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BIMODULES AND HYPERGROUPS ASSOCIATED WITH ACTIONS OF A PAIR OF GROUPS

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Abstract. The present paper presents conditions on the set of equivalence classes of bimodules associated with actions of a pair of finite groups on a von Neumann algebra to have the structure of a fusion rule algebra.

1. Introduction

After V. F. R. Jones [16] introduced the notion of index for subfactors, Galois theory and the theory of paragroups for inclusions of subfactors have been developed, which one initiated by A. Ocneanu's idea [21] by applying induced representations and restrictions of representations of a pair of groups. On the other hand V. S. Sunder and N. J. Wildberger [27] investigated the constructing of fusion rule algebras associated with Dynkin diagrams appearing in the principal graphs of the inclusions of subfactors. There are many works related to bimodules and fusion rule algebras in tensor categories, for examples, V. S. Sunder [25] and [26], A. K. Vijayarajan [28], R. Schaflitzel [22] and [23], S. Yamagami [29], H. Kosaki and S. Yamagami [18] and [19], D. Evans and Y. Kawahigashi [4], M. Izumi [14] and the graph theoretical approach in [5].

Hypergroups are locally compact spaces on which the convolution and the involution of bounded measures are given analogous to the group case, where the convolution of two point measures is a probability measure with compact support (not necessarily a point measure). There exists an axiomatic approach to hypergroups initiated by C. F. Dunkl ([2], [3], 1973), R. I. Jewett ([15], 1975) and R. Spector ([24], 1975) which leads to an extensive harmonic analysis of hypergroups. For the historical background of the theory we just refer to R. I. Jewett's fundamental paper [15] and the monograph [1] by W. R. Bloom and H. Heyer. In fact, the hypergroup was developed to be of significant applicability in probability theory where the hypergroup convolution of measure reflects a stochastic operation in the basic space of the hypergroup. Nowadays hypergroup structures are studied within various frameworks from non-commutative duality of groups to quantum groups, deformations of hypergroups [17] and bimodules.

The present work stands against a background of the above works together with our results [6], [7], [8], [9], [10], [11], [12], and [13], which are joint works with Herbert Heyer.

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Let G be a finite group and $\widehat{G}$ the dual of G, namely the set of equivalence classes of irreducible representations of G. We will obtain a fusion rule algebra $\mathcal{F}(\widehat{G})$ through the tensor category of $\widehat{G}$ and the character hypergroup $\mathcal{K}(\widehat{G})$ of G.

Let α be an action of G on a von Neumann algebra M. We denote the fixed point algebra by $A := M^G$. In the present paper we introduce a notion of dual unitary property on the action α of G on M.

In Section 3 we consider irreducible A-A bimodules and G-modules in M. Let $\mathcal{F}(M, G, \alpha)$ be the set of equivalence classes of such irreducible A-A bimodules and G-modules. Then we show that if the action α has dual unitary property, then $\mathcal{F}(M, G, \alpha)$ has a structure of a fusion rule algebra and $\mathcal{F}(M, G, \alpha) \cong \mathcal{F}(\widehat{G})$. By normalizing $\mathcal{F}(M, G, \alpha)$ with the dimension function we obtain a hypergroup $\mathcal{K}(M, G, \alpha)$ and $\mathcal{K}(M, G, \alpha) \cong \mathcal{K}(\widehat{G})$.

In Section 4 we discuss the case $M = B(\ell^2(G))$ and $\alpha_g = \mathrm{Ad}\lambda_g$ where λ is the regular representation of G. We show that the action α has dual unitary property so that $\mathcal{F}(B(\ell^2(G)), G, \mathrm{Ad}\lambda) \cong \mathcal{F}(\widehat{G})$. By this fact we see that $\mathcal{F}(M, G, \alpha) \cong \mathcal{F}(\widehat{G})$ for an outer action α of G on a hyperfinite II_1-factor M.

In Section 5 we study fusion rule algebras and hypergroups associated with a pair (G, G_0) where G_0 is a subgroup of G. We give the convolution on $\mathcal{F}(M^G \subset M^{G_0})$ which is the disjoint union $\mathcal{F}(M, G, \alpha)$ and $\mathcal{F}(M, G_0, \alpha)$ by inductions and restrictions of modules. Then we show that if the pair (G, G_0) is admissible, then $\mathcal{F}(M^G \subset M^{G_0})$ is a fusion rule algebra. By normalizing $\mathcal{F}(M^G \subset M^{G_0})$ we obtain a hypergroup $\mathcal{K}(M^G \subset M^{G_0})$ and show that $\mathcal{K}(M^G \subset M^{G_0}) \cong \mathcal{K}(\widehat{G} \cup \widehat{G_0})$ where the hypergroup $\mathcal{K}(\widehat{G} \cup \widehat{G_0})$ is introduced in [10] and [12].

2. Preliminaries

For a finite set $K = \{c_0, c_1, \ldots, c_n\}$, we denote by $\mathbb{C}K$ the algebraic complex linear space based on K, namely

$$\mathbb{C}K := \left\{ \sum_{j=0}^{n} a_j c_j \ : \ a_j \in \mathbb{C} \ (j = 0, 1, 2, \ldots, n) \right\}$$

and

$$(\mathbb{C}K)_1 := \left\{ \sum_{j=0}^{n} a_j c_j \ : \ a_j \geq 0 \ (j = 0, 1, 2, \ldots, n), \ \sum_{j=0}^{n} a_j = 1 \right\}.$$

For $\mu = a_0 c_0 + a_1 c_1 + \cdots + a_n c_n \in \mathbb{C}K$, the *support* of μ is

$$\mathrm{supp}(\mu) := \{c_j \in K \ : \ a_j \neq 0 \ (j = 0, 1, 2, \cdots, n)\}.$$

2.1. Finite hypergroups. A *finite hypergroup* $K = (K, \mathbb{C}K, \circ, *)$ consists of a finite set $K = \{c_0, c_1, \ldots, c_n\}$ together with an associative product (called convolution) $\circ$ and an involution $*$ in $\mathbb{C}K$ satisfying the following conditions.

- (H1) The space $(\mathbb{C}K, \circ, *)$ is an associative $*$-algebra with unit c_0.
- (H2) For $c_i, c_j \in K$, the convolution $c_i \circ c_j$ belongs to $(\mathbb{C}K)_1$.
- (H3) $K^* = K$ i.e. $c_i^* \in K$ for $c_i \in K$. Moreover, $c_j = c_i^*$ if and only if $c_0 \in \mathrm{supp}(c_i \circ c_j)$.

2.2. Finite fusion rule algebras. A *finite fusion rule algebra* $F = (F, \mathbb{C}F, \circ, *)$ consists of a finite set $F = \{X_0, X_1, \dots, X_n\}$ together with an associative product (called convolution) $\circ$ and an involution $*$ in $\mathbb{C}F$ satisfying the following conditions.

