International Journal of Mathematical Sciences
June 2002, Volume 1, No. 1, pp. 95–107

GENERALIZED (F, ρ) - CONVEXITY AND DUALITY THEOREM FOR NONDIFFERENTIABLE PROGRAMS INVOLVING SQUARE ROOT TERMS

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ABSTRACT

A Multi Objective Programming Problem in which each of the objective function is the sum of a nondifferentiable function and a term involving square root of a positive semi-definite quadratic form. Duality results are proved under (F, ρ)- convexity assumptions on functions involved. Later fractional versions of above problem are studied with different dual problems and duality theorem are proved.

1. INTRODUCTION

Bhatia and Jain [7] established duality results under F - convexity assumptions for a scalar nonlinear program of which the objective function is the sum of a nondifferentiable function and a term involving square root of a positive semi–definite quadratic form. The idea is extended to the following multi objective programming problem:

(P) Minimize $\theta(x) = \left(f_1(x) + (x^t B_1 x)^{1/2}, f_2(x) + (x^t B_2 x)^{1/2} + ... f_k(x) + (x^t B_k x)^{1/2}\right)^t$

subject to

$$g_j(x) \leqq 0, \qquad j = 1, 2, ..., m$$

$$x \in X$$

where X is an open convex subset of $\mathbf{R}^n$; f_i, i = 1, 2, . . ., k ; g_j, j = 1,2,...,m are real valued functions defined on X and B_i, i = 1, 2, . . ., k are n ´ n symmetric positive semi–definite matrices.

Mond Weir type dual to (P) is introduced and duality results are established under (F, ρ)– convexity assumptions. These results are then extended for the following multi objective fractional program (FP):

(FP) Minimize $\dfrac{\theta(x)}{h(x)} = \left(\dfrac{f_1(x) + (x^t B_1 x)^{1/2}}{h_1(x)}, ..., \dfrac{f_k(x) + (x^k B_k x)^{1/2}}{h_k(x)}\right)^t$

subject to

$$g_j(x) \leqq 0, \quad j = 1, 2, \ldots, m$$

$$x \in X$$

where f_i, h_i, $i = 1, 2, \ldots, k$; g_j, $j = 1, 2, \ldots, m$ are real valued functions defined on an open convex subset X of $\mathbf{R}^n$ with $f_i(.) \geqq 0$, $h_i(.) > 0$; $i = 1,2,\ldots,k$; B_i, $i = 1, 2, \ldots, k$ are $n \times n$ symmetric positive semi–definite matrices.

Assumption (A)

The convex sets ri (dom f_i); ri (dom $(x^t B_i x)^{1/2}$), $i = 1, 2, \ldots, k$ have a point in common so that

$$\partial(f_i(x) + (x^t B_i x)^{1/2}) = \partial f_i(x) + \partial(x^t B_i x)^{1/2} \quad \forall x \text{ and } \forall i = 1, 2, \ldots, k.$$

Assumption (B)

We assume the following constraint qualification of Slater's type:

(i) Let x^0 be an efficient solution of (P). For each $r \in \{1, 2, \ldots, k\}$, suppose that there exists $x^r \in X$ such that

$$g_j(x^r) < 0 \qquad \forall j = 1,2, \ldots, m \text{ and}$$

$$\theta_i(x^r) < \theta_i(x^0) \qquad \forall i \neq r,$$

where $\theta_i(x) = f_i(x) + (x^t B_i x)^{1/2}, \qquad i = 1, 2, \ldots, k.$

(ii) Let x^0 be an efficient solution of (FP). For each $r \in \{1, 2, \ldots, k\}$, suppose that there exists $x^r \in X$ such that

$$g_j(x^r) < 0 \qquad \forall j = 1, 2, \ldots, m; \qquad \text{and}$$

either

$$\theta_i(x^r) < \theta_i(x^0) \qquad \forall i \neq r$$

$$h_i(x^r) > h_i(x^0) \qquad \forall = 1, 2, \ldots, k$$

or

$$\theta_i(x^r) < \theta_i(x^0) \qquad \forall = 1, 2, \ldots, k$$

$$h_i(x^r) > h_i(x^0), \qquad i \neq r$$

$$(\theta_i(x) = f_i(x) + (x^t B_i x)^{1/2}, \qquad i = 1, 2, \ldots, k)$$

Kanniappan [11] established that the following Kuhn Tucker type conditions (correct version by B. Lemaire (Math. Reviews # 90150, 1984)) and Fritz John type conditions are necessary for x^0 to be an efficient solution of (P).

