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ON THE UNSTEADY LIFT OF FLAPPING WING-A MATHEMATICAL APPROACH

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Abstract: This theoretical paper discusses recent advances in the fluid dynamics of insect flight and considers theoretical analyses necessary for their future development. Theoretically, the fluid dynamic description is based on: (i) the superposition of the unsteady contributions of wing pitching, plunging and sweeping; and (ii) adding corrections due to the bound leading-edge vortex and wake distortion. Viscosity is accounted for indirectly by imposing the Kutta condition on the trailing edge and including the influence of the vertical structure on the leading edge. In this paper, mathematically, an analytic approach is proposed. It derives all the quantities of interest from the notion of circulation and leads to tractable integral equation. This is an application of the von Karman-Sears unsteady wing theory and its nonlinear extensions due to McCune and Taveres; the latter can account for the bound leading-edge vortex and wake distortion. The paper also discusses connections of the proposed analytical approach with aeroelasticity.

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1. INTRODUCTION

The fluid mechanics of insect flight has a rich and most interesting history [1]. Among other things, its study has helped to discover sophisticated mechanisms for lift generation as that of Weis-Fogh [2, 3] and the paring of downward moving two-dimensional vortices [4]. Apart from its biological interest, the understanding of the details of insect flight has also technological interest, for instance, for the development of flying microvehicles [5-7]. In particular, of considerable importance are themechanisms associated to hovering at a fixed position. For some insect such as small wasps, hovering seems to depend strongly on the interaction of two wings, as in the Weis-Fogh mechanism [2]. For many other insects, however, single wing effects are the most important [8]. In the past few years detailed experiments with scaled insect model wing were performed, in which a careful determination was made of forces resulting form imposed complex movements [9, 10].

Micro air vehicles (MAVs) are defined as flying vehicles of approximately six inches in size (hand held) and are developed to reconnoitre in confined spaces (inside buildings, tunnels, etc.). This requires power-efficient, highly manoeuvrable, low-speed flight with stable hover. Such performance is routinely exhibited by flying insects, hence the focus on emulating insect-like flapping by engineering means.

From both the insect-flight-analysis and MAV-design perspectives there is a need for an analytic framework for aerodynamic modelling of flapping wings. It should offer qualitative and quantitative interpretations of the main phenomena involved, while avoiding the extremes of mathematical oversimplification and intractable complexity. This problem is the main motivation for the developments presented here.

The flow involved in insect flight is incompressible, laminar, unsteady and occurs at low Reynolds numbers. However, our understanding of the resulting aerodynamics is incomplete even on the phenomenological level. Not only is the qualitative picture unfinished, but also the quantitative analysis is wanting. In fact, the few mathematical approaches attempted involve either simple algebra or advanced computational fluid dynamics (CFD). The CFD approach has not produced very satisfactory results yet, due to the complicated kinematics of wing motion and the inadequacy of experimental data for full verification. Also, the CFD route is very expensive in terms of code development and running and therefore cannot be used as the main analytical and/or designtool.

This theoretical paper discusses recent advances in the fluid dynamics of insect flight and considers theoretical analyses necessary for their future development. A new conceptual framework is proposed as is, within this framework, an analytic approaches to aerodynamic modelling of an insect-like flapping wing in hover in the context of MAVs. It is assumed that, the wing is thin, rigid and of symmetrical section, while the flow is incompressible, of low Reynolds number and Laminar.

The paper starts with a summery of the basics of insect flight in the section 2, focusing on the wing kinematics, and the observed aerodynamic phenomena. Their interpretation and proposed mathematical modeling and are described in the section 3. Mathematically, an analytical approach is out lined. It is based on the von karman-sears unsteady wing theory and its nonlinear extensions due to McCune and Tavares and leads to tractable integral equation.

2. BASICS OF INSECT FLIGHT

This section summarizes the basics of insect flight focusing on two issues: (1) flapping wing kinematics, and (2) the main aerodynamic phenomena involved.