(F1) The space $(\mathbb{C}F, \circ, *)$ is an associative involutive algebra with unit X_0.

(F2) For $X_i, X_j \in F$, the convolution of X_i and X_j is given by

$$X_i \circ X_j = \sum_{k=0}^{n} a_{ij}^k X_k$$

where the a_{ij}^k are non-negative integers.

(F3) $F^* = F$ i.e. $X_i^* \in F$ for $X_i \in F$. Moreover, $X_j = X_i^*$ if and only if $X_0 \in \operatorname{supp}(X_i \circ X_j)$ and

$$X_i \circ X_j = X_0 + \sum_{k=1}^{n} a_{ij}^k X_k.$$

For a finite fusion rule algebra $F = \{X_0, X_1, \dots, X_n\}$ there exists the unique dimension function $d : F \to \mathbb{R}_+^\times$. For $X_j \in F$, put

$$c_j := \frac{1}{d(X_j)} X_j.$$

Then $K = \{c_0, c_1, \dots, c_n\}$ becomes a hypergroup. We call this K the hypergroup obtained by *normalizing* F, denoted K by $\mathcal{K}_d(F)$.

2.3. Hypergroup joins. Let $H = \{h_0, h_1, \dots, h_n\}$ and $L = \{\ell_0, \ell_1, \dots, \ell_m\}$ be finite hypergroups where $\circ_H$, $\circ_L$ are the convolutions of H and L respectively. On the set $K = H \vee_K L = \{h_0, h_1, \dots, h_n, \ell_1, \dots, \ell_m\}$, we define the convolution $\circ_K$ by

$$\begin{aligned}
&h_i \circ_K h_j := h_i \circ_H h_j, \\
&h_i \circ_K \ell_j := \ell_j, \\
&\ell_i \circ_K \ell_j := \ell_i \circ_L \ell_j \text{ if } \ell_j \neq \ell_i^*, \\
&\ell_i \circ_K \ell_j := a_{ij}^0 \omega_H + \sum_{k=1}^{m} a_{ij}^k \ell_k \text{ if } \ell_j = \ell_i^*,
\end{aligned}$$

where ω_H is the normalized Haar measure of H and

$$\ell_i \circ_L \ell_j = \sum_{k=0}^{m} a_{ij}^k \ell_k.$$

We also define the involution $*_K$ by

$$h_i^{*_K} := h_i^{*_H} \text{ and } \ell_i^{*_K} := \ell_i^{*_L}.$$

Then $(H \vee_K L, \circ_K, *_K)$ becomes a finite hypergroup, which we call the hypergroup *join* of H and L.

2.4. Fusion rule algebra joins ([20], related to near-group in [14]). Let $H = \{X_0, X_1, \dots, X_n\}$ be finite fusion rule algebra and $L = \{Y_0, Y_1\}$ the cyclic group of order two where $\circ_H$, $\circ_L$ are the convolutions of H and L respectively. We assume that $d(X_i)$ is a natural number for each $X_i \in H$. On the set $F = H \vee_F L = \{X_0, X_1, \dots, X_n, Y_1\}$, we define the convolution $\circ_F$ by

$$\begin{aligned} X_i \circ_F X_j &:= X_i \circ_H X_j, \\ X_i \circ_F Y_1 &:= d(X_i)Y_1, \\ Y_1 \circ_F Y_1 &:= R(H) \end{aligned}$$

where

$$R(H) = \sum_{k=0}^{n} d(X_k)X_k.$$

We also define the involution $*_F$ by

$$X_i^{*_F} := X_i^{*_H} \text{ and } Y_1^{*_F} := Y_1^{*_L}.$$

Then $(H \vee_F L, \circ_F, *_F)$ becomes a finite fusion rule algebra, which we call the fusion rule algebra *join* of H and L.

3. Bimodules Associated with Actions of a Finite Group

Let α be an action of a finite group G on a von Neumann algebra M. We denote the fixed point algebra of M under the action α by

$$A := M^G := \{x \in M : \alpha_g(x) = x \text{ for all } g \in G\}.$$

Then M is interpreted as an A-A bimodule and a G-module. We call a closed subspace X of M an $(A\text{-}A, G)$*-module* if X is an A-A bimodule and a G-module, i.e., if $x \in X$, then $axb \in X$ and $\alpha_g(x)$ belong to X for $a, b \in A$ and $g \in G$. For two $(A\text{-}A, G)$-modules X and Y, a linear map T from X to Y is called a $(A\text{-}A, G)$*-module map* if T satisfies

$$T(axb) = aT(x)b \text{ and } T(\alpha_g(x)) = \alpha_g(T(x))$$

for $a, b \in A$ and $g \in G$. If there is a bijective $(A\text{-}A, G)$-module map from X onto Y, we say that X is *equivalent* to Y, written by $X \cong Y$.

For an $(A\text{-}A, G)$-module X, we denote by $\mathrm{End}(X)$ the space of all $(A\text{-}A, G)$-module maps on X. The module X is called *irreducible* if $\mathrm{End}(X) = \mathbb{C} \cdot I$, where I is the identity on X.

For $x \in M$ we denote by $V(x)$ the linear span of $\alpha_h(x)$, $h \in G$:

$$V(x) := \left\{ \sum_{h \in G} c_h \alpha_h(x) : c_h \in \mathbb{C} \right\}.$$

Definition 3.1. (Dual unitary property) An action α of G on M has *dual unitary property* if there exists a unitary operator $u(\pi) \in M$ for each $\pi \in \widehat{G}$ such that $\alpha_g(u(\pi)) = \pi(g)u(\pi)$ on $V(\pi) := V(u(\pi))$.

Throughout this section we assume that the action α of G on M has the dual unitary property. Let $H(\pi)$ be the representation space of $\pi \in \widehat{G}$.

Lemma 3.2. *$\alpha_g(x) = \pi(g)x$ for all $x \in V(\pi)$. Moreover, $V(\pi) \cong H(\pi)$ as a G-module.*

Proof. We only show that

$$\alpha_g(\alpha_h(u(\pi))) = \pi(g)(\alpha_h(u(\pi)))$$

for $g, h \in G$. Indeed

$$\begin{aligned}\alpha_g(\alpha_h(u(\pi))) &= \alpha_{gh}(u(\pi)) \\ &= \pi(gh)u(\pi) \\ &= \pi(g)\pi(h)u(\pi) \\ &= \pi(g)(\alpha_h(u(\pi)).\end{aligned}$$

□

Lemma 3.3. *For $\pi \in \widehat{G}$, $AV(\pi)A = AV(\pi) = V(\pi)A$.*

Proof. For $axb \in AV(\pi)A$ $(a, b \in A,\ x \in V(\pi))$, since

$$axbu(\pi)^* \in A \text{ and } \ u(\pi) \in V(\pi)$$

we have

$$axb = (abxu(\pi)^*)u(\pi) \in AV(\pi).$$

Hence we see that $AV(\pi)A = AV(\pi)$. The formula $AV(\pi)A = V(\pi)A$ follows in a similar way. □

For $\pi \in \widehat{G}$ we write $X(\pi) := AV(\pi)A$.