Theorem A (Kuhn Tucker type necessary conditions)

If x^0 is an efficient solution of (P) and if we assume the above constraint qualification of Slater's type (B) (i), then there exist $\alpha^0 = (\alpha_1^0, \alpha_2^0, ..., \alpha_k^0) \in \mathbf{R}^k$ and $\lambda^0 = (\lambda_1^0 \lambda_2^0 ... \lambda_m^0) \in \mathbf{R}^m$, such that

$$\lambda_j^0 g_j(x^0) = 0, \quad j = 1, 2, \ldots, m,$$

$$0 \in \sum_{i=1}^{k} \alpha_i^0 \partial\theta_i(x^0) + \sum_{j=1}^{n} \lambda_j^0 \partial g_j(x^0) + N_X(x^0)$$

$$\alpha_i^0 > 0, \quad i = 1, 2, \ldots, k; \ \lambda_j^0 \geqq 0, \ j = 1, 2, \ldots, m.$$

Theorem B (Fritz John type necessary conditions)

If x^0 is an efficient solution of (P), then there exist $\alpha^0 = (\alpha_1^0, \alpha_2^0, ..., \alpha_k^0) \in \mathbf{R}^k$, $\lambda^0 = (\lambda_1^0, \lambda^0, ..., \lambda_m^0) \in \mathbf{R}^m$, such that

$$\lambda_j^0 g_j(x^0) = 0, \quad j = 1, 2, \ldots, m$$

$$0 \in \sum_{i=1}^{k} \alpha_i^0 \partial\theta_i(x^0) + \sum_{j=1}^{m} \lambda_j^0 \partial g_j(x^0) + N_X(x^0),$$

$$\left(\alpha^0, \lambda^0\right) \geq 0 .$$

The following results are needed in the sequel.

Lemma 1 [10]

Let $\phi(x) = (x^t B x)^{1/2}$, B is a positive semi–definite symmetric matrix. Then $\phi(x)$ is convex and $w \partial \phi(x)$ if and only if $w = Bz$, $z^t B z \leqq 1$, $x^t B z = (x^t B x)^{1/2}$.

Lemma 2 [7]

Let $\phi(x) = (x^t B x)^{1/2}$. Then $\phi(x)$ is locally Lipschitz.

Throughout the paper, we shall assume that f_i, $-h_i$; i = 1, 2, . . ., k and g_j, j = 1, 2, . . ,m are Lipschitz and regular, and the set X is an open convex subset of $\mathbf{R}^n$.

Mond Weir type dual for the problem (P) is

(D) Maximize $H(u, \alpha, \lambda, z) = (f_1(u) + u^t B_i z_i, \ \ldots, \ f_k(u) + u^t B_k z_k)^t$

subject to

$$0 \in \sum_{i=1}^{k} \alpha_i (\partial f_i(u) + B_i z_i) + \sum_{j=1}^{m} \lambda_j \partial g_j(u) + N_X(u) \tag{1}$$

$$\lambda_j g_j(u) \geqq 0 \ , \quad j = 1, 2, \ldots, m \tag{2}$$

$$z_i^t B_i z_i \leqq 1 \ , \quad i = 1, 2, \ldots, k \tag{3}$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_k, \lambda_1, \lambda_2, \ldots, \lambda_m) \geqq 0 \tag{4}$$

Theorem 1 (Weak Duality)

Let x be feasible for (P) and (u, α, λ, z) be feasible for (D). Assume that the functions f_i, i = 1, 2, …, k ; g_j, j = 1, 2, . . .,m ; $(. - x)^t B_i z_i$ ($\forall z_i \in \mathbf{R}^n$), i = 1, 2, . . ., k, $(. - x)^t u^*$ ($\forall u^* \in N_X(u)$) are F–convex and assume that a > 0.

