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CHARACTERIZATION OF LINEAR 2-NORMED SPACES

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ABSTRACT

In this paper we provide some characterization theorems in the context of linear 2-normed space.

Linear 2-normed space was first introduced by S.Gähler and was extended by C. Diminnie, S.Gähler and A. White.

Key words: *best approximation, linear 2-normed space, proximinal in linear 2-normed space.*

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Definition 1 [5]: Let X be a real linear space of dimension greater than 1 and let $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following conditions

(1) $\|x, y\| = 0$ iff x and y are linearly dependent,

(2) $\|x, y\| = \|y, x\|, \forall x, y \in X$.

(3) $\|\alpha x, y\| = |\alpha| \ \|x, y\|, \forall x, y \in X$ and α a real number,

(4) $\|x, y + z\| \leq \| x, y \| + \| x, z \|, \forall x, y, z \in X$.

$\|.,.\|$ is called a 2 norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space.

Example 1: Let $X = R^3$ with the vector addition and scalar multiplication defined component wise and with 2-norm defined as follows:

for $x = (a_1, b_1, c_1)$, $y = (a_2, b_2, c_2)$

$\|x, y\| = \max \{|a_1b_2 - a_2b_1| , |b_1c_2 - b_2c_1| , |a_1c_2 - a_2c_1|\}$

Then $\|x, y\|$ is a 2-norm and $(X, \|.,.\|)$ is a linear 2-normed space.

Let X be a linear 2-normed space over real number R and the mappings

$<\cdot, \cdot|\cdot>_{i}$, $<\cdot , \cdot|\cdot>_s$ be the 2-normed derivatives defined by

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$$< x, y | z >_i = \lim_{t\to 0-} \frac{\|y + tx, z\|^2 - \|y, z\|^2}{2t}$$

and

$$< x, y | z >_s = \lim_{t\to 0+} \frac{\|y + tx, z\|^2 - \|y, z\|^2}{2t}$$

for $x, y \in X$ and $z \in X\backslash V(x, y)$, where $V(x, y)$ is the subspace of X generated x and y in X.

$x \perp_B y$ —(x is orthogonal to y in the sense of Birkhoff) iff

$$\| x + ky, z\| \geq \| x, z \| \text{ for all } k \in R,\ x, y, z \in X.$$

Example 2: Let $X = R^2$ with $\| (x_1, x_2) \| = |x_1| + |x_2|$ and let $e_1 = (1, 0)$, $e_2 = (0,1)$. Then the set of linear functionals satisfying

$\{\max (|f(e_1)| ,|f(e_2)|) \leq 1\}$which implies that $\|e_1, e_2\| = 2$,

since $\|(a_1, a_2), (b_1, b_2)\| = 2| a_1b_2 - a_2b_1|$, for all real α

$\|e_1 + \alpha e_2 \| = 1 + | \alpha | \geq \| e_1 \|$ and hence $e_1 \perp_B e_2$.

Lemma 1: $\|x + ky, z\| \geq \| x, z \|$ iff

$$<y, x |z>_i \leq 0 \leq <y, x |z>_s$$

Proof: Given that $\| x + ky, z\| \geq \| x, z \|$

$$< y, x | z >_s = \lim_{t\to 0+} \frac{\|x + ty, z\|^2 - \|x, z\|^2}{2t} \geq 0$$

and

$$< y, x | z >_i = \lim_{t\to 0-} \frac{\|x + ty, z\|^2 - \|x, z\|^2}{2t} \leq 0$$

Thus $< y, x |z >_i \leq 0 \leq < y, x |z >_s$

Lemma 2: $x \perp_B -\alpha x + y$ iff

$< y, x|z >_i \ \leq\ \alpha \|x, z\|^2 \ \leq < y , x|z >_s$

Proof: By Lemma 1,

$$< y - \alpha x, x|z >_i \ \leq\ 0 \leq < y - \alpha x,\ x|z >_s$$

i.e. $< y, x|z >_i - \alpha\|x, z\|^2 \leq y, x|z >_s - \alpha \|x, z\|^2$

$\Rightarrow$ $<y, x|\, z>_i \leq \alpha \, \|x, z\|^2 \leq <y, x|z>_s$

Let G be a subspace of a linear 2-normed space X.

write $P_{G,z}(x) \quad = \{g_o \in G : \|g_o - x, z\|\}$

$= \inf_{g \in G} \{\|g - x, z\|\}$, the set of best

approximation elements to $x \in X \setminus \overline{G}$ in G.