2.1 Wing Kinematics

Insects fly by oscillating (plunging) and rotating (pitching) their wings through large angles, while sweeping them forwards and backwards. The wing beat cycle (typical frequency range

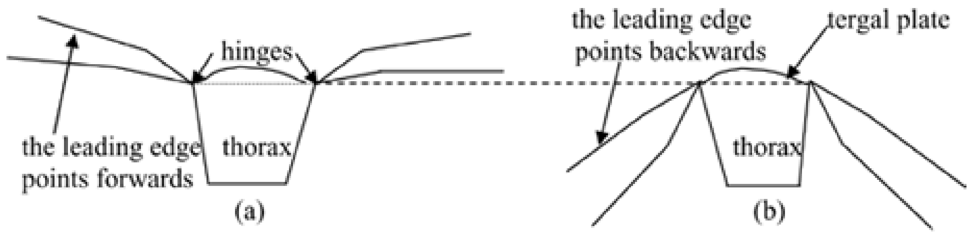

Figure 1: Insect-wing flapping is a Periodic Motion and Each Cycle is Composed of the Downstroke and Upstroke. The Leading Edge Always Leads, Irrespective of the Direction of Wing Motion: (a) Beginning of Downstroke and (b) Beginning of Upstroke

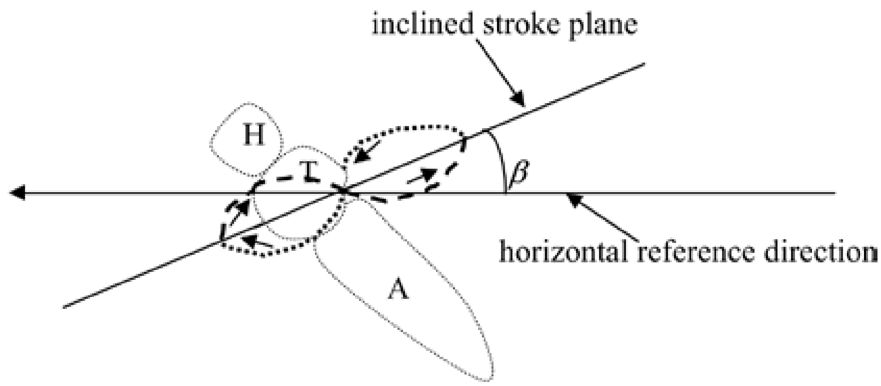

Figure 2: Typical Motions of an Insect Wing in Hover. The Insect Body is Orientated Almost Vertically, While the Wing Tip Traces a Flat Figure of Eight Around the Stroke Plane. The Stroke Plane is inclined by the Angle $\beta \approx 15°$. The Dashed Line Represents the Upstroke nd the Dotted Line Represents the Downstroke; 'H' is the Head, 'T' is the Thorax and 'A' is the Abdomen

of 5-200 Hz) can be divided into two phases: downstroke and upstroke (see Figures 1 and 2). At the beginning of downstroke, the wing (as seen from the front of the insect) is in the uppermost and rearmost position with the leading edge pointing forward. The wing is then pushed downwards (plunged) and forwards (swept) and rotated (pitched) continuously with considerable change of the angle of attack. At the end of the downstroke, the wing is twisted rapidly, so that the leading edge points backwards and the upstroke begins. During the upstroke the wing is pushed upwards and backwards and rotated again, which changes the angle of attack throughout this motion. At the highest point, the wing is twisted again, so that the leading edge points forward and the next downstroke begin.

Insect-wing flapping occurs in a stroke plane that generally remains at the same orientation to the body and is either horizontal or inclined, see Figure 2. In forward flight the downstroke lasts longer than the upstroke, because of the need to generate thrust. In hover they are equal, resulting in the wing-tip tracing a flat figure of eight (as seem from the insect's side).

Since each half-cycle starts from rest and comes to a stop, the velocity distribution in non-uniform. In hover, the motion of the wing tip seems to be adequately described by the first three harmonics, so does not differ dramatically from a purely sinusoidal motion [13].

2.2 Main Aerodynamic Phenomena in Insect Flight

The kinematics of insect wings makes the analysis of the associated aerodynamics a non-trivial task, not yet completed, especially in terms of its mathematical description. The classical approach was based on the quasi-steady assumption that the instantaneous forces on the flapping wing area equivalent to those for steady motion at the same instantaneous velocity and angle of attack. However, Ellington in his seminal work [11-16] showed that this framework is inadequate to explain the high lift generated by insects, especially in hover (underestimated by a factor of three).