Then

$$\theta(x) \not\leq H(u, \alpha, \lambda, z) .$$

Proof. Since (u, α, λ, z) is feasible for (D), we have

$$0 = \sum_{i=1}^{k} \alpha_i (\xi_i + B_i z_i) + \sum_{j=1}^{m} \lambda_j \eta_j + u^*$$

where $\xi_i \in \partial f_i(u)$, i = 1, 2, . . ., k; $\eta_j \in \partial g_j(u)$, j = 1, 2, . . ., m and $u^* \in N_X(u)$

$$\Rightarrow \qquad F\left(x, u; \sum_{i=1}^{k} \alpha_i (\xi_i + B_i z_i) + \sum_{j=1}^{m} \lambda_j \eta_j + u^* \right) = 0 \tag{5}$$

Again Since the functions f_i, i = 1, 2, . . ., k; g_j, j = 1, 2, . . ., m; $(. - x)^t B_i z_i$,

I = 1, 2, . . . , k and $(. - x)^t u^*$ are F - convex, a < 0, $\lambda \geqq 0$, we have

$$\sum_{i=1}^{k} \alpha_i (f_i(x) - f_i(u)) \geqq \sum_{i=1}^{k} \alpha_i F(x, u; \xi_i)$$

$$\sum_{j=1}^{m} \lambda_j (g_j(x) - g_j(u)) \geqq \sum_{j=1}^{m} \lambda_j F(x, u; \eta_j)$$

$$\sum_{i=1}^{k} \alpha_i (x - u)^t B_i z_i \geqq \sum_{i=1}^{k} \alpha_i F(x, u; B_i z_i)$$

$$(x - u)^t u^* \geqq F(x, u; u^*)$$

Adding the above four inequalities and using the definition of sublinear function, we get

$$\sum_{i=1}^{k}\alpha_i\left(f_i(x)-f_i(u)+(x-u)^t B_i z_i\right)$$

$$+\sum_{j=1}^{m}\lambda_j\left(g_j(x)-g_j(u)\right)+(x-u)^t u^*$$

$$\geqq F\left(x,u;\sum_{i=1}^{k}\alpha_i(\xi_i+B_i z_i)+\sum_{j=1}^{m}\lambda_j\eta_j+u^*\right)$$

$$= 0 \text{ (by using (5))} \qquad (6)$$

Now, consider

$$\alpha^t\left(\theta(x)-H(u,\alpha,\lambda,z)\right)$$

$$=\sum_{i=1}^{k}\alpha_i\left(f_i(x)+(x^t B_i x)^{1/2}-f_i(u)-u^t B_i z_i\right)$$

$$\geqq\sum_{i=1}^{k}\alpha_i\left(f_i(x)-f_i(u)+(x^t B_i x)^{1/2}(z_i^t B_i z_i)^{1/2}-u^t B_i z_i\right) \qquad \text{(by (3))}$$

$$\geqq\sum_{i=1}^{k}\alpha_i\left(f_i(x)-f_i(u)+(x-u)^t B_i z_i\right)$$

(using Schwarz's inequality)

$$\geqq-\sum_{j=1}^{m}\lambda_j(g_j(x)-g_j(u))-(x-u)^t u^* \qquad \text{(by (6))}$$

$$\geqq 0.$$

The last inequality holds because of $u^*\in N_X(u)$, feasibility of x for (P), (2) and (4).

Hence $\alpha^t\theta(x)\geqq\alpha^t H(u,\alpha,\lambda,z)$

and thus

$$\theta(x)\nleq H(u,\alpha,\lambda,z).$$

Theorem 2 (Strong Duality)

Let x^0 be an efficient solution of (P) then there exist $\alpha^0\in\mathbf{R}^k$, $\lambda^0\in\mathbf{R}^n$, $z^0=(z_1^0,z_2^0,\dots,z_k^0)$ such that $(x^0,\alpha^0,\lambda^0,z^0)$ is feasible for (D) and the two problems have the same extremal values. Further, if the conditions of Weak Duality Theorem 1 hold then $(x^0,\alpha^0,\lambda^0,z^0)$ is properly efficient for (D).

