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STUDY ON CONVEXITY AND INEQUALITIES ASSOCIATED WITH GENERALIZED EXTENDED BETA FUNCTION AND CONFLUENT HYPERGEOMETRIC FUNCTION

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Abstract: The aim of the present paper is to establish the logarithmic convexity and some inequalities for the generalized extended beta function and, by using these inequalities for the generalized extended beta function, find the logarithmic convexity and the monotonicity for the generalized extended confluent hyper geometric function.

1. PRELIMINARIES

The extended beta function [3] is defined by:

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-p/t(1-t)}dt \quad (\Re(p),\Re(x),\Re(y)>0), \tag{1.1}$$

Clearly, if $p=0,$ then $B(x,y;0)=B(x,y)$, the classical beta function [11].

The generalized extended beta function $B_p^{(\xi,\zeta;\eta)}(x,y)$ defined by (see [10])

$$B_p^{(\xi,\zeta;\eta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\xi;\zeta;-\frac{p}{t^\eta(1-t)^\eta})dt), \tag{1.2}$$

$$(\Re(p)\geq 0;\min\left[\Re(x),\Re(y),\Re(\xi),\Re(\zeta)\right]>0;\Re(\eta)>0).$$

The classical Beta function $B(x,y)$ is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (\Re(x)>0,\Re(y)>0) \tag{1.3}$$

It is clear that there is following relationship between the classical Beta function $B(x,y)$ and its extensions:

$$B(x,y)=B_0(x,y)=B_0^{(\xi,\zeta)}(x,y)=B_0^{(\xi;\zeta;1)}(x,y)$$

The extended confluent hyper geometric function [4], is defined as:

$$\Phi_p(\zeta,\gamma;z)=\sum_{0}^{\infty}\frac{B(\zeta+n,\gamma-\zeta;p)}{B(\zeta,\gamma-\zeta)}\frac{z^n}{n!},$$

Where $\Re(p)>0$ and $\zeta,\gamma\in\mathbb{C}$ with $\gamma\neq 0,-1,-2,......$

An integral representation of $\phi_p(\zeta,\gamma;z)$ as given in [4, Eq. (3.7)] by

$$\Phi_p(\zeta,\gamma;z)=\frac{1}{B(\zeta,\gamma-\zeta)}\int_0^1 t^{\zeta-1}(1-t)^{\gamma-\zeta-1}\exp(zt-\frac{p}{t(1-t)})dt \tag{1.4}$$

$$p\geq 0,\Re(\gamma)>\Re(\zeta)>0.$$

Another generalized form of the extended confluent hyper geometric function $\Phi_p^{\xi,\zeta,\eta}(x,y;t)$ is defined by:

$$\Phi_p^{\xi,\zeta,\eta}(x,y;t)=\frac{1}{B(x,y-x)}\int_0^1 t^{x-1}(1-t)^{y-x-1}\exp(zt)\,{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})dt \tag{1.5}$$

$$(\Re(p)\geq 0;\min\left[\Re(x),\Re(y),\Re(\xi),\Re(\zeta)\right]>0\,.$$

In the present paper, we present some inequalities for generalized extended beta functions $B_p^{(\xi,\zeta;\eta)}(x,y)$ defined in (1.4). Also, we find the monotonicity and the logarithmic convexity for functions related to extended confluent hyper geometric functions $\Phi_p^{\xi,\zeta,\eta}(x,y;t)$ defined in (1.5).

2. DEFINITIONS AND LEMMAS

Now we recall definitions of some convex functions and recite several lemmas.

Definition 1: ([2,6]). Let X be a convex set in real vector space and let $g:X\to\mathbb{R}$ be a function. The function g is said to be convex on X if the inequality

$$g(\xi x_1+(1-\xi)x_2)\leq\alpha g(x_1)+(1-\xi)g(x_2)$$

Is valid for any $x_1,x_2\in X$ and $\xi\in[0,1]$.

A function g is said to be concave if $-g$ is convex.

A function g is said to be logarithmically convex (or logarithmically concave respectively) on X if $g > 0$ and $\ln g$ (or $-\ln g$ respectively) in convex (or concave respectively) on X.

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Lemma 1 (Chebychev's integral inequality [7, 9, 5, 8]. Let $f, g, h : I \subseteq \Re \to \Re$ be mapping such that $h(x) \geq 0, h(x) f(x) g(x), h(x) f(x),$ and $h(x) g(x)$ are integrable on I. if $f(x)$ and $g(x)$ are synchronous (or asynchronous respectively) on I, that is,

$$[f(x) - f(y)][g(x) - g(y)] \gtreqless 0$$

For all $x, y \in I$, then

$$\int_I h(x) dx \int_1 h(x) f(x) g(x) dx \gtreqless \int_1 h(x) f(x) dx \int_1 h(x) g(x). \tag{2.1}$$

Lemma 2: (Holder's inequality [12,14]). Let θ_1 and θ_2 be positive numbers such that $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$ and let $f, g : [a,b] \to \mathbb{R}$ be integrable functions. Then

$$\left| \int_a^b f(x) g(x) dx \right| \leq \left(\int_a^b \left| f(x) \right|^{\theta_1} dx \right)^{\frac{1}{\theta_1}} \left(\int_a^b \left| g(x)^{\theta_2} \right| dx \right)^{\frac{1}{\theta_2}}. \tag{2.2}$$

Lemma 3: ([1]). Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, with $a_n \in \mathbb{R}$ and $b_n > 0$ for all n, converge on $(-\xi, \xi)$. If sequence $\left\{ \frac{a_n}{b_n} \right\}_{n \geq 0}$ is increasing (or decreasing respectively), then $x \mapsto \frac{f(x)}{g(x)}$ is also increasing (or decreasing respectively) on $(0, \xi)$.