Lemma 3.4. *For $\pi \in \widehat{G}$, $X(\pi)$ is an irreducible $(A\text{-}A, G)$-module.*

Proof. For $x \in V(\pi)$ and $T \in \mathrm{End}(X)$,

$$T(\alpha_g(x)) = T(\pi(g)x) = \pi(g)T(x),$$

which gives $T\pi(g) = \pi(g)T$. Since π is irreducible, i.e. $\pi(G)' = \mathbb{C} \cdot I$ where

$$\pi(G)' = \{S \in B(\ell^2(G)) : S\pi(G) = \pi(G)S \text{ for all } g \in G\},$$

we see that $T(x) = c \cdot x$ $(c \in \mathbb{C})$ for all $x \in V(\pi)$. For $axb \in X(\pi)$ $(a, b \in A)$,

$$T(axb) = aT(x)b = a(c \cdot x)b = c \cdot axb,$$

which means that $T = c \cdot I$. Hence $X(\pi)$ is an irreducible $(A\text{-}A, G)$-module. □

For a finite group G, write $\widehat{G} := \{\pi_0, \pi_1, \ldots, \pi_\ell\}$ where π_0 is the trivial representation of G, and for $\pi \in \widehat{G}$, put

$$Ch(\pi)(g) := \mathrm{tr}(\pi(g)) \text{ and } ch(\pi) := \frac{1}{\dim \pi} Ch(\pi).$$

Then

$$\mathcal{F}(\widehat{G}) = \{Ch(\pi) : \pi \in \widehat{G}\}$$

is a fusion rule algebra with unit $Ch(\pi_0)$ and

$$\mathcal{K}(\widehat{G}) = \{ch(\pi) : \pi \in \widehat{G}\}$$

is a hypergroup called the character hypergroup of G. We note that $\mathcal{K}_d(\mathcal{F}(\widehat{G})) = \mathcal{K}(\widehat{G})$ and $\mathcal{K}_d(H \vee_F L) = \mathcal{K}_d(H) \vee \mathcal{K}_d(L)$.

A finite representation π of G is decomposed as

$$\pi \cong \sum_{j=1}^{\ell}{}^{\oplus} m(\pi_j)\pi_j.$$

Then we define $X(\pi)$ by

$$X(\pi) = \sum_{j=1}^{\ell}{}^{\oplus} m(\pi_j)X(\pi_j)$$

where

$$m(\pi_j)X(\pi_j) = \overbrace{X(\pi_j) \oplus \cdots \oplus X(\pi_j)}^{m(\pi_j)}.$$

Proposition 3.5. *For $\pi_i, \pi_j \in \widehat{G}$, $X(\pi_i)_A\otimes_A X(\pi_j) = X(\pi_i \otimes \pi_j)$.*

Proof.

$$\begin{aligned} X(\pi_i)_A\otimes_A X(\pi_j) &= AV(\pi_i) \otimes V(\pi_j)A \\ &= AV(\pi_i \otimes \pi_j)A \\ &= X(\pi_i \otimes \pi_j). \end{aligned}$$

□

For an action α of G on M we denote the set of the equivalence classes of irreducible $(A\text{-}A, G)$-modules by $\mathcal{F}(M, G, \alpha)$. For X_i and $X_j \in \mathcal{F}(M, G, \alpha)$, we define the convolution $X_i * X_j$ by

$$X_i * X_j := X_{iA}\otimes_A X_j$$

and the sum of $X_i + X_j$ by

$$X_i + X_j := X_i \oplus X_j.$$

Theorem 3.6. *If an action α of a finite group G on a von Neumann algebra M has the dual unitary property, then $\mathcal{F}(M, G, \alpha)$ is an fusion rule algebra and $\mathcal{F}(M, G, \alpha) \cong \mathcal{F}(\widehat{G})$.*

Proof. This statement follows from

$$\mathcal{F}(M, G, \alpha) = \{X(\pi) : \pi \in \widehat{G}\}$$

together with Proposition 3.5. □

Let $\mathcal{K}(M, G, \alpha)$ be the hypergroup obtained by normalizing $\mathcal{F}(M, G, \alpha)$.

Corollary 3.7. $\mathcal{K}(M, G, \alpha) \cong \mathcal{K}(\widehat{G})$.

4. Bimodules Associated with the Regular Actions

Let λ be the left regular representation of a finite group G and α an action of G on $M = B(\ell^2(G))$ given by $\alpha_g := \mathrm{Ad}\lambda_g$ for $g \in G$.

Proposition 4.1. *The action α of G on $B(\ell^2(G))$ has the dual unitary property and $\mathcal{F}(B(\ell^2(G)), G, \alpha) \cong \mathcal{F}(\widehat{G})$.*

Proof. There exists the minimal projection $p \in B(\ell^2(G))$ such that $\{\alpha_h(p) : h \in G\}$ are mutual orthogonal projections and $\sum_{h\in G} \alpha_h(p) = 1$. We write $p_h := \alpha_h^{-1}(p)$. Then we see that $\alpha_g(p_h) = p_{hg^{-1}}$. We denote by $\ell^2(\widehat{G})$ the space of $B(H_\pi)$-valued functions on each $\pi \in \widehat{G}$.

Let W be Fourier transform from $\ell^2(G)$ onto $\ell^2(\widehat{G})$ given by

$$(W\xi)(\pi) = \hat{\xi}(\pi) = \frac{1}{\sqrt{|G|}} \sum_{h\in G} \pi(h)^* \xi(h)$$

for $\xi \in \ell^2(G)$. Define

$$q_h := W p_h W^*$$

for $h \in G$. Then q_h, $h \in G$ are also mutual orthogonal projections on $\ell^2(\widehat{G})$ such that

$$\alpha_g(q_h) = q_{hg^{-1}} \text{ and } \sum_{h\in G} q_h = 1.$$

We consider the unitary operator $\tilde{\pi}$ on $\ell^2(\widehat{G})$ for $\pi \in \widehat{G}$ defined by

$$(\tilde{\pi}(g)\hat{\xi})(\tau) := \begin{cases} \hat{\xi}(\tau) & \text{if } \tau \neq \pi, \\ \pi(g)\hat{\xi}(\pi) & \text{if } \tau = \pi. \end{cases}$$