Proof. Since x^0 is efficient for (P), so by Theorem B, there exist $\alpha^0 = (\alpha_1^0 \alpha_2^0 ... \alpha_k^0) \in \mathbf{R}^k$, $\lambda^0 = (\lambda_1^0 \lambda_2^0 ... \lambda_m^0) \in \mathbf{R}^m$ such that

$$\lambda_j^0 g_j(x^0) = 0 , \quad j = 1, 2, \ldots, m$$

$$0 \in \sum_{i=1}^{k} \alpha_i^0 \partial\theta_i(x^0) + \sum_{j=1}^{m} \lambda_j^0 \partial g_j(x^0) + N_X(x^0) \tag{7}$$

$$(\alpha^0, \lambda^0) \geq 0$$

Under Assumption A, condition (7) reduces to

where $z_i^0 \in \mathbf{R}^n$, $z_i^{0^t} B_i z_i^0 \leqq 1$, $\quad x^{0^t} B_i z_i^0 = (x^{0^t} B_i x^0)^{1/2}$, $\forall$ i = 1,2,...,k

These conditions show that $(x^0, \alpha^0, \lambda^0, z^0)$ is feasible for (D). Further, since $x^{0^t} B_i z_i^0 = (x^{0^t} B_i x^0)^{1/2}$, i = 1, 2, . . ., k, the two problems have the same extremal values, so by a result from [2] and Weak Duality Theorem, $(x^0, \alpha^0, \lambda^0, z^0)$ is properly efficient for (D).

For fractional programming problem (FP) we establish duality results between (FP) and Bhatia and Pandev [8] type of dual under the assumptions of generalized (F,r) - convexity.

Bhatia and Pandey [8] type of dual for (FP) is

(FD) Maximize $\dfrac{\beta}{\alpha} = \left(\dfrac{\beta_1}{\alpha_1}, \dfrac{\beta_2}{\alpha_2}, ..., \dfrac{\beta_k}{\alpha_k} \right)^t$

subject to

$$0 \in \sum_{i=1}^{k} (\alpha_i(\partial f_i(u) + B_i z_i) - \beta_i \partial h_i(u))$$

$$+ \sum_{j=1}^{m} \lambda_j \partial g_j(u) + N_X(u) \tag{8}$$

$$\sum_{i=1}^{k} (\alpha_i(f_i(u) + u^t B_i z_i) - \beta_i h_i(u)) \geqq 0 \tag{9}$$

$$\lambda^t g(u) \geqq 0 \tag{10}$$

$$z_i^t B_i z_i \leqq 1 , \qquad i = 1, 2, \ldots, k \tag{11}$$

$$\alpha = (\alpha_1 \alpha_2 ... \alpha_k) > 0 , \qquad \alpha^t e = 1 \tag{12}$$

$$\beta = (\beta_1 \beta_2 ... \beta_k) \geqq 0 , \qquad \lambda = (\lambda_1 \lambda_2 ... \lambda_m) \geqq 0 . \tag{13}$$

Theorem 3 (Weak Duality)

Suppose for feasible x to (FP) and feasible $(u, \alpha, \beta, \lambda, z)$ to (FD)

(i) f_i is (F, ρ_{1i}) - convex; $i = 1, 2, \ldots, k$,

(ii) $-h_i$ is (F, ρ_{2i}) - convex; $i = 1, 2, \ldots, k$,

(iii) g_j is (F, ρ_{3j}) - convex; $j = 1, 2, \ldots, m$,

(iv) $(\,.-x)^t B_i z_i$ is (F, ρ_{4i}) - convex; $\forall\, z_i \in \mathbf{R}^n$, $i = 1, 2, \ldots, k$,

(v) $(\,.-x)^t u^*$ is (F, ρ^*) - convex; $\forall\, u^* \in N_X(u)$ and

(vi) $\sum_{i=1}^{k}(\alpha_i\rho_{1i} + \beta_i\rho_{2i} + \alpha_i\rho_{4i}) + \sum_{j=1}^{m}\lambda_j\rho_{3j} + \rho^* \geqq 0$.

