to minimize the mean integral square error $\varepsilon=\int_0^1 (y(t)-a^T\ \Psi_m(t))^2\ dt$ and are given by $a_i = 2^j\int_0^1 y(t)\,\psi_i(t)\ dt$.

The matrix H_m can be defined as

$$H_m = [\Psi_m(t_0),\Psi_m(t_1),\cdots,\Psi_m(t_{m-1})\,] \tag{2}$$

where $i/m \leq t_i < {i+1}/m$, we get $[\hat{y}(t_0)\ \hat{y}(t_1)\ \ldots\ \hat{y}(t_{m-1})] = a^T\ H_m$.

The integration of the vector $\Psi_m(t)$ can be approximated by

$$\int_0^t \Psi_m(t)\,dt = P_m\ \Psi_m(t) \tag{3}$$

where the matrix $P_m =< \int_0^t \Psi_m(\tau)\ d\tau,\Psi_m(t) >= \int_0^1\int_0^t\ \Psi_m(r)\,dr\ \Psi_m^T(t)\,dt$ represents the integral operator matrix for PCBFs on the interval $[0,1)$ at the resolution m. For HWs, the square matrix P_m satisfies the following recursive formula [6]:

$$P_m = \frac{1}{2m}\begin{bmatrix}2m\,P_{\frac{m}{2}} & -H_{\frac{m}{2}}\\ H_{\frac{m}{2}}^{-1} & 0_{\frac{m}{2}}\end{bmatrix} \tag{4}$$

with $P_1 = \frac{1}{2}$ and $H_m^{-1} = \dfrac{1}{m} H_m^T \; diag\,(r)$ where the matrix H_m defined in (2) and also the vector r is represented by $r := \underbrace{(1,1,2,2,4,4,4,4,\cdots,(\frac{m}{2}),(\frac{m}{2}),\cdots,(\frac{m}{2}))^T}_{(\frac{m}{2})\ elements}$ for $m > 2$. For example, at resolution scale $j = 3$, the matrices H_8 and P_8 are represented as

$$H_8 = \begin{bmatrix} \psi_0(t_0) & \psi_0(t_1) & & \psi_0(t_7) \\ \psi_1(t_0) & \psi_1(t_1) & & \psi_1(t_7) \\ \psi_2(t_0) & \psi_2(t_1) & & \psi_2(t_7) \\ \psi_3(t_0) & \psi_3(t_1) & \cdots & \psi_3(t_7) \\ \psi_4(t_0) & \psi_4(t_1) & & \psi_4(t_7) \\ \psi_5(t_0) & \psi_5(t_1) & & \psi_5(t_7) \\ \psi_6(t_0) & \psi_6(t_1) & & \psi_6(t_7) \\ \psi_7(t_0) & \psi_7(t_1) & & \psi_7(t_7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

and

$$P_8 = \frac{1}{16}\begin{bmatrix} \begin{matrix} 8 & -4H_1 \\ 4H_1^{-1} & 0 \end{matrix} & -2H_2 & \\ 4H_2^{-1} & 0 & -H_4 \\ H_4^{-1} & & 0 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 16P_4 & -H_4 \\ H_4^{-1} & 0 \end{bmatrix}$$

$$= \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 & -4 & 4 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for further information see [6].

In the study of time-varying systems, it is usually necessary to evaluate the product of two Haar function vectors [6]. Let us define $R_m(t) := \Psi_m(t)\Psi_m^T(t)$ where $R_m(t)$ satisfies the following recursive formula

$$R_m(t)=\frac{1}{2m}\begin{bmatrix} R_{\frac{m}{2}}(t) & H_{\frac{m}{2}}\ diag(\Psi_b(t)) \\ (H_{\frac{m}{2}}\ diag(\Psi_b(t)))^T & diag(H_{\frac{m}{2}}^{-1}\Psi_a(t)) \end{bmatrix} \tag{5}$$

with $R_1(t)=\psi_0(t)\psi_0^T(t)$ and

$$\begin{cases} \Psi_a(t):=\left[\psi_0(t),\psi_1(t),\cdots,\psi_{\frac{m}{2}-1}(t)\right]^T = \Psi_{\frac{m}{2}}(t) \\ \Psi_b(t):=\left[\psi_{\frac{m}{2}}(t),\psi_{\frac{m}{2}+1}(t),\cdots,\psi_{m-1}(t)\right]^T. \end{cases}$$