Lemma 3: Let G be a linear subspace of a linear 2-normed space X, $x_o \in X \setminus \overline{G}$ and $g_o \in G$. Then $g_o \in P_G(x_o, z)$ iff $x_o - g_o \perp G$.

Proof: For the sake of completeness we provide the proof.

Let $x_o \in X \setminus \overline{G}$, $g_o \in G$. let $g_o \in P_G(x_o, z)$.

Then $\|x_o - g_o + \alpha\, (g_o - g), z\| \geq \|x_o - g_o, z\|$

$\Rightarrow$ $x_o - g_o \perp G$.

Assume that $x_o - g_o \perp G$. Then

$\|x_o - g_o + \alpha(g_o - g), z\| \geq \|x_o - g_o, z\|$

When $\alpha = 1$, $\|x_o - g, z\| \geq \|x_o - g_o, z\|$

$\Rightarrow$ $g_o \in P_G(x_o, z)$.

Lemma 4: Let X be a linear 2-normed space and $b \in X$. Then f be a non-zero continuous functional on $X \times [b]$, $x_o \in X \setminus \ker(f)$.

Then $g_o \in \ker(f)$ $[\ker(f) = \{g_o \in X\text{: } f\,(g_o, z) = 0,\ z \in X\}]$ iff the following estimate holds.

$$\|f\| < x, \frac{\lambda_o (x_o - g_o)}{\|x_o - g_o, z\|} \,|\, z >_i \;\leq f\,(x, z) \leq \|f\| < x, \frac{\lambda_o (x_o - g_o)}{\|x_o - g_o, z\|} \,|\, z >_s \tag{1}$$

for all $x \in X$ and z is independent of x and x_o; $\lambda_o = \operatorname{sgn} f(x_o, z)$

Proof : Let $g_o \in P_{\ker(f)}(x_o, z)$. Then

$w_o = x_o - g_o \perp \ker(f)$ and $[f(x, z)\, w_o - f(w_o, z)x] \in \ker(f)$ for all $x \in X$, $z \in X \setminus \ker(f)$.

since $w_o \perp [f(x, z)\, w_o - f(w_o, z)x]$, We have

$< f(x,z)\, w_o - f(w_o, z)x\,,\, w_o \,|\, z >_i \;\leq 0$

$$\leq\; < f(x,z)\, w_0 - f(w_0, z)x\,,\, w_0 \,|\, z >_s \tag{2}$$

for all $x \in X$, $z \in X \setminus \ker(f)$.

Using the properties of norm derivatives $< \cdot\, ,\, \cdot| \cdot >_s$ and $< \cdot\, ,\cdot|\cdot>_i$

we have $< f(x,z)\, w_o - f(w_o, z)x\, ,\, w_o \mid z >_p =$

$$f(x, z)\, \|w_o, z\|^2 - < x, f(w_0, z)\, w_0 \mid z >_q$$

where p = s , i and q = i , s.