Then $\dfrac{\theta(x)}{h(x)} \nleq \dfrac{\beta}{\alpha}$.

Proof. The constraint (8) ensures the existence of $\xi_i \in \partial f_i(u)$; $\eta_i \in \partial h_i(u)$, $i = 1, 2, \ldots, k$; $\psi_j \in \partial g_j(u)$, $j = 1, 2, \ldots, m$; $u^* \in N_X(u)$ such that

$$0 = \sum_{i=1}^{k}\left(\alpha_i(\xi_i + B_i z_i)\right) + \sum_{j=1}^{m}\lambda_j\psi_j + u^*$$

and hence

$$F\left(x, u; \sum_{i=1}^{k}(\alpha_i(\xi_i + B_i z_i) - \beta_i\eta_i) + \sum_{j=1}^{m}\lambda_j\psi_j + u^*\right) = 0 \tag{14}$$

Now suppose, on the contrary,

$$\frac{\theta(x)}{h(x)} \leq \frac{\beta}{\alpha}$$

i.e. $\dfrac{f_1(x) + (x^t B_1(x)^{1/2}}{h_1(x)} \leqq \dfrac{\beta_i}{\alpha_i}$ $\forall\, i = 1, 2, \ldots, k$,

and $\dfrac{f_r(x) + (x^t B_r x)^{1/2}}{h_r(x)} < \dfrac{\beta_r}{\alpha_r}$ for some $r \in \{1, 2, \ldots, k\}$

$$\Rightarrow \quad \alpha_i\left(f_1(x) + (x^t B_i x)^{1/2}\right) - \beta_i h_i(x) \leqq 0 \qquad \forall\, i = 1, 2, \ldots, k,$$

and $\alpha_r\left(f_r(x)+\left(x^tB_rx\right)^{1/2}\right)-\beta_rh_r(x)<0$ for some r $\in$ {1, 2, . . ., k}

Adding the above inequalities over i = 1, 2, . . ., k, we get

$$\sum_{i=1}^{k}(\alpha_i(f_i(x)+(x^tB_ix)^{1/2})-\beta_ih_i(x))<0$$

This inequality, together with (9), gives

$$\sum_{i=1}^{k}(\alpha_i(f_i(x)+(x^tB_ix)^{1/2})-\beta_ih_i(x))$$

$$<\sum_{i=1}^{k}(\alpha_i(f_i(u)+u^tB_iz_i)-\beta_ih_i(u))$$

$$\Rightarrow \quad \sum_{i=1}^{k}\alpha_i(f_i(x)-f_1(u))-\sum_{i=1}^{k}\beta_i(h_i(x)-h_i(u))$$

$$<\sum_{i=1}^{k}\alpha_i(u^tB_iz_i-(x^tB_ix)^{1/2}) \tag{15}$$

In view of assumptions (i) to (v), (12) and (13), we have

$$\sum_{i=1}^{k}\alpha_i(f_i(x)-f_i(u)) \geqq \sum_{i=1}^{k}\alpha_iF(x,u,\xi_i)+\sum_{i=1}^{k}\alpha_i\rho_{1i}d^2(x,u)$$

$$-\sum_{i=1}^{k}\beta_i(h_i(x)-h_i(u)) \geqq \sum_{i=1}^{k}\beta_iF(x,u;-\eta_i)+\sum_{i=1}^{k}\beta_i\rho_{2i}d^2(x,u)$$

$$\sum_{j=1}^{m}\lambda_j(g_j(x)-g_j(u)) \geqq \sum_{j=1}^{m}\lambda_jF(x,u;\psi_j)+\sum_{j=1}^{m}\lambda_j\rho_{3j}d^2(x,u)$$