Moreover, the following relation can be obtained based on the relation above:

$$R_m(t)\,a_m=\tilde{a}_m\Psi_m(t) \tag{6}$$

where $\tilde{a}_1=a_0$ and

$$\tilde{a}_m=\begin{bmatrix} \tilde{a}_{\frac{m}{2}} & H_{\frac{m}{2}}\,diag(a_b) \\ diag(a_b)H_{\frac{m}{2}}^{-1} & diag(a_a^T\,H_{\frac{m}{2}}) \end{bmatrix}$$

with

$$\begin{cases} a_a:=\left[a_0,a_1,\cdots,a_{\frac{m}{2}-1}\right]^T = a_{\frac{m}{2}}(t) \\ a_b:=\left[a_{\frac{m}{2}}(t),a_{\frac{m}{2}+1}(t),\cdots,a_{m-1}(t)\right]^T. \end{cases}$$

3. ALGEBRAIC SOLUTION OF SYSTEM EQUATIONS

In this section, we study the problem of solving the second-order differential equations of the following system in terms of the input control and exogenous disturbance using HWs and develop appropriate algebraic equations.

$$\begin{cases} M\,\ddot{x}(t)+C\,\dot{x}(t)+K\,x(t)=B_f\ f(t)+B_d\ d_e(t), \qquad t\in\left[0,T_f\right] \\ z(t)=\begin{bmatrix} C_1\,x(t) \\ C_2\,\dot{x}(t) \\ C_3\ f(t) \end{bmatrix} \end{cases} \tag{7}$$

where $x(t)\in\Re^n$ is the state; $f(t)\in\Re$ is the control input; $d_e(t)\in\Re$ is the disturbance input which belongs to $L_2[0,\infty)$; and $z(t)\in\Re^{m_1+m_2}$ is the controlled output with $C_1\in\Re^{m_1\times n}$, $C_2\in\Re^{m_2\times n}$ and C_3 is a positive scalar.

In this paper, a state feedback controller is to be determined computationally such that the following requirements are satisfied:

(I) the closed-loop system is asymptotically stable;

(II) under zero initial condition, the closed-loop system satisfies $\| z(t)\|_2 < \gamma \| d_e(t)\|_2$ for any non-zero $d_e(t) \in [0, \infty)$ where γ > 0 is a prescribed scalar.

Based on HWs definition on the interval time $[0,1]$, we need to rescale the finite time interval $\lfloor 0, T_f \rfloor$ into $[0,1]$ by considering $t = \mathrm{T}_f\ \sigma$; normalizing the system Eq. (7) with the time scale would be as follows

$$M\,\ddot{x}(\sigma) + C\,\dot{x}(\sigma) + K\,x(\sigma) = B_f\ f(\sigma) + B_d\ d_e(\sigma). \tag{8}$$

Now by integrating the system above in an interval [0, σ], we obtain

$$\begin{aligned} &M\,(\dot{x}(\sigma) - \dot{x}(0)) + \mathrm{T}_f\,C\,(x(\sigma) - x(0)) + \mathrm{T}_f^2\,K \int_0^\sigma x(\tau)\ d\tau \\ &= \mathrm{T}_f^2\,B_f \int_0^\sigma f(\tau)\ d\tau + \mathrm{T}_f^2\,B_d \int_0^\sigma d_e(\tau)\ d\tau. \end{aligned} \tag{9}$$

To avoid the differentiation of wavelets, we take again the integration of (9) in the interval [0, σ] as follows:

$$\begin{aligned} &M\,(x(\sigma) - x(0)) + \mathrm{T}_f\,C \int_0^\sigma x(\tau)\,d\tau + \mathrm{T}_f^2\,K \int_0^\sigma\!\!\int_0^\xi x(\tau)\ d\tau\,d\xi = \mathrm{T}_f^2\,B_f \int_0^\sigma\!\!\int_0^\xi f(\tau)\,d\tau\,d\xi \\ &+ \mathrm{T}_f^2\,B_d \int_0^\sigma\!\!\int_0^\xi d_e(\tau)\,d\tau\,d\xi + \int_0^\sigma (M\,\dot{x}(0) + \mathrm{T}_f\,C\,x(0))\,d\xi. \end{aligned} \tag{10}$$