3. INEQUALITIES FOR GENRALISED EXTENDED BETA FUNCTIONS

Now we establish inequalities for functions involving generalized extended beta function (1.4).

Theorem 1: if x, y, x_1, y_1 are positive number such that $(x-x_1)(y-y_1)\geq 0$, then

$$B_p^{(\xi,\zeta,\eta)}(x,y_1)B_p^{(\xi,\zeta,\eta)}(x_1,y)\leq B_p^{(\xi,\zeta,\eta)}(x_1,y_1)B_p^{(\xi,\zeta,\eta)}(x,y) \tag{3.1}$$

Proof. Consider the mapping $f,g,h:[0,1]\rightarrow[0,\infty]$ given by

$$f(t)=t^{x-x_1},\, g(t)=(1-t)^{y-y_{1,}}$$

And

$$h(t)=t^{x_1-1}(1-t)^{y_1-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})$$

Since

$$f^{'}(t)=(x-x_1)t^{x-x_1-1} \text{ and } g^{'}(t)=(y_1-y)(1-t)^{y-y_1-1}$$

By virtue of Lemma 1, we observe that the mapping *f* and *g* are synchronous (asynchronous) 0n [0,1] and h is non-negative [0,1]. Thus, using Chebychev's integral inequality for the function *f*, *g* and *h*, we arrive at

$$\begin{aligned}&\int_0^1(t^{x_1-1}(1-t)^{y_1-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})dt\\&\times\int_0^1(t^{x-x_1}(1-t)^{y-y_1}t^{x_1-1}(1-t)^{y_1-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})dt\end{aligned} \tag{3.2}$$

$$\leq \begin{aligned}&\int_0^1(t^{x_1-1}t^{x-x_1}(1-t)^{y_1-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})dt\\&\times\int_0^1(t^{x_1-1}(1-t)^{y_1-1}(1-t)^{y-y_1}{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})dt \qquad (x\in(0,1))\end{aligned}$$

Which implies

$$\int_0^1 (t^{x_1-1}(1-t)^{y_1-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})dt$$
$$\times\int_0^1 (t^{x-1}(1-t)^{y-1}{}_1F_1\left(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta}\right)dt$$
$$\le \int_0^1 t^{x-1}(1-t)^{y_1-1}{}_1F_1\left(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta}\right)dt \tag{3.3}$$
$$\times\int_0^1 (t^{x_1-1}(1-t)^{y-1}{}_1F_1\left(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta}\right)dt \qquad (x\in(0,1))$$

Further using (1.4) , we easily arrive at the result (3.1)

Corollary 1. For $x, x_1 > 0$, we have

$$B_p^{(\xi,\zeta,\eta)}(x,x_1) \ge \sqrt{B_p^{(\xi,\zeta,\eta)}(x,x)B_p^{(\xi,\zeta,\eta)}(x_1,x_1)} \tag{3.4}$$

Proof. This follows immediately form setting $y = x$ and $y_1 = x_1$ in Theorem 1 directly.

For $\eta = 1$ in Theorem (1) corollary (3.4), we have

Corollary 2. If x, y, x_1, y_1 are positive numbers such that $(x-x_1)(y-y_1) \ge 0$, then

$$B_p^{\xi,\zeta}(x,y_1)B_p^{\xi,\zeta}(x_1,y) \le B_p^{\xi,\zeta}(x_1,y_1)B_p^{\xi,\zeta}(x,y). \tag{3.5}$$

Corollary 3. For $x, x_1 > 0$, we have

$$B_p^{\xi,\zeta}(x,x_1) \ge \sqrt{B_p^{\xi,\zeta}(x,x)B_p^{\xi,\zeta}(x_1,x_1)}\,. \tag{3.6}$$

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Theorem 2. The function $(x,y) \mapsto B_p^{\xi,\zeta,\eta}(x,y)$ is logarithmically convex on $(0,\infty)$ x $(0,\infty)$ for all $p \ge 0$ and $\lambda > 0$. Consequently,

$$\left[B_p^{(\xi,\zeta,\eta)}(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2})\right]^2 \le B_p^{(\xi,\zeta,\eta)}(x_1,y_1)B_p^{(\xi,\zeta,\eta)}(x_2,y_2) \tag{3.7}$$

Proof. Let $(x_1,y_1),(x_2,y_2)\in(0,\infty)^2$ and let $c,d\ge 0$ with $c+d=1$. Then

$$B_p^{(\xi,\zeta,\eta)}(c(x_1,y_1)+d(x_2,y_2)) = B_p^{(\xi,\zeta,\eta)}(cx_1+dx_2,cy_1+dy_2)$$

By definitions, we have

$