Ellington concluded that unsteady aerodynamics must be involved; however, the nature of the unsteadiness was not clear. Further difficult experimental work by Ellington *et al.*, followed [17, 18] and led to the remarkable discovery of a spiraling leading-edge vortex in a large insect. This is a bound vortex, its position on the wing remains constant during a half-cycle, despite the wing's pitching, plunging and sweeping, while its size fluctuates. Inside the vertical structure, spanwise flow (along the leading edge, from the wing base to the tip) was observed, an apparent cause of spiralling out of the vortex. In hover, at the end of the downstroke the vortex is shed by a sudden wing twist and a new one is created symmetrically during the upstroke and shed when the wing flips again.

This persisting leading-edge vortex was discovered through three-dimensional flow visualization for a tethered hawkmoth *Manduca sexta* [19] and confirmed with a better resolution on an aerodynamically scaled, large, mechanical model of the hawkmoth [20, 21], powered by electric servomotors. Recent experiments on a mechanical model of the fruit fly Drosophila melanogaster wing [9] seem to suggest that a bound leading-edge vortex also occurs in smaller insects. However, the spanwise spiralling out, detected by Ellington et al. for the hawkmoth, was not observed.

As explained previously, insect-wing kinematics involve two translational phases (upstroke and downstroke) and two turning phases (at the end of each half stroke) when the wings rapidly reverse direction. Dickinson *et al*., [9] pointed out that the leading-edge vortex is a plausible translational mechanism. However, contributions to the lift from wing reversal and the wing's interaction with the flow pattern of the previous stroke are yet to be explained.

3. PROPOSED MATHEMATICAL MODELLING

Based on the discussion of insect flight kinematics and aerodynamics in previous section, the tentative conclusion, adopted here, is that the main likely aerodynamic phenomena occurring in insect flapping are:

(i) bound leading-edge vortex, persisting during each half-cycle and shed at the end of it,

(ii) effects (other than the vortex) of wing pitching, plunging and sweeping present all the time, and

(iii) wing interaction with its own convected wake (caused by previous wingbeats) due to its forward-backward sweeping (re-entering the wake).

The flow is assumed incompressible, of low Reynolds number and laminar, while the wing is treated as rigid, thin and of symmetrical section. These postulates have good support in experimental observation of insect flight, with the exception of wing rigidity. Because the flow is laminar, it is susceptible to separation and it is hypothesized here that insects deliberately provoke separation at the leading edge to exploit the vertical lift thus obtained. It is also postulated that no further separation occurs during each half-cycle and that the vortex is shed at the end of it, due to a sudden wing flip. Hence, we will focus to interpret phenomenon (i) as accounting for the separated part of the flow, while treating (ii) as responsible for the attached part of the flow interacting with (iii), no interaction between (i) and (iii).

The vortical lift due to (i) is interpreted as essentially identical to the leading-edge vortex on sharp-edged delta wings. It should be emphasized that it is a nonlinear phenomenon [22], because of the interaction of viscosity and the velocity field which result in separation of the boundary layer and rolling up of the separated vortex sheet into a spiral vortex. In every half-cycle on insect flapping, α (the angle of attack) increases continually well above $20°$ and the leading-edge vortex is still bound. This is because separation occurs at the beginning of the motion and keeps generating stable vortical lift throughout the half-cycle of the motion. Over a half-cycle, it is not a transient phenomenon leading to a catastrophic loss of lift. Moreover, this controlled separation is localized at the leading edge and occurs nowhere else on the wing, so that the rest of the wing flow is attached. Therefore, it seems plausible to assume that the classical unsteady thin aerofoil theory can be used to interpret (ii), interacting with (iii), despite the large angles of attack involved. However, wake distortion may be present and care must be taken to include such effects when describing the interaction.

The non-vortical part of the flow will be treated using the unsteady aerodynamics methods for helicopter blades in attached flow. Due to the flow attachment, this theory is linear and inviscid (with the Kutta condition imposed on the trailing edge). This means that the contributions of each type of the motion and the interaction with the wake can be treated separately and then superimposed.

On the other hand, the bound leading-edge vortex is a nonlinear phenomenon and will be treated as such. Special methods, different from thin aerofoil theory, must be used, because of the viscous character of the vertical structure. The computed contribution of the vortex will provide a nonlinear correction to the inviscid results for pitching, plunging and sweeping motions. In this way, the whole flow can be handled with transparent interpretation of its components and analytic formulae for each of them. One actual realization of the above programme is sketched below.