By using the Haar wavelet expansion, we express the solution of Eq. (10), input force $f(\sigma)$ and the disturbance $d_e(\sigma)$ in terms of HWs in the forms

$$x(\sigma) = X\ \Psi_m(\sigma), \tag{11}$$

$$f(\sigma) = F\ \Psi_m(\sigma), \tag{12}$$

$$d_e(\sigma) = D_e\ \Psi_m(\sigma), \tag{13}$$

where $X \in \Re^{n\times m}$, $F \in \Re^{1\times m}$ and $D_e \in \Re^{1\times m}$ denote the wavelet coefficients of $x(\sigma)$, $f(\sigma)$ and $d_e(\sigma)$, respectively. The initial conditions of $x(0)$ and $\dot{x}(0)$ are also represented by $x(0) = X_0\ \Psi_m(\sigma)$ and $\dot{x}(0) = \overline{X}_0\ \Psi_m(\sigma)$, where the matrices $\{X_0, \overline{X}_0\} \in \Re^{n\times m}$ are defined, respectively, as

$$X_0 := \left[x(0)\ \ \underbrace{0_{n\times 1}\ \ \ldots\ \ 0_{n\times 1}}_{(m-1)} \right], \tag{14}$$

$$\overline{X}_0 := \left[\dot{x}(0)\ \ \underbrace{0_{n\times 1}\ \ \ldots\ \ 0_{n\times 1}}_{(m-1)} \right]. \tag{15}$$

Therefore, using the wavelet expansions (11)-(13), the relation (10) becomes

$$M(X - X_0)\Psi_m(\sigma) + \mathrm{T}_f\, C\, X \int_0^{\sigma} \Psi_m(\tau)d\tau + \mathrm{T}_f^2\, K\, X \int_0^{\sigma}\int_0^{\xi} \Psi_m(\tau)d\tau d\xi = \mathrm{T}_f^2\, B_f\, F \int_0^{\sigma}\int_0^{\xi} \Psi_m(\tau)d\tau d\xi$$
$$+ \mathrm{T}_f^2\, B_d\, D_e \int_0^{\sigma}\int_0^{\xi} \Psi_m(\tau)d\tau d\xi + (M\,\overline{X}_0 + \mathrm{T}_f C\, X_0) \int_0^{\sigma} \Psi_m(\xi)\, d\xi \tag{16}$$

Moreover, using the wavelet integral operational matrix P_m in Eq. (4), we can rewrite Eq. (16) as

$$M(X - X_0) + \mathrm{T}_f\, C\, X\, P_m + \mathrm{T}_f^2\, K\, X\, P_m^2 = \mathrm{T}_f^2\, B_f\, F\, P_m^2 + \mathrm{T}_f^2\, B_d\, D_e\, P_m^2 + (M\,\overline{X}_0 + \mathrm{T}_f C\, X_0)\, P_m \tag{17}$$

For calculating the matrix X, we apply the operator $vec(.)$ to Eq. (17) and according to the property of the Kronecker product, i.e. $vec(ABC) = (C^T \otimes A)\, vec(B)$, we have:

$$\begin{aligned}(I_m \otimes M)(vec(X) - vec(X_0)) + \mathrm{T}_f\,(P_m^T \otimes C)vec(X) + \mathrm{T}_f^2\,(P_m^{2T} \otimes K)vec(X) \\ = \mathrm{T}_f^2\,(P_m^{2T} \otimes B_f)vec(F) + \mathrm{T}_f^2\,(P_m^{2T} \otimes B_d)vec(D_e) \\ + \mathrm{T}_f\,(P_m^T \otimes C)vec(X_0) + (P_m^T \otimes M)vec(\overline{X}_0).\end{aligned} \tag{18}$$