We define a unitary operator $u(\pi)$ by

$$u(\pi) := \sum_{h\in G} \tilde{\pi}(h) q_h.$$

Then $\alpha_g(u(\pi)) = \pi(g)u(\pi)$ on $V(\pi)$. Indeed for $u(\pi) \in V(\pi)$

$$\begin{aligned} \alpha_g(u(\pi)) &= \sum_{h\in G} \alpha_g(\tilde{\pi}(h) q_h) \\ &= \sum_{h\in G} \alpha_g(\tilde{\pi}(h))\alpha_g(q_h) \\ &= \sum_{h\in G} \lambda_g \tilde{\pi}(h) \lambda_g^* \alpha_g(q_h) \\ &= \sum_{h\in G} \tilde{\pi}(ghg^{-1}) q_{hg^{-1}} \\ &= \sum_{k\in G} \tilde{\pi}(g)\tilde{\pi}(k) q_k \;\; k = hg^{-1} \\ &= \tilde{\pi}(g) \sum_{k\in G} \tilde{\pi}(k) q_k \text{ on } \ell^2(\widehat{G}) \\ &= \pi(g)u(\pi). \end{aligned}$$

Since $B(\ell^2(G)) \cong B(\ell^2(\widehat{G}))$ through AdW, then there exists a unitary operator $u(\pi) \in B(\ell^2(G))$ such that $\alpha_g(u(\pi)) = \pi(g)u(\pi)$ on $V(\pi)$. By Theorem 3.6 we see that $\mathcal{F}(B(\ell^2(G)), G, \alpha) \cong \mathcal{F}(\widehat{G})$. □

It is easy to see the following.

Theorem 4.2. *Let α be an action of a finite group G on a von Neumann algebra M. If $(M, G, \alpha) \cong (B(\ell^2(G)) \otimes M_0, G, \mathrm{Ad}\lambda \otimes \alpha_0)$, then $\mathcal{F}(M, G, \alpha) \cong \mathcal{F}(\widehat{G})$.*

Proof. Since by Proposition 4.1 $B(\ell^2(G))$ contains the unitary operator $u(\pi)$ for each $\pi \in \widehat{G}$ such that $\alpha_g(u(\pi)) = \pi_g(u(\pi))$, we have $\tilde{u}(\pi) = u(\pi) \otimes I$ is a unitary operator on $M = B(\ell^2(G)) \otimes M_0$. Then

$$\begin{aligned}
\alpha_g(\tilde{u}(\pi)) &= (\mathrm{Ad}\lambda_g \otimes \alpha_g^0)(u(\pi) \otimes I) \\
&= (\mathrm{Ad}\lambda_g)(u(\pi)) \otimes \alpha_g^0(I) \\
&= \pi_g(u(\pi)) \otimes I \\
&= \pi_g(u(\pi) \otimes I) \\
&= \pi_g(\tilde{u}(\pi)).
\end{aligned}$$

This implies that the action $\alpha = (\mathrm{Ad}\lambda) \otimes \alpha^0$ on $B(\ell^2(G)) \otimes M_0$ has the dual unitary property, leading to the desired conclusion. □

Corollary 4.3. *If α is an outer action of G on a hyperfinite II_1-factor M, then $(M, G, \alpha) \cong (B(\ell^2(G)) \otimes M_0, G, \mathrm{Ad}\lambda \otimes \alpha_0)$ so that $\mathcal{F}(M, G, \alpha) \cong \mathcal{F}(\widehat{G})$.*

Conjecture If an outer action α of a finite group G on a factor has Rohlin property i.e. there exists a projection p such that $\alpha_h(p)$, $h \in G$ are mutually orthogonal projections and $\sum_{h\in G} \alpha_h(p) = 1$, then $\mathcal{F}(M, G, \alpha) \cong \mathcal{F}(\widehat{G})$.

5. Bimodules Associated with a Pair (G, G_0) of Finite Groups

Let G_0 be a subgroup of a finite group G and α an action of G on a von Neumann algebra M. In this section we assume that the action α has dual unitary property and the restriction of α to G_0 also has the dual unitary property. Then by Theorem 3.6 $\mathcal{F}(M, G, \alpha)$ and $\mathcal{F}(M, G_0, \alpha)$ are fusion rule algebras and $\mathcal{F}(M, G, \alpha) \cong \mathcal{F}(\widehat{G})$ and $\mathcal{F}(M, G_0, \alpha) \cong \mathcal{F}(\widehat{G_0})$.

Definition 5.1. For $X(\pi) \in \mathcal{F}(M, G, \alpha)$ ($\pi \in \widehat{G}$) we define the *restriction* of $X(\pi)$ to G_0 by

$$\mathrm{res}_{G_0}^G X(\pi) := Y(\mathrm{res}_{G_0}^G \pi),$$

and the *induced module* of $Y(\tau) \in \mathcal{F}(M, G_0, \alpha)$ ($\tau \in \widehat{G_0}$) to G by

$$\mathrm{ind}_{G_0}^G Y(\tau) := X(\mathrm{ind}_{G_0}^G \tau).$$

Definition 5.2. The *convolution* $*$ of $\mathcal{F}(M^G \subset M^{G_0}) = \{(X(\pi), \circ), (Y(\tau), \bullet) : \pi \in \widehat{G},\ \tau \in \widehat{G_0}\}$ is defined by

$$(X(\pi_i), \circ) * (X(\pi_j), \circ) := (X(\pi_i) * X(\pi_j), \circ),$$

$$(X(\pi), \circ) * (Y(\tau), \bullet) := (\mathrm{res}_{G_0}^G X(\pi) * Y(\tau), \bullet),$$

$$(Y(\tau),\bullet)*(X(\pi),\circ):=(Y(\tau)*\mathrm{res}^G_{G_0}X(\pi),\bullet),$$
$$(Y(\tau_i),\bullet)*(Y(\tau_j),\bullet):=(\mathrm{ind}^G_{G_0}(Y(\tau_i)*Y(\tau_j)),\circ).$$

Definition 5.3. Let (G,G_0) be a pair of finite groups. We call (G,G_0) an *admissible pair* if the following conditions are satisfied:

(1) For $\pi\in\widehat{G}$ and $\tau\in\widehat{G_0}$,
$$\mathrm{ind}^G_{G_0}((\mathrm{res}^G_{G_0}X(\pi))*Y(\tau))=X(\pi)*\mathrm{ind}^G_{G_0}Y(\tau).$$

(2) For $\tau\in\widehat{G_0}$,
$$\mathrm{res}^G_{G_0}(\mathrm{ind}^G_{G_0}Y(\tau))=Y(\tau)*\mathrm{res}^G_{G_0}(\mathrm{ind}^G_{G_0}Y(\tau_0)),$$
where τ_0 is the trivial representation of G_0.

$\mathcal{K}(M^G\subset M^{G_0})$ is the hypergroup obtained by normalizing $\mathcal{F}(M^G\subset M^{G_0})$, and the hypergroup $\mathcal{K}(\widehat{G}\cup\widehat{G_0})$ is introduced in [10].