$$\sum_{i=1}^{k}\alpha_i(x-u)^tB_iz_i \geqq \sum_{i=1}^{k}\alpha_iF(x,u;B_iz_i)+\sum_{i=1}^{k}\alpha_i\rho_{4i}d^2(x,u)$$

$$(x-u)^tu^* \geqq F(x,u;u^*)+\rho^*d^2(x,u)$$

Adding the above five inequalities and using sub-linearity of F, we have

$$\sum_{i=1}^{k}\alpha_i(f_i(x)-f_i(u))-\sum_{i=1}^{k}\beta_i(h_i(x)-h_i(u))$$

$$+\sum_{j=1}^{m}\lambda_j(g_j(x)-g_j(u))+\sum_{i=1}^{k}\alpha_i(x-u)^tB_iz_i+(x-u)^tu^*$$

$$\geqq F(x,u;\sum_{i=1}^{k}(\alpha_i(\xi_i+B_iz_i)-\beta_i\eta_i)+\sum_{j=1}^{m}\lambda_j\psi_j+u^*)$$

$$+(\sum_{i=1}^{k}(\alpha_i\rho_{1i}+\beta_i\rho_{2i}+\alpha_i\rho_{4i})+\sum_{j=1}^{m}\lambda_j\rho_{3j}+\rho^*)d^2(x,u) \tag{16}$$

or

$$F(x,u;\sum_{i=1}^{k}(\alpha_i(\xi_i+\beta_iz_i)-\beta_i\eta_i)+\sum_{j=1}^{m}\lambda_j\psi_j+u^*)$$

$$<\sum_{i=1}^{k}\alpha_i(u^tB_iz_i-(x^tB_ix)^{1/2})+\sum_{j=1}^{m}\lambda_j(g_j(x)-g_j(u))$$

$$+\sum_{i=1}^{k}\alpha_i(x-u)^tB_iz_i+(x-u)^tu^*$$

(by assumption (vi), (15) and definition of pseudo-metric)

$$\leqq \sum_{i=1}^{k}\alpha_i(-(x^tB_ix)^{1/2}(z_i^tB_iz_i)^{1/2})+\sum_{j=1}^{m}\lambda_jg_j(x)+\sum_{i=1}^{k}\alpha_ix^tB_iz_i$$ (by (10), (11) and the fact that u^* $N_X(u)$)

$$\leqq -\sum_{i=1}^{k}\alpha_ix^tB_iz_i+\sum_{j=1}^{m}\lambda_jg_j(x)+\sum_{i=1}^{k}\alpha_ix^tB_iz_i$$

(By Schwarz's Inequality)

$$=\sum_{j=1}^{m}\lambda_jg_j(x)$$

$\leqq 0$, (by feasibility of x for (FP) and (13))

i.e. $F(x,u;\sum_{i=1}^{k}(\alpha_i(\xi_i+B_iz_i)-\beta_i\eta_i)+\sum_{j=1}^{m}\lambda_j\psi_j+u^*)<0$

a contradiction to (14).

Hence

$$\frac{\theta(x)}{h(x)}\not\leq\frac{\beta}{\alpha}.$$

Theorem 4 (Strong Duality)

Let x^0 be an efficient solution of (FP) and assume that the Slater's type constraint qualification Assumption B(ii) is satisfied at x^0, then there exist $\alpha^0 = (\alpha_1^0\alpha_2^0 \ldots \alpha_k^0) \in \mathbf{R}^k$, $\beta^0 = (\beta_1^0\beta_2^0 \ldots \beta_k^0) \in \mathbf{R}^k$, $\lambda^0 = (\lambda_1^0\lambda_2^0 \ldots \lambda_m^0) \in \mathbf{R}^m$ and $z^0 = (z_1^0 z_2^0, \ldots, z_k^0)$ with each $z_i^0 \in \mathbf{R}^n$ such that $(x^0, \alpha^0, \beta^0, \lambda^0, z^0)$ is efficient for (FD).