It should be mentioned that some of the analytical tools used below were alluded to by Ellington in his seminal work from 1984 [11], and also in his more recent work [6, 21]. Similar hints can also be found in [23].

3.1 Circulation and Induced Velocity

The methodology outlined here is a generalization of the classical, linear, thin aerofoil theory to unsteady wing motion. This theory assumes a wing of negligible thickness and camber which are valid for most standard wings at small angles of attack. Insect wings are made of round spars spanned by a membrane whose thickness is much smaller than the diameter of the spars. In fact, they can be reasonably approximated by a thin flat plate with a cylinder at the leading edge, acting as a trip.

In the developments below the following convention will be used. Let the aerofoil chord length be c and the semi-chord $b = c/2$. The origin of the reference line is placed at the mid-point of the chord, so that the leading edge is given by $x = -b$ and the trailing edge by $x = b$. Finally, the right-hand rule convention is followed for both the vorticity and the circulation, as in McCune & Tavares (1993); this is different from von Karaman & Sears (1938).

In the steady theory the camber line is replaced by a line of vorticity which varies along the aerofoil, but not in time. This is expressed by the dependence of the chordwise vorticity distribution γ_a on the coordinate x along the reference line, but not the time t, i.e. $\gamma_a = \gamma_a(x)$ with $-b \le x \le b$. Hence, the total circulation about the chord is the sum of the vortex elements

$$\Gamma_a = \int_{-b}^{b} \gamma_a(z)\, dz \tag{3.1}$$

and is constant both in space and time. The induced velocity at point on the aerofoil is given by

$$v(x) = \frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma_a(z)}{z-x}\,dz\,. \tag{3.2}$$

3.2 Von Karman-Sears Theory and Its Nonlinear Extentions

An important consequence of aerofoil motion is the presence of vorticity in the wake and its influence on the aerofoil vorticity. For unsteady flow [24, 25] both the vorticity distribution bound to the aerofoil γ_a and its wake counterpart γ_w are time-varying, i.e. $\gamma_a = \gamma_a(x, t)$ with $-b \le x \le b$, and $\gamma_w = \gamma_w(x, t)$ with $b\, x \le R(t)$, where $R = R(t)$ is a time-varying wake extent. The wake is interpreted as arising due to the aerofoil's motion, which results in shedding the aerofoil's line of vorticity from the trailing edge. The unsteady Kutta condition at the trailing edge is

$$\gamma_a(b, t) \equiv 0, \tag{3.3}$$

which is shorthand for $\gamma_a(b, t) \equiv 0$ all t. Since the flow is assumed inviscid, the total circulation of the aerofoil-wake system must be equal to its initial value $\Gamma = const$. This means that the point vortices of the aerofoil and the wake must form vortex pairs, so that the computation of γ_a and γ_w is interdependent. In other words, vorticity in the wake influences aerofoil vorticity and hence its circulation.

The total circulation about the chord will now be time-varying, $\Gamma_a = \Gamma_a(\mathrm{t})$, and so will be the induced vorticity at point x on the aerofoil:

$$v(x, t) = \frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma_a(z,t)}{z-x}\,dz + \frac{1}{2\pi}\int_{b}^{R(t)}\frac{\gamma_w(z,t)}{z-x}\,dz \tag{3.4}$$

Where the wake influence is expressed by the second term, see equation (3.2).

Thus, the vorticity distribution bound to the aerofoil γ_a will be a superposition of two contributions:

$$\gamma_a(x, t) = \gamma_0(x, t) + \gamma_1(x, t) \tag{3.5}$$

(The superposition expresses linearity of the theory.) Here γ_0 is the quasi-steady part (existing when no wake is present) and γ_1 is induced by the wake. The total circulation of the aerofoil-wake system must be zero, so that

$$\Gamma_a(t) + \Gamma_w(t) = \Gamma = const \tag{3.6}$$

where

$$\Gamma_a(t) = \Gamma_0(t) + \Gamma_1(t) \tag{3.7}$$

see equation (3.5). In the right-hand side of (3.7),

$$\Gamma_0(t) = \int_{-b}^{b} \gamma_0(z,t)\, dz \tag{3.8}$$

where γ_0 is obtained by considering for each t the corresponding steady flow in the absence of the wake, see equation (3.1), and

$$\Gamma_1(t) = \int_{b}^{R(t)} \gamma_w(z,t)\left(\sqrt{\frac{z+b}{z-b}} - 1\right) dz \tag{3.9}$$