Theorem 5.4. *$\mathcal{F}(M^G\subset M^{G_0})$ is a fusion rule algebra if and only if (G,G_0) is admissible. Moreover, $\mathcal{F}(M^G\subset M^{G_0})\cong\mathcal{F}(\widehat{G}\cup\widehat{G_0})$ and $\mathcal{K}(M^G\subset M^{G_0})\cong\mathcal{K}(\widehat{G}\cup\widehat{G_0})$.*

Proof. If a finite group G together with a subgroup G_0 forms an admissible pair, then the following associativity relations hold. For $\pi_i,\pi_j,\pi_k,\pi\in\widehat{G}$ and $\tau_i,\tau_j,\tau_k,\tau\in\widehat{G_0}$,

$(A1)$ $((X(\pi_i),\circ)*(X(\pi_j),\circ))*(X(\pi_k),\circ)=(X(\pi_i),\circ)*((X(\pi_j),\circ)*(X(\pi_k),\circ))$,
$(A2)$ $((Y(\tau),\bullet)*(X(\pi_i),\circ))*(X(\pi_j),\circ)=(Y(\tau),\bullet)*((X(\pi_i),\circ)*(X(\pi_j),\circ))$,
$(A3)$ $((Y(\tau_i),\bullet)*(Y(\tau_j),\bullet))*(X(\pi),\circ)=(Y(\tau_i),\bullet)*((Y(\tau_j),\bullet)*(X(\pi),\circ))$,
$(A4)$ $((Y(\tau_i),\bullet)*(Y(\tau_j),\bullet))*(Y(\tau_k),\bullet)=(Y(\tau_i),\bullet)*((Y(\tau_j),\bullet)*(Y(\tau_k),\bullet))$.

$(A1)$ is clear by the fact that $\mathcal{F}(\widehat{G})$ is a fusion rule algebra.
$(A2)$ For $\tau\in\widehat{G_0}$ and $\pi_i,\pi_j\in\widehat{G}$,
$$\begin{aligned}&((Y(\tau),\bullet)*(X(\pi_i),\circ))*(X(\pi_j),\circ)\\&=(Y(\tau)*\mathrm{res}^G_{G_0}X(\pi_i),\bullet)*(X(\pi_j),\circ)\\&=(Y(\tau)*(\mathrm{res}^G_{G_0}X(\pi_i))*(\mathrm{res}^G_{G_0}X(\pi_j)),\bullet).\end{aligned}$$
On the other hand,
$$\begin{aligned}&(Y(\tau),\bullet)*((X(\pi_i),\circ)*(X(\pi_j),\circ))\\&=(Y(\tau),\bullet)*(X(\pi_i)*X(\pi_j),\circ)\\&=(Y(\tau)*\mathrm{res}^G_{G_0}(X(\pi_i)*X(\pi_j)),\bullet)\\&=(Y(\tau)*(\mathrm{res}^G_{G_0}X(\pi_i))*(\mathrm{res}^G_{G_0}X(\pi_j)),\bullet).\end{aligned}$$
$(A3)$ For $\tau_i,\tau_j\in\widehat{G_0}$ and $\pi\in\widehat{G}$,
$$\begin{aligned}&((Y(\tau_i),\bullet)*(Y(\tau_j),\bullet))*(X(\pi),\circ)\\&=(\mathrm{ind}^G_{G_0}(Y(\tau_i)*Y(\tau_j)),\circ)*(X(\pi),\circ)\end{aligned}$$

$$=(\mathrm{ind}_{G_0}^G(Y(\tau_i) * Y(\tau_j)) * X(\pi), \circ),$$

and applying the condition (1) of the definition of an admissible pair,

$$\begin{aligned}
&=(\mathrm{ind}_{G_0}^G((Y(\tau_i) * Y(\tau_j)) * \mathrm{res}_{G_0}^G X(\pi)), \circ) \\
&=(\mathrm{ind}_{G_0}^G(Y(\tau_i) * (Y(\tau_j) * \mathrm{res}_{G_0}^G X(\pi)), \circ) \\
&=(Y(\tau_i), \bullet) * (Y(\tau_j) * \mathrm{res}_{G_0}^G X(\pi), \bullet) \\
&=(Y(\tau_i), \bullet) * ((Y(\tau_j), \bullet) * (X(\pi), \circ)).
\end{aligned}$$

$(A4)$ For $\tau_i, \tau_j, \tau_k \in \widehat{G_0}$

$$\begin{aligned}
&((Y(\tau_i), \bullet) * (Y(\tau_j), \bullet)) * (Y(\tau_k), \bullet) \\
&=(\mathrm{ind}_{G_0}^G(Y(\tau_i) * Y(\tau_j)), \circ) * (Y(\tau_k), \bullet) \\
&=(\mathrm{res}_{G_0}^G(\mathrm{ind}_{G_0}^G(Y(\tau_i) * Y(\tau_j))) * Y(\tau_k), \bullet),
\end{aligned}$$

and applying the condition (2) of the definition of an admissible pair,

$$= (\mathrm{res}_{G_0}^G(\mathrm{ind}_{G_0}^G Y(\tau_0)) * (Y(\tau_i) * Y(\tau_j) * Y(\tau_k)), \bullet)$$

This implies the associativity:

$$((Y(\tau_i), \bullet) * (Y(\tau_j), \bullet)) * (Y(\tau_k), \bullet) = (Y(\tau_i), \bullet) * ((Y(\tau_j), \bullet) * (Y(\tau_k), \bullet)).$$

The other conditions of a fusion rule algebra are easy to check.

On the other hand, if $\mathcal{F}(M^G \subset M^{G_0})$ is a fusion rule algebra, the associativity relations (A_1), (A_2), (A_3) and (A_4) hold. We obtain the conditions (1) and (2) of the admissible pair from the associativity relations (A_3) and (A_4). □

6. Examples

Example 6.1. $G = \mathbb{Z}_2 = \{e, g\}$, $G_0 = \{e\}$, $\widehat{G} = \{\pi_0, \pi_1\}$, $\widehat{G_0} = \{\tau_0\}$, $M = M(2, \mathbb{C})$.