Proof. Since x^0 is efficient for (FP), so it is efficient for the following multi-objective program [5] :

(EP) Minimize $(f_1(x) + (x^t B_1 x)^{1/2}, \ldots, f_k(x) + (x^t B_k x)^{1/2}, -h_1(x), \ldots, -h_k(x))^t$

subject to

$$g(x) \leqq 0$$

$$x \in X$$

and hence by [8], there exist scalars $\alpha_i > 0$, $\beta_i \geqq 0$, i = 1, 2, . . . , k and $\lambda_j \geqq 0$, j = 1, 2, . . ., m, satisfying

$$0 \in \sum_{i=1}^{k} (\alpha_i \partial\theta_i(x^0) - \beta_i \partial h_i(x^0)) + \sum_{i=1}^{m} \lambda_j \partial g_j(x^0) + N_X(x^0) \tag{17}$$

$$\sum_{i=1}^{k} (\alpha_i \theta_i(x^0) - \beta_i h_i(x^0)) + \sum_{j=1}^{m} \lambda_j g_j(x^0) \geqq 0 \tag{18}$$

$$\lambda_j g_j(x^0) = 0\ , \qquad j = 1,2,\ldots,m \tag{19}$$

Under Assumption (A), (17) becomes

$$0 \in \sum_{i=1}^{k} (\alpha_i (\partial f_i(x^0) + B_i z_i^0) - \beta_i \partial h_i(x^0)) + \sum_{j=1}^{m} \lambda_j \partial g_j(x^0) + N_X(x^0) \tag{20}$$

where $z_i^0 \in \mathbf{R}^n$, $z_i^{0^t} B_i z_i^0 \leqq 1$, $x^{0^t} B_i z_i^0 = (x^{0^t} B_i x^0)^{1/2}$, i = 1, 2, . . ., k. (21)

From (18) and (19), we have

$$\sum_{i=1}^{k} (\alpha_i (f_i(x^0) + (x^{0^t} B_i x^0)^{1/2} - \beta_i h_i(x^0)) \geqq 0 \tag{22}$$

Dividing (19), (20) and (22) by $\sum_{i=1}^{k} \alpha_i > 0$ and setting

$$\alpha_i^0 = \frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i} > 0 \quad ; \quad \beta_i^0 = \frac{\beta_i}{\sum_{i=1}^{k}\alpha_i}; \; i = 1, 2, ..., k \quad ;$$

$$\lambda_j^0 = \frac{\lambda_j}{\sum_{i=1}^{k}\alpha_i} \geqq 0 \quad , \quad j = 1, 2, \ldots, m$$

and using the definition of normal cone, we get

$$\lambda_j^0 g_j(x^0) = 0 \;, \quad j = 1, 2, \ldots, m$$

$$0 \in \sum_{i=1}^{k}(\alpha_i^0(\partial f_i(x^0) + B_i z_i^0) - \beta_i^0 \partial h_i(x^0) \; + \sum_{j=1}^{m}\lambda_j^0 \partial g_j(x^0) + N_X(x^0)$$

$$\sum_{i=1}^{k}(\alpha_i^0(f_i(x^0) + x^{0^t} B_i z_i^0) - \beta_i^0 h_i(x^0) \geqq 0 \qquad \text{(by (21))}$$

$$\alpha^0 = (\alpha_1^0, \alpha_2^0, ..., \alpha_k^0) > 0, \; \alpha^{0^t} e = 1,$$

$$\beta^0 = (\beta_1^0 \beta_2^0 ... \beta_k^0) \geqq 0, \; \lambda^0 = (\lambda_1^0 \lambda_2^0 ... \lambda_m^0) \geqq 0.$$

Also $\quad z_i^{0^t} B_i z_i^0 \leqq 1$, $\quad i = 1, 2, \ldots, k$ (by (21))

Thus $(x^0, \alpha^0, \beta^0, \lambda^0, z^0)$ is feasible for (FD). Efficiency of $(x^0, \alpha^0, \beta^0, \lambda^0, z^0)$ follows on the lines of proof of the Theorem 2 [8]

Theorem 5 (Strict Converse Duality)