Then (2) becomes

$$< x, \frac{f(w_o,z)\,w_o}{\|w_o,z\|^2} \mid z >_i \quad \le f(x, z)$$

$$\le < x, \frac{f(w_o,z)w_o}{\|w_o,z\|^2} \mid z >_s \tag{3}$$

where $x \in X$ and $z \in X \setminus \ker(f)$

Let

$$u = \frac{f(w_o,z)\,w_o}{\|w_o,z\|^2} \tag{4}$$

Then $f(x,z) \le < x\, ,\, u|z >_s \le \|x, z\|\, \|u, z\|$, for all $x \in X$,

$z \in X \setminus V(x, u)$ and $f(x, z) \quad \ge <x\, ,\, u|z >_i = - < x\, ,\, u|z >_s$

$$\ge - \|x, z\|\, \|u, z\|$$

$$\therefore \quad \|u, z\| \ge \frac{f(x,z)}{\|x,z\|} \ge -\|u, z\|$$

for all $x \in X$ and $z \in X \setminus V(x, u)$

$\Rightarrow \|f\| \le \|u, z\|$

On the other hand $\| f \| \ge \dfrac{f(u,z)}{\|u,z\|} \ge \dfrac{< u, u \mid z >_i}{\|u,z\|} = \|u, z\|$

Then $\|f\| = \|u, z\| = \dfrac{|f(w_o,z)|}{\|w_o,z\|}$

But $f(w_o ,z) = f(x_o ,z) \ne 0$

Hence $\| f \| = \dfrac{|f(x_o,z)|}{\|x_o - g_o, z\|} = \dfrac{f(x_o,z)}{\lambda_o \|x_o - g_o, z\|}$

$\Rightarrow$ f $(x_o, z) = \lambda_o \| f \| \|x_o - g_o, z\|$. Then, by (3)

$$\|f\| < x, \frac{f(w_o,z)\,w_o}{|f(w_o,z)|\,\|w_o,z\|} \mid z >_i \leq f(x,z)$$

$$\leq \|f\| < x, \frac{f(w_0,z)w_0}{|f(w_0,z)|\,\|w_0,z\|} \mid z >_s$$

which is equivalent to (1).

Conversely if (1) holds, then

$< x, x_o - g_o \mid z >_i \leq 0 \leq < x, x_o - g_o \mid z >_s$ for all $x \in \ker(f)$.

$\Rightarrow x_o - g_o \perp \ker(f)$.

$\Rightarrow g_o \in P_{\ker(f)}(x_o, z)$, $z \in X \setminus V(x_o, \ker(f))$

Hence the result.

Theorem 1: Let f be a non-zero continuous linear functional on X x [b] where X is a linear 2-normed space and $b \in X$. Then the following statements are equivalent.

(i) ker(f) is proximinal

(ii) There exists at least one u_f in X with $\| u_f, z \|z\|$ such that

$\| f \| <x, u_f \mid z >_i \leq f(x, z) \leq <x, u \mid z >_s$ (5)

holds, for all $x, z \in X$ such that and z is independent of x and u_f

Proof: (i) $\Rightarrow$ (ii)

Assume that ker(f) is proximinal, then $\exists\, w_o \in X \setminus \ker(f)$ such that $w_o \perp \ker(f)$ (as in lemma 3) for all $x \in X$. we obtain (as in lemma 4)

$$< x, \frac{f(w_o,z)w_o}{\|w_o,z\|^2} \mid z >_i \leq f(x,z)$$

$$\leq \; < x, \frac{f(w_o,z)w_o}{\|w_o,z\|^2} \mid z >_s \qquad (6)$$

where $z \in X$ such that z is independent of x and w_0.

$$\|f\| = \frac{|f(w_o,z)|}{\|w_o,z\|}$$

Let $\lambda_o = \dfrac{f(w_o, z)}{|f(w_o, z)|}$

and put $u_f = \dfrac{\lambda_o w_o}{\|w_o, z\|} = \dfrac{f(w_o, z) w_o}{|f(w_o, z)| \, \|w_o, z\|}$

Then by (6) we obtain,

$\| f \| <x , u_f | z >_i \leq f(x, z) \leq <x , u_f | z >_s$

(ii) $\Rightarrow$ (i)

Assume that there exists atleast one $u_f \in X$ with $\|u_f , z\| = 1$ such that (i) holds.

Then for all $x \in \ker(f)$, $<x , u_f |z >_i \leq 0 \leq <x , u_f |z >_s$

$\Rightarrow$ $u_f \perp \ker(f)$. Then by lemma 3, ker(f) is proximinal.

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