The action α is defined by

$$\alpha_g(x) = \lambda_g x \lambda_g^*$$

for $x \in M$ and $g \in G$, where λ is the regular representation of $\mathbb{Z}_2$, namely

$$\lambda_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \lambda_g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{C})$, we obtain $\alpha_g(x) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$. It is easy to see that

$$u(\pi_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ u(\pi_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } u(\tau_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A = X(\pi_0) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

By $X(\pi_1) = A \cdot u(\pi_1)$, we obtain

$$X(\pi_1) = \left\{ \begin{pmatrix} c & -d \\ d & -c \end{pmatrix} : c, d \in \mathbb{C} \right\}.$$

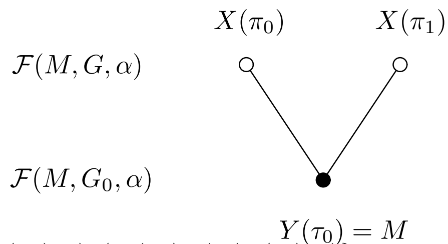
It is trivial that $Y(\tau_0) = M$. Moreover,

$$\mathcal{F}(M, G, \alpha) = \{X(\pi_0), X(\pi_1)\},\ \mathcal{F}(M, G_0, \alpha) = \{Y(\tau_0)\},$$

and

$$\begin{aligned}&\mathrm{res}^G_{G_0} X(\pi_0) = Y(\tau_0),\ \mathrm{res}^G_{G_0} X(\pi_1) = Y(\tau_0) \text{ and}\\ &\mathrm{ind}^G_{G_0} Y(\tau_0) = X(\pi_0) + X(\pi_1) = M.\end{aligned}$$

Frobenius diagram of the inclusion $G \supset G_0$ is Dynkin diagram of type A_3:



$\mathcal{F}(M^G \subset M^{G_0}) = \{(X(\pi_0), \circ), (X(\pi_1), \circ), (Y(\tau_0), \bullet)\}$, where $(X(\pi_0), \circ)$ is unit of $\mathcal{F}(M^G \subset M^{G_0})$. Then

$$\begin{aligned}&(X(\pi_1), \circ) * (X(\pi_1), \circ) = (X(\pi_0), \circ),\\ &(X(\pi_1), \circ) * (Y(\tau_0), \bullet) = (Y(\tau_0), \bullet) * (X(\pi_1), \circ) = (Y(\tau_0), \bullet),\\ &(Y(\tau_0), \bullet) * (Y(\tau_0), \bullet) = (X(\pi_0), \circ) + (X(\pi_1), \circ).\end{aligned}$$

We remark that $\mathcal{F}(M^{\mathbb{Z}_2} \subset M) \cong \mathbb{Z}_2 \vee_F \mathbb{Z}_2$ where $\vee_F$ is a fusion rule algebra join. The above Frobenius diagram coincides with the principle graph of the inclusion of $M^G \subset M$ by an outer action α on a factor M.

We have $\mathcal{K}(M^{\mathbb{Z}_2} \subset M) = \{c(\pi_0), c(\pi_1), c(\tau_0)\}$, where $c(\pi_0) = (X(\pi_0), \circ)$, $c(\pi_1) = (X(\pi_1), \circ)$ and $c(\tau_0) = \frac{1}{2}(Y(\tau_0), \bullet)$. Then $\mathcal{K}(M^{\mathbb{Z}_2} \subset M)$ is a hypergroup isomorphic with $\mathcal{K}(\widehat{\mathbb{Z}_2} \cup \widehat{\{e\}}) = \mathbb{Z}_2 \vee \mathbb{Z}_2$.

Example 6.2. $G = \mathbb{Z}_3 = \{e, g, g^2\}$, $G_0 = \{e\}$, $\widehat{G} = \{\pi_0, \pi_1, \pi_2\}$, $\widehat{G_0} = \{\tau_0\}$, $M = M(3, \mathbb{C})$.

In a similar way to the above, we obtain

$$u(\pi_0) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ u(\pi_1) = \begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix},$$

$$u(\pi_2) = \begin{pmatrix}1 & 0 & 0\\ 0 & \omega^2 & 0\\ 0 & 0 & \omega\end{pmatrix},\ u(\tau_0) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

$$A = X(\pi_0) = \left\{ \begin{pmatrix}a_1 & a_3 & a_2\\ a_2 & a_1 & a_3\\ a_3 & a_2 & a_1\end{pmatrix} : a_1, a_2, a_3 \in \mathbb{C} \right\},$$

$$X(\pi_1) = \left\{ \begin{pmatrix} b_1 & \omega b_3 & \omega^2 b_2 \\ b_2 & \omega b_1 & \omega^2 b_3 \\ b_3 & \omega b_2 & \omega^2 b_1 \end{pmatrix} : b_1, b_2, b_3 \in \mathbb{C} \right\},$$

$$X(\pi_2) = \left\{ \begin{pmatrix} c_1 & \omega^2 c_3 & \omega c_2 \\ c_2 & \omega^2 c_1 & \omega c_3 \\ c_3 & \omega^2 c_2 & \omega c_1 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{C} \right\},$$

$$Y(\tau_0) = M.$$

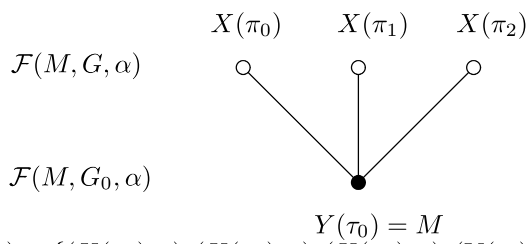
$$\mathcal{F}(M, G, \alpha) = \{X(\pi_0), X(\pi_1), X(\pi_2)\},\ \mathcal{F}(M, G_0, \alpha) = \{Y(\tau_0)\}.$$

$$\begin{aligned} &\mathrm{res}^G_{G_0} X(\pi_0) = Y(\tau_0),\ \mathrm{res}^G_{G_0} X(\pi_1) = Y(\tau_0),\ \mathrm{res}^G_{G_0} X(\pi_2) = Y(\tau_0), \\ &\mathrm{ind}^G_{G_0} Y(\tau_0) = X(\pi_0) + X(\pi_1) + X(\pi_2), \end{aligned}$$

and

$$M = X(\pi_0) + X(\pi_1) + X(\pi_2).$$

The Frobenius diagram of $G \supset G_0$ is a Dynkin diagram of type D_4:



Here $\mathcal{F}(M^G \subset M^{G_0}) = \{(X(\pi_0), \circ), (X(\pi_1), \circ), (X(\pi_2), \circ), (Y(\tau_0), \bullet)\}$, where $(X(\pi_0), \circ)$ is unit of $\mathcal{F}(M^G \subset M^{G_0})$.

$$\begin{aligned} &(X(\pi_1), \circ) * (X(\pi_1), \circ) = (X(\pi_2), \circ), \\ &(X(\pi_2), \circ) * (X(\pi_2), \circ) = (X(\pi_1), \circ), \\ &(X(\pi_1), \circ) * (X(\pi_2), \circ) = (X(\pi_0), \circ), \\ &(X(\pi_1), \circ) * (Y(\tau_0), \bullet) = (X(\pi_2), \circ) * (Y(\tau_0), \bullet) = (Y(\tau_0), \bullet), \\ &(Y(\tau_0), \bullet) * (Y(\tau_0), \bullet) = (X(\pi_0), \circ) + (X(\pi_1), \circ) + (X(\pi_2), \circ). \end{aligned}$$

We remark that $\mathcal{F}(M^{\mathbb{Z}_3} \subset M) \cong \mathbb{Z}_3 \vee_F \mathbb{Z}_2$ and $\mathcal{K}(M^{\mathbb{Z}_3} \subset M) \cong \mathbb{Z}_3 \vee \mathbb{Z}_2$.