Let x and (u, α, β , λ, z) be efficient solutions to (FP) and (FD) respectively with

$$\frac{\theta(x)}{h(x)} = \frac{\beta}{\alpha} \tag{23}$$

Assume, further, for all feasible solutions $\bar{x}$ for (FP) and $(\bar{u}, \bar{\alpha}, \bar{\beta}, \bar{\lambda}, \bar{z})$ for (FD),

(i) f_i is strict (F, ρ_{1i}) - convex; $\quad i = 1, 2, \ldots, k,$

(ii) $-h_i$ is strict (F, ρ_{2i}) - convex; $\quad i = 1, 2, \ldots, k,$

(iii) g_j is strict (F, ρ_{3j}) - convex; $\quad j = 1, 2, \ldots, m,$

(iv) $(. - \bar{x})^t B_i z_i$ is strict (F, ρ_{4i}) - convex $\quad \forall\, z_i \in \mathbf{R}^n, \; i = 1, 2, \ldots, k,$

(v) $(. - \bar{x})^t u^*$ is strict (F, ρ^*) - convex; $\quad \forall\, u \in N_X(u)$ and

(vi) $\sum_{i=1}^{k}(\alpha_i\rho_{1i} + \beta_i\rho_{2i} + \alpha_i\rho_{4i}) + \sum_{j=1}^{m}\lambda_j\rho_{3j} + \rho^* > 0$

Then u is efficient for (FP).

Proof. It is sufficient to prove that x = u.

Suppose, on the contrary, $x \neq u$

Now, feasibility of (u, α, β, λ, z), as in the proof of Theorem (3), gives the relation (14) viz.

$$F(x, u; \sum_{i=1}^{k}(\alpha_i(\xi_i - \beta_i z_i) - \beta_i\eta_i) + \sum_{j=1}^{m}\lambda_j\psi_j + u^*) = 0$$

Again, as in the proof of Theorem (3), assumptions (i) to (v) lead to inequality (16) which reduces to (using (9), (10) and (23))

$$F(x, u; \sum_{i=1}^{k}(\alpha_i(\xi_i + B_i z_i) - \beta_i\eta_i) + \sum_{j=1}^{m}\lambda_j\psi_j + u^*)$$

$$+ (\sum_{i=1}^{k}(\alpha_i\rho_{1i} + \beta_i\rho_{2i} + \alpha_i\rho_{4i}) + \sum_{j=1}^{m,}\lambda_j\rho_{3j} + \rho^*)d^2(x, u)$$

$$\leqq -\sum_{i=1}^{k}\alpha_i(x^t B_i x)^{1/2} + \sum_{i=1}^{k}\alpha_i(u^t B_i z_i) + \sum_{j=1}^{m}\lambda_j g_j(x)$$

$$+\sum_{i=1}^{k}\alpha\ (x-u)^t B_i z_i + (x-u)^t u^*$$

$$\leqq -\sum_{i=1}^{k}\alpha_i(x^t B_i x)^{1/2}(z_i^t B_i z_i)^{1/2} + \sum_{j=1}^{m}\lambda_j g_j(x) + \sum_{i=1}^{k}\alpha_i(x^t B_i z_i)$$

(by (11) and $(x-u)^t u^* \leqq 0$, as $u^* \in N_X(u)$)

$$\leqq -\sum_{i=1}^{k}\alpha_i(x^t B_i z_i) + \sum_{j=1}^{m}\lambda_j g_j(x) + \sum_{i=1}^{k}\alpha_i(x^t B_i z_i)$$

(using Schwarz ‘s Inequality)

$\leqq 0$, by feasibility of x for (FP) and (13).

Hence, by assumption (vi) and definition of strict pseudo-metric, we have

$$F(x, u; \sum_{i=1}^{k} (\alpha_i (\xi_i + B_i z_i) - \beta_i \eta_i) + \sum_{j=1}^{m} \lambda_j \psi_j + u^*) < 0$$

This contradicts (14) and hence u = x.

ACKNOWLEDGEMENT

The Authors thank Professor ***R.N. Kaul***, University of Delhi, for his valuable guidance throughout the preparation of this paper.

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