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STRONG ARCS AND MAXIMUM SPANNING TREES IN A FUZZY GRAPH

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Abstract

In this paper we show that an arc is strong iff it is in at least one maximum spanning tree of a fuzzy graph.

1. INTRODUCTION

This paper is a continuation of the study of the concept of strong arcs in a fuzzy graph[4].

A fuzzy graph [7] is a pair G : (σ , μ) where σ is a fuzzy subset of a set S and μ is a fuzzy relation on σ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. We assume that S is finite and nonempty, μ is reflexive and symmetric [7]. Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in S : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in SXS : \mu(u, v) > 0\}$. A fuzzy graph $H : (\tau, \nu)$ is called a partial fuzzy subgraph of $G : (\sigma, \mu)$, if $\tau \subseteq \sigma$ and $\nu \subseteq \mu$ [6]. In particular, we call H : (τ , ν), a fuzzy subgraph of $G : (\sigma, \mu)$, if $\tau(u) = \sigma(u)$ for every $u \in \tau^*$ and $\nu(u, v) = \mu(u, v)$ for every $(u, v) \in \nu^*$. A path P of length n is a sequence of distinct nodes $u_0, u_1, \ldots\ldots u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2 \ldots\ldots n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle and P is called a fuzzy cycle if it contains more than one weakest arc [5]. $G : (\sigma, \mu)$ is called a complete fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all u, v. The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$. An $x - y$ path P is called a strongest $x - y$ path if its strength equals $CONN_G(x, y)$ [7].

A fuzzy graph $G: (\sigma, \mu)$ is connected if for every x, y in σ^*, $CONN_G(x, y) > 0$. An arc of a fuzzy graph is called strong if its weight is at least as great as the connectedness of its end nodes when it is deleted and an $x - y$ path P is called a

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strong path if P contains only strong arcs [4].

An arc is called a fuzzy bridge of G if its removal reduces the strength of connectedness between some pair of nodes in G. A connected fuzzy graph G: (σ, μ) is a fuzzy tree if it has a partial fuzzy spanning subgraph T: (σ, ν), which is a tree, where for all arcs (x, y) not in T there exists a path from x to y in T whose strength is more than $\mu(x, y)$ [6]. A maximum spanning tree of a connected fuzzy graph G: (σ, μ) is a fuzzy spanning subgraph T: (σ, ν) such that T^* is a tree and for which $\Sigma_{u \neq v} \nu(u, v)$ is maximum [1].

2. ARCS IN A MAXIMUM SPANNING TREE

Note that if G is a fuzzy tree, then it has a unique maximum spanning tree T [9] and all arcs in T are just the strong arcs in G[4]. The propositions 4, 5 and 6 of [4] can be obtained as corollaries to the following theorem.

Theorem 1: An arc in a fuzzy graph G is strong if and only if it is an arc of atleast one maximum spanning tree of G.

Proof: Let (x, y) be a strong arc in a fuzzy graph G: (σ, μ). Then by definition, $\mu(x, y) > 0$ and $\mu(x, y) \geq CONN_{G-(x,y)}(x, y)$. Consider the following two cases.

Case I

$\mu(x, y) > CONN_{G-(x,y)}(x, y)$.

Then, removal of (x, y) reduces the strength of connectedness between x and y, which shows that (x, y) is a fuzzy bridge in G. Then (x, y) is an arc of every maximum spanning tree of G[8].

Case II

$\mu(x, y) = CONN_{G-(x,y)}(x, y)$.

Then, (x, y) is an arc of some cycle C in G which contains an $x - y$ path, say P, of strength $\mu(x, y)$. This implies that C contains an arc (u, v) with $\mu(u, v) = \mu(x, y)$ and (x, y) and (u, v) are weakest arcs of C. Then it follows that there are atleast two maximum spanning trees, say, T_1 and T_2 with the property that (x, y) is in T_1, (u, v) is not in T_1 and (x, y) is not in T_2, (u, v) is in T_2, which completes the proof.

Conversely, assume that (x, y) is an arc of a maximum spanning tree T: (σ, ν) of G and that (x, y) is not strong. Then by definition, $\mu(x, y) < CONN_{G-(x,y)}(x, y)$.

This implies that there exists an $x - y$ path, say P, whose arcs having strength greater than that of (x, y). Then replacing the arc (x, y) by the path P in T results in another spanning tree whose total weight exceeds that of T, contradicting our assumption that T is a maximum spanning tree of G. Thus (x, y) should be strong.

Note that it also follows that all arcs in a maximum spanning tree are strong.

Corollary 1: If G is a fuzzy tree, then an arc of G is strong if and only if it is an arc of the maximum spanning tree of G.

Corollary 2: G is a fuzzy tree if and only if there is a unique strong path in G between any two nodes of G.

Corollary 3: In a fuzzy tree, a strong path between any two nodes u, v is a strongest $u–v$ path.

Remark: Note that all arcs in a fuzzy cycle [2] and a complete fuzzy graph [3] are strong, but a fuzzy graph with all its arcs strong and whose underlying graph is complete, need not be a complete fuzzy graph as in the following example.

Example: Consider the fuzzy graph G:$(\sigma, \mu)$with $\sigma^* = \{u, v, w, x\}$ and $\sigma(u) = \sigma(w) = 1$, $\sigma(v) = .8$, $\sigma(x) = .5$, $\mu(u, v) = \mu(v, w) = .8$, $\mu(u, x) = \mu(x, w) = \mu(x, v) = .4$, $\mu(u, w) = 1$. It can be easily verified that all arcs are strong. But G is not a complete fuzzy graph.

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