Solving Eq. (18) for $vec(X)$ leads to

$$vec(X) = \Delta_1 vec(F) + \Delta_2\, vec(D_e) + \Delta_3\, vec(X_0) + \Delta_4\, vec(\overline{X}_0) \tag{19}$$

where the matrices $\{\Delta_1, \Delta_2\} \in \Re^{nm\times m}$ and $\{\Delta_3, \Delta_4\} \in \Re^{nm\times nm}$ are defined as

$$\begin{cases}\Delta_1 = \mathrm{T}_f^2\,(\mathrm{T}_f\,(P_m^T \otimes C) + \mathrm{T}_f^2\,(P_m^{2T} \otimes K) + I_m \otimes M)^{-1}(P_m^{2T} \otimes B_f) \\ \Delta_2 = \mathrm{T}_f^2\,(\mathrm{T}_f\,(P_m^T \otimes C) + \mathrm{T}_f^2\,(P_m^{2T} \otimes K) + I_m \otimes M)^{-1}(P_m^{2T} \otimes B_d) \\ \Delta_3 = (\mathrm{T}_f\,(P_m^T \otimes C) + \mathrm{T}_f^2\,(P_m^{2T} \otimes K) + I_m \otimes M)^{-1}(I_m \otimes M + \mathrm{T}_f\, P_m^T \otimes C) \\ \Delta_4 = (\mathrm{T}_f\,(P_m^T \otimes C) + \mathrm{T}_f^2\,(P_m^{2T} \otimes K) + I_m \otimes M)^{-1}(P_m^T \otimes M).\end{cases} \tag{20}$$

Consequently, using the properties of the Kronecker product, the solution of system (8) is

$$x(\sigma) = (\Psi_m^T(\sigma) \otimes I_n)\, vec(X), \tag{21}$$

and it is also clear that to find the approximated solution of the system, we have to calculate the inverse of the matrix $\mathrm{T}_f\,(P_m^T \otimes C) + \mathrm{T}_f^2\,(P_m^{2T} \otimes K) + I_m \otimes M$ with dimension $nm \times nm$ only once.

4. ROBUST OPTIMAL CONTROL DESIGN

The control objective is to find the approximated robust optimal control $f(t)$ with an H_∞ performance such $f(t)$ guarantees desired L_2 gain performance. Next, we shall establish the H_∞ performance of the system (8) under zero initial condition. To this end, we introduce

$$J = \tfrac{1}{2} x^T(\mathrm{T}_f) S_1\, x(\mathrm{T}_f) + \frac{1}{2}\dot{x}^T(\mathrm{T}_f) S_2\, \dot{x}(\mathrm{T}_f) + \tfrac{1}{2}\int_0^{T_f} (z^T(t) z(t) - \gamma^2 d_e^2(t))\, dt. \tag{22}$$

It is well known that a sufficient condition for achieving robust disturbance attenuation is that the inequality $J < 0$ for every $d_e(t) \in L_2[0, \infty)$ [14]. Therefore, we will establish conditions under which

$$\underset{vec(F)}{Inf}\ \underset{vec(D_e)}{Sup}\ J(vec(F), vec(D_e)) \le 0. \tag{23}$$

The Eq. (22) can be represented as

$$J = \tfrac{1}{2}(x^T(\mathrm{T}_f) \quad \dot{x}^T(\mathrm{T}_f))\tilde{S}\begin{pmatrix} x(\mathrm{T}_f) \\ \dot{x}(\mathrm{T}_f) \end{pmatrix} + \tfrac{1}{2}\int_0^{\mathrm{T}_f} ((x^T(t) \quad \dot{x}^T(t))\tilde{C}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + C_3^2\ f^2(t) - \gamma^2 d_e^2(t))\, dt \tag{24}$$

where $\tilde{S} = diagonal(S_1, S_2)$ and $\tilde{C} = diagonal(C_1^T C_1, C_2^T C_2)$. Normalizing (24) with the time scale $t = \mathrm{T}_f\ \sigma$ yields

$$\begin{aligned} J &= \tfrac{1}{2}(x^T(1) \quad \frac{1}{\mathrm{T}_f}\dot{x}^T(1))\tilde{S}\begin{pmatrix} x(1) \\ \mathrm{T}_f^{-1}\dot{x}(1) \end{pmatrix} \\ &+ {}^{\mathrm{T}_f}\!/_2 \int_0^1 ((x^T(\sigma) \quad \mathrm{T}_f^{-1}\dot{x}^T(\sigma))\tilde{C}\begin{pmatrix} x(\sigma) \\ \mathrm{T}_f^{-1}\dot{x}(\sigma) \end{pmatrix} + C_3^2\ f^2(\sigma) - \gamma^2 d_e^2(\sigma))\, d\sigma. \end{aligned} \tag{25}$$