Example 6.3. $G = S_3 = \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2 = \{h_0, h_1, h_2, g, h_1 g, h_2 g\}$, where $\mathbb{Z}_3 = \{h_0, h_1, h_2\}$ and $\mathbb{Z}_2 = \{e, g\}$, $G_0 = \mathbb{Z}_2$, $\widehat{G} = \{\pi_0, \pi_1, \pi_2\}$ ($\dim \pi_0 = \dim \pi_1 = 1$, $\dim \pi_2 = 2$), $\widehat{G_0} = \{\tau_0, \tau_1\}$ and $M = M(6, \mathbb{C}) = B(\ell^2(G))$.

Let λ be the regular representation of S_3 which is given by

$$\lambda_{h_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix}, \ \lambda_g = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $x = (x_{ij}) \in M$,

$$\alpha_{h_1}(x) = \begin{pmatrix} x_{11} & \omega^2 x_{12} & \omega x_{13} & x_{14} & \omega^2 x_{15} & \omega x_{16} \\ \omega x_{21} & x_{22} & \omega^2 x_{23} & \omega x_{24} & x_{25} & \omega^2 x_{26} \\ \omega^2 x_{31} & \omega x_{32} & x_{33} & \omega^2 x_{34} & \omega x_{35} & x_{36} \\ x_{41} & \omega^2 x_{42} & \omega x_{43} & x_{44} & \omega^2 x_{45} & \omega x_{46} \\ \omega x_{51} & x_{52} & \omega^2 x_{53} & \omega x_{54} & x_{55} & \omega^2 x_{56} \\ \omega^2 x_{61} & \omega x_{62} & x_{63} & \omega^2 x_{64} & \omega x_{65} & x_{66} \end{pmatrix},$$

$$\alpha_g(x) = \begin{pmatrix} x_{44} & x_{46} & x_{45} & x_{41} & x_{43} & x_{42} \\ x_{64} & x_{66} & x_{65} & x_{61} & x_{63} & x_{62} \\ x_{54} & x_{56} & x_{55} & x_{51} & x_{53} & x_{52} \\ x_{14} & x_{16} & x_{15} & x_{11} & x_{13} & x_{12} \\ x_{34} & x_{36} & x_{35} & x_{31} & x_{33} & x_{32} \\ x_{24} & x_{26} & x_{25} & x_{21} & x_{23} & x_{22} \end{pmatrix}.$$

Then we see that

$$u(\pi_0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ u(\pi_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$u(\pi_2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$u(\tau_0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } u(\tau_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Moreover, it is easy to see that

$$V(\pi_2) = \left\{ \begin{pmatrix} 0 & b & a & 0 & b & a \\ a & 0 & b & a & 0 & b \\ b & a & 0 & b & a & 0 \\ 0 & b & a & 0 & b & a \\ a & 0 & b & a & 0 & b \\ b & a & 0 & b & a & 0 \end{pmatrix} : a, b \in \mathbb{C} \right\},$$

$$A = X(\pi_0) = \left\{ \begin{pmatrix} a_1 & 0 & 0 & a_4 & 0 & 0 \\ 0 & a_2 & 0 & 0 & a_5 & 0 \\ 0 & 0 & a_3 & 0 & 0 & a_6 \\ a_4 & 0 & 0 & a_1 & 0 & 0 \\ 0 & a_5 & 0 & 0 & a_2 & 0 \\ 0 & 0 & a_6 & 0 & 0 & a_3 \end{pmatrix} : a_i \in \mathbb{C} \right\},$$

$$X(\pi_1) = \left\{ \begin{pmatrix} b_1 & 0 & 0 & -b_4 & 0 & 0 \\ 0 & b_2 & 0 & 0 & -b_5 & 0 \\ 0 & 0 & b_3 & 0 & 0 & -b_6 \\ b_4 & 0 & 0 & -b_1 & 0 & 0 \\ 0 & b_5 & 0 & 0 & -b_2 & 0 \\ 0 & 0 & b_6 & 0 & 0 & -b_3 \end{pmatrix} : b_i \in \mathbb{C} \right\},$$

$$X(\pi_2) = A \cdot V(\pi_2) = \left\{ \begin{pmatrix} 0 & c_3 & c_5 & 0 & c_9 & c_{11} \\ c_1 & 0 & c_6 & c_7 & 0 & c_{12} \\ c_2 & c_4 & 0 & c_8 & c_{10} & 0 \\ 0 & c_9 & c_{11} & 0 & c_3 & c_5 \\ c_7 & 0 & c_{12} & c_1 & 0 & c_6 \\ c_8 & c_{10} & 0 & c_2 & c_4 & 0 \end{pmatrix} : c_i \in \mathbb{C} \right\},$$

$$X_1(\pi_2) = \left\{ \begin{pmatrix} 0 & d_3 & d_5 & 0 & -d_9 & -d_{11} \\ d_1 & 0 & d_6 & -d_7 & 0 & -d_{12} \\ d_2 & d_4 & 0 & -d_8 & -d_{10} & 0 \\ 0 & d_9 & d_{11} & 0 & -d_3 & -d_5 \\ d_7 & 0 & d_{12} & -d_1 & 0 & -d_6 \\ d_8 & d_{10} & 0 & -d_2 & -d_4 & 0 \end{pmatrix} : d_i \in \mathbb{C} \right\}.$$

Here we note that $X_1(\pi_2) \cong X(\pi_2)$ as (A-A,G)-module and

$$M = X(\pi_0) \oplus X(\pi_1) \oplus X(\pi_2) \oplus X_1(\pi_2),$$

$$Y(\tau_0) = \left\{ \begin{pmatrix} e_1 & e_4 & e_7 & e_{10} & e_{16} & e_{13} \\ e_2 & e_5 & e_8 & e_{12} & e_{18} & e_{15} \\ e_3 & e_6 & e_9 & e_{11} & e_{17} & e_{14} \\ e_{10} & e_{13} & e_{16} & e_1 & e_7 & e_4 \\ e_{11} & e_{14} & e_{17} & e_3 & e_9 & e_6 \\ e_{12} & e_{15} & e_{18} & e_2 & e_8 & e_5 \end{pmatrix} : e_i \in \mathbb{C} \right\},$$

$$Y(\tau_1) = \left\{ \begin{pmatrix} f_1 & f_4 & f_7 & -f_{10} & -f_{16} & -f_{13} \\ f_2 & f_5 & f_8 & -f_{12} & -f_{18} & -f_{15} \\ f_3 & f_6 & f_9 & -f_{11} & -f_{17} & -f_{14} \\ f_{10} & f_{13} & f_{16} & -f_1 & -f_7 & -f_4 \\ f_{11} & f_{14} & f_{17} & -f_3 & -f_9 & -f_6 \\ f_{12} & f_{15} & f_{18} & -f_2 & -f_8 & -f_5 \end{pmatrix} : f_i \in \mathbb{C} \right\}.$$