Finally, the wake circulation is simply

$$\Gamma_w(t) = \int_{b}^{R(t)} \gamma_w(z,t)\, dz \tag{3.10}$$

Collecting equations (3.6)-(3.10) yields the linear integral equation [26]

$$\int_{-b}^{b} \gamma_0(z,t)\, dz - \Gamma = \int_{b}^{R(t)} \gamma_w(z,t)\sqrt{\frac{z+b}{z-b}}\, dz \tag{3.11}$$

For a specified unsteady motion and initial conditions, the left-hand side of (3.11) is readily computed, so that the problem reduces to finding the wake vorticity distribution $\gamma_w(x, t)$ on $b < x \le R(t)$, consistent with (3.11).

The above reasoning yields a transparent formula for unsteady lift,

$$L(t) = L_0(t) + L_1(t) + L_2(t) \tag{3.12}$$

Where, with $V = V(t)$ being the time-varying incident velocity accounting for wing sweeping [25],

$$L_0(t) = -\rho V(t)\,\Gamma_0(t) \tag{3.13}$$

$$L_1(t) = \rho \frac{d}{dt}\left(\int_{-b}^{b} z\gamma_0(z,t)\, dz\right) \tag{3.14}$$

$$L_2(t) = -\rho V(t)\, b \frac{d}{dt} \int_{b}^{R(t)} \frac{\gamma_w(z,t)}{\sqrt{z^2 - b^2}}\, dz \tag{3.15}$$

Here the right-hand rule convention for the circulation was used, as [27].The term L_0, given by equation (3.13), is the quasi-steady contribution (see equation (3.8)). Equation (3.14) for L_1 expresses the apparent mass contribution and does not depend on the incident velocity V, unlike L_0 and L_2. This term exists even if the aerofoil executes its motion without producing circulation, i.e. when for all t there is no quasi-steady lift, $\gamma_0(x, t) = 0$, and no wake, $\gamma_w(x, t) = 0$. (Note that $\gamma_0(x, t) = 0$ for all t does not nullify L_1, because the integral in

(3.14) is differentiated with respect to *t*). This non-circulatory contribution is due to the inertia of the fluid accelerated by the aerofoil motion and is computed from the unsteady Bernoulli equation [28]. It is interesting to note that it is possible that during certain phases of the wing motion its lift is increased by L_1, corresponding to extraction of the energy from the flow (due to the reaction of the surrounding fluid). This would give support to the view expressed by Ennos that in flies the kinematics is helped by aerodynamics [29], the motion need not be forced all the time. Indeed, the corresponding aerodynamic moment is

$$M_1 = \frac{\rho}{2}\frac{d}{dt}\left(\int_{-b}^{b}(z^2 - b^2/2)\right)\gamma_0(z,t)\,dz \tag{3.16}$$

and acts at the quarter chord [26].

Finally, the term L_2, given by (3.15), is the wake-induced contribution. In particular, if the aerofoil experiences an impulsive start, then no fluid mass is accelerated yet, so $L_1 = 0$. However, the wing does not immediately attain its full steady value, which would be expressed by L_0 alone. This gradual lift increase is precisely the influence of the wake and is a result of the vortex pairs mentioned at the beginning of this section. It also gives a clear interpretation of the existenceof the starting vortex in impulsive motion, also known as the Wagner effect [30].

If this general theory is applied to an aerofoil in pitching or plunging oscillations, or starting (stopping) impulsively, or entering a gust, the integrals can be evaluated in closed form [25]. Since the theory is linear, any superposition of these is also allowed. Hence, this approach can handle insect flapping kinematics (including the wing starting and stopping) if wing interaction with its own convected wake is interpreted as a gust encounter [25]; the pressure distribution can also be computed [31]. Thus, analytic formulae for the non-vortical part of the flow can be obtained, provided the wake is planar and moves at the incident velocity *V*.