Using the relation $\dot{x}(\sigma) = \bar{X}\ \Psi_m(\sigma)$, where $\bar{X} : n \times m$ denotes the wavelet coefficients of $\dot{x}(\sigma)$ after its expansion in terms of Haar wavelet basis functions, we read

$$\begin{bmatrix} x(\sigma) \\ \mathrm{T}_f^{-1}\dot{x}(\sigma) \end{bmatrix} = \begin{bmatrix} X \\ \mathrm{T}_f^{-1}\bar{X} \end{bmatrix}\Psi_m(\sigma) := X_{aug}\ \Psi_m(\sigma) \tag{26}$$

where $X_{aug} = \begin{bmatrix} X \\ \mathrm{T}_f^{-1}\bar{X} \end{bmatrix}$ and

$$vec(X_{aug}) = \left[vec^T(X) \quad \mathrm{T}_f^{-1} vec^T(\bar{X})\right]^T. \tag{27}$$

Moreover, the following relation is already satisfied between $vec(X)$ and $vec(\bar{X})$

$$vec(X) - vec(X_0) = (P_m^T \otimes I_n)\ vec(\bar{X}). \tag{28}$$

Therefore, we have

$$\begin{aligned} J &= \tfrac{1}{2}(tr(M_{mf}\ X_{aug}^T\ \tilde{S}\ X_{aug})) \\ &+ {}^{\mathrm{T}_f}\!/_2\ (tr(M_m\ X_{aug}^T\ \tilde{C}\ X_{aug}) + tr(C_3^2\ M_m\ F^T F) - \gamma^2\ tr(M_m\ D_e^T\ D_e)) \end{aligned} \tag{29}$$

where the matrices $\{M_m, M_{mf}\} \in \Re^{m \times m}$ are defined as $M_m := \int_0^1 \Psi_m(\sigma)\Psi_m^T(\sigma)\, d\sigma$ and $M_{mf} := \Psi_m(1)\Psi_m^T(1)$, respectively.

Using the property of the Kronecker products, we can write (29) as

$$J = \frac{1}{2}(vec^T(X_{aug})\Pi_{m1}\, vec(X_{aug}) + C_3^2\ vec^T(F)\Pi_{m2}\ vec(F) - \gamma^2\, vec^T(D_e)\Pi_{m2}\, vec(D_e)) \quad (30)$$

where the matrices $\Pi_{m1} \in \Re^{2nm \times 2nm}, \Pi_{m2} \in \Re^{m \times m}$ are defined as $\Pi_{m1} = M_{mf} \otimes \tilde{S} + {}^{T_f}\!/_2\,(M_m \otimes \tilde{C})$ and $\Pi_{m2} = {}^{T_f}\!/_2\, M_m$, respectively.

It is easy to show that the worst-case disturbance in Eq. (29) occurs when

$$vec^*(D_e) = \gamma^{-2}\,\Pi_{m2}^{-1}\left\lfloor \Delta_2^T \quad \mathrm{T}_f^{-1}\,\Delta_2^T(P_m^{-1} \otimes I_4)\right\rfloor \Pi_{m1}\ vec(X_{aug}) := \gamma^{-2}\,\Pi_{md}\ vec(X_{aug}). \quad (31)$$

By substituting Eq. (31) into Eq. (30) we obtain

$$\underset{vec(F)}{Inf}\ \underset{vec(D_e)}{Sup}\ J(vec(F), vec(D_e)) = \underset{vec(F)}{Inf}\ J(vec(F), vec^*(D_e)). \quad (32)$$

Minimizing the right-hand side of Eq. (32) results in the algebraic relation between wavelet coefficients of the robust optimal control and of the optimal state trajectories in the following closed form

$$\begin{aligned} vec(F) &= -C_3^{-2}\Pi_{m2}^{-1}\left\lfloor \Delta_1^T \quad \mathrm{T}_f^{-1}\,\Delta_1^T(P_m^{-1} \otimes I_4)\right\rfloor (\Pi_{m1} - \gamma^{-2}\,\Pi_{md}^T\,\Pi_{m2}\,\Pi_{md})\ vec(X_{aug}) \\ &:= \Pi_{mf}\ vec(X_{aug}). \end{aligned} \quad (33)$$

As a result we have

$$\begin{aligned} &\underset{vec(F)}{Inf}\ \underset{vec(D_e)}{Sup}\ J(vec(F), vec(D_e)) \\ &\le vec^T(X_{aug})\,(\Pi_{m1} + R\,\Pi_{mf}^T\,\Pi_{m2}\,\Pi_{mf} - \gamma^2\,\Pi_{md}^T\,\Pi_{m2}\,\Pi_{md})vec(X_{aug}). \end{aligned} \quad (34)$$

Consequently, if there exists positive scalar γ to the matrix inequality

$$\Pi_{m1} + C_3^2\,\Pi_{mf}^T\,\Pi_{m2}\,\Pi_{mf} - \gamma^2\,\Pi_{md}^T\,\Pi_{m2}\,\Pi_{md} \le 0 \quad (35)$$

then inequality (23) is concluded.

From the relations (28), (31) and (33) we obtain the robust optimal vectors of $vec(X)$ and $vec(F)$ after some matrix calculations, respectively, in the following forms

$$\begin{aligned} vec(X) = (I_{4m} - (\Delta_1\,\Pi_{mf} + \gamma^{-2}\Delta_2\,\Pi_{md})\begin{bmatrix} I_{4m} \\ \mathrm{T}_f^{-1}(P_m^T \otimes I_4)^{-1}\end{bmatrix})^{-1}((\Delta_3 - (\Delta_1\,\Pi_{mf} + \gamma^{-2}\Delta_2\,\Pi_{md}) \\ \times\begin{bmatrix} 0_{4m} \\ \mathrm{T}_f^{-1}(P_m^T \otimes I_4)^{-1}\end{bmatrix})\ vec(X_0) + \Delta_4\ vec(\overline{X}_0)), \end{aligned} \quad (36)$$

and

$$\begin{aligned} vec(F) = \Pi_{mf}\ \{(\begin{bmatrix} I_{4m} \\ \mathrm{T}_f^{-1}(P_m^T \otimes I_4)^{-1}\end{bmatrix}((I_{4m} - (\Delta_1\,\Pi_{mf} + \gamma^{-2}\Delta_2\,\Pi_{md})\begin{bmatrix} I_{4m} \\ \mathrm{T}_f^{-1}(P_m^T \otimes I_4)^{-1}\end{bmatrix})^{-1} \\ \times(\Delta_3 - (\Delta_1\,\Pi_{mf} + \gamma^{-2}\Delta_2\,\Pi_{md})\begin{bmatrix} 0_{4m} \\ \mathrm{T}_f^{-1}(P_m^T \otimes I_4)^{-1}\end{bmatrix}) - \begin{bmatrix} 0_{4m} \\ \mathrm{T}_f^{-1}(P_m^T \otimes I_4)^{-1}\end{bmatrix})\ vec(X_0) \\ + \begin{bmatrix} I_{4m} \\ \mathrm{T}_f^{-1}(P_m^T \otimes I_4)^{-1}\end{bmatrix}(I_{4m} - (\Delta_1\,\Pi_{mf} + \gamma^{-2}\Delta_2\,\Pi_{md})\begin{bmatrix} I_{4m} \\ \mathrm{T}_f^{-1}(P_m^T \otimes I_4)^{-1}\end{bmatrix})^{-1}\Delta_4\, vec(\overline{X}_0))\}. \end{aligned} \quad (37)$$

Finally, the Haar wavelet-based robust optimal trajectories and robust optimal control are obtained approximately from Eq. (21) and $f(t) = \Psi_m^T(t)\, vec(F)$, respectively.

5. CONCLUSION

This paper investigated a computational methodology to solve the problem of H_∞ control design for second-order linear systems using Haar wavelets. The main focus was on development of an H_∞ controller for the system under consideration computationally that guarantees both the closed-loop stability and satisfies a desired H_∞ performance. The Haar wavelet properties were utilized to find the approximate solutions of trajectories and robust optimal control by solving only algebraic equations instead of solving the Riccati differential equation.

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