We see that

$$\mathcal{F}(M, G, \alpha) = \{X(\pi_0), X(\pi_1), X(\pi_2)\},\ \mathcal{F}(M, G_0, \alpha) = \{Y(\tau_0), Y(\tau_1)\},$$

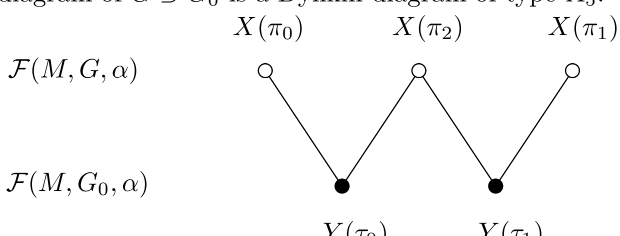
and

$$\mathrm{res}^G_{G_0} X(\pi_0) = Y(\tau_0),\ \mathrm{res}^G_{G_0} X(\pi_1) = Y(\tau_1),\ \mathrm{res}^G_{G_0} X(\pi_2) = Y(\tau_0) + Y(\tau_1),$$
$$\mathrm{ind}^G_{G_0} Y(\tau_0) = X(\pi_0) + X(\pi_2) \text{ and } \mathrm{ind}^G_{G_0} Y(\tau_1) = X(\pi_1) + X(\pi_2).$$

We note that

$$M = X(\pi_0) \oplus X(\pi_1) \oplus X(\pi_2) \oplus X_1(\pi_2) \cong X(\pi_0) + X(\pi_1) + 2X(\pi_2).$$

The Frobenius diagram of $G \supset G_0$ is a Dynkin diagram of type A_5:



Now $\mathcal{F}(M^{S_3} \subset M^{\mathbb{Z}_2}) = \{(X(\pi_0), \circ), (X(\pi_1), \circ), (X(\pi_2), \circ), (Y(\tau_0), \bullet), (Y(\tau_1), \bullet)\}$, where $(X(\pi_0), \circ)$ is unit of $\mathcal{F}(M^{S_3} \subset M^{\mathbb{Z}_2})$. We remark that $\mathcal{F}(M^{S_3} \subset M^{\mathbb{Z}_2}) \cong \mathcal{F}(\widehat{S_3} \cup \widehat{\mathbb{Z}_2})$, refer to [20] and $\mathcal{K}(M^{S_3} \subset M^{\mathbb{Z}_2}) \cong \mathcal{K}(\widehat{S_3} \cup \widehat{\mathbb{Z}_2})$, refer to [10].

Example 6.4. $G = S_3$, $G_0 = \mathbb{Z}_2$, $\widehat{G} = \{\pi_0, \pi_1, \pi_2\}$, $\widehat{G_0} = \{\tau_0, \tau_1\}$, $M = M(2, \mathbb{C})$.

We define an action action α of S_3 on $M(2, \mathbb{C})$ by

$$\alpha_s(x) = \pi_2(s) x \pi_2(s)^* \ (s \in S_3).$$

It is easy to see that

$$u(\pi_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ u(\pi_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ u(\pi_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$u(\tau_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ u(\tau_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$V(\pi_2) = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

Moreover, we have

$$A = X(\pi_0) = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} : a_1 \in \mathbb{C} \right\},\ X(\pi_1) = \left\{ \begin{pmatrix} a_2 & 0 \\ 0 & -a_2 \end{pmatrix} : a_2 \in \mathbb{C} \right\},$$

$$X(\pi_2) = A \cdot V(\pi_2) = \left\{ \begin{pmatrix} 0 & a_4 \\ a_3 & 0 \end{pmatrix} : a_3, a_4 \in \mathbb{C} \right\},$$

$$Y(\tau_0) = \left\{ \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix} : b_1, b_2 \in \mathbb{C} \right\},\ Y(\tau_1) = \left\{ \begin{pmatrix} b_3 & -b_4 \\ b_4 & -b_3 \end{pmatrix} : b_3, b_4 \in \mathbb{C} \right\},$$

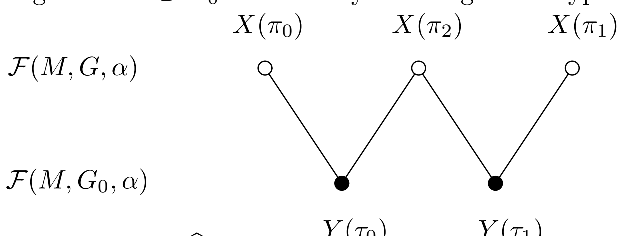
$$\mathcal{F}(M, G, \alpha) = \{X(\pi_0), X(\pi_1), X(\pi_2)\},\ \mathcal{F}(M, G_0, \alpha) = \{Y(\tau_0), Y(\tau_1)\},$$

$$\mathrm{res}_{G_0}^{G} X(\pi_0) = Y(\tau_0),\ \mathrm{res}_{G_0}^{G} X(\pi_1) = Y(\tau_1),\ \mathrm{res}_{G_0}^{G} X(\pi_2) = Y(\tau_0) + Y(\tau_1),$$
$$\mathrm{ind}_{G_0}^{G} Y(\tau_0) = X(\pi_0) + X(\pi_2),\ \mathrm{ind}_{G_0}^{G} Y(\tau_1) = X(\pi_1) + X(\pi_2)$$

and

$$M = X(\pi_0) + X(\pi_1) + X(\pi_2).$$

The Frobenius diagram of $G \supset G_0$ is also a Dynkin diagram of type A_5:



Counterexample 1 $G = S_3$, $\widehat{G} = \{\pi_0, \pi_1, \pi_2\}$, $M = M(3, \mathbb{C})$.

We define an action α of S_3 on $M(3, \mathbb{C})$ by

$$\alpha_s(x) := \pi(s)x\pi(s)^*\ (x \in M)$$

where $\pi = \pi_0 \oplus \pi_2$. This action α does not have dual unitary property and $\mathcal{F}(M, S_3, \alpha)$ does not have a structure of fusion rule algebra.

Counterexample 2 $G = S_3 = \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$, $G_0 = \mathbb{Z}_3$, $M = M(6, \mathbb{C})$, $\alpha = \mathrm{Ad}\lambda$.

Since the pair (G, G_0) is not admissible, $\mathcal{F}(M^G \subset M^{G_0})$, does not have a structure of a fusion rule algebra by Theorem 5.4.

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