McCune *et al.*, [32] proposed an extension of the von Karaman-sears linear theory to the situation when the wake can be deformed and rolled up, while maintaining the transparency and spirit of the classical approach. In essence, it is a nonlinear version of integral equation (3.11), so that a more involved wake influence can be accounted for. In terms of lift the new formula is

$$L(t) = L_0^n(t) + L_1^n(t) + L_2^n(t) + \overline{L}(t) \tag{3.17}$$

Where the first three terms correspond conceptually to L_0, L_1 and L_2 in (3.12) and their actual expressions [27] do not differ much from (3.13)-(3.15). However, the term $\overline{L}$ is new and is a nonlinear correction which vanishes if the linear theory assumptions are restored. It is also given by a simple integral expression, but cannot be evaluated in closed form.

Instead, it must be integrated numerically, which poses no real algorithmic difficulties and can be done quickly [32].

McCune's nonlinear generalization of the von Karman-sears approach enabled Tavares [33] to apply a similar approach to a delta wing in unsteady motion. In particular, he was able to analyse the leading-edge vortex contribution. This opens a way to accountfor the vortical part of the flow in insect flapping. Thus, the circulation approach, based on the von Karman-Sears linear theory and its nonlinear extensions due to McCune and Tavares, offers a coherent, insightful and mathematically tractable method ofthe flow analysis. The formulae involved are either in closed form or require simple numerical evaluation; the latter can be done quickly and efficiently.

4. FURTHER REMARKS AND CONCLUSIONSG

The model described above is valid under the assumption of wing rigidity. However, in insects aeroelasticity is significant, as can be seen from photographs [34]. In fact, their remarkable manoeuvrability is enabled by active control of the three-dimensional shape of the wings during the beat cycle. This is achieved by deformability of the wings; the observed patterns of deformation (both active and passive) include torsion, camber change and transverse bending.

In principle, if for a wing element at a point x the corresponding time-varying (unsteady) aerodynamic loading F is available (see, for example, Theodorsen's formula for lift [36]), then the generic equation

$$m\ddot{x} + c\dot{x} + kx = F(x, \dot{x}, \ddot{x}, t) \tag{4.1}$$

Should account for the structural response and its influence on the loading. This is because x appears on both sides of equation (4.1), so that there is constant feedback between the inertia $m\ddot{x}$, damping $c\dot{x}$, elasticity kx and theaerodynamic force $F(x, \dot{x}, \ddot{x}, t)$. However, for large deformations (observed in insects) the parameters may depend on x and/or $\dot{x}$, so that equation (4.1) is no longer linear. Even if it is, then obtaining realistic values for m, c and k may be difficult experimentally. However, when accurate measurements of x, $\dot{x}$ and $\ddot{x}$ are available and $F(x, \dot{x}, \ddot{x}, t)$ is known, then finding parameters m, c and k is a standard estimation problem [36]. This is because the left-hand side of (4.1) is linear in these parameters, so the linear least-squares framework is applicable.

Design and construction of flapping MAVs inspired by flying insects is a nontrivial task, further complicated by the lack of in-depth physical and mathematical understanding of insect flight. Progress in fundamental analysis is unlikely to be quick, if only because of considerable experimental difficulties in investigation of insects in free flight. However, it is possible to move forward in the engineering design by adopting a plausible, analytic framework and this was the main motivation for this paper.The history of aeronautics, and in particular of helicopter technology [35], gives some credence to this approach. When faced with cumbersome complexity, the engineer has to simplify judiciously and his design is then a spur to further fundamental analyses, which, in turn, lead to valuable insights producing a better design.

It should be emphasized that despite the impressive progress in the insect aerodynamics research, must remains to be done. Especially needed are accurate kinematic data and instantaneous force measurements on the whole wing. These would have to be done ona multipoint grid to provide a field of positions, velocities and accelerations (forces), as opposed to the observations of the wing tip and mean forces at a single point. Finally, the data available so far come either from tethered insects (when they donot fly naturally) and/or approximate mechanical models and thus cannot be considered definitive. Detailed information from free-flying insects would be ideal but is very difficult to obtain experimentally.

For insect-like flapping-wing MAVs, both the analysis from first principles and the synthesis of workable prototypes should go together. In this paper, we have tried to show, for the example of mathematical/aerodynamic modelling, that there is a considerable body of engineering knowledge this process can draw upon in a systematic and transparent way. Verification of the proposed method is currently in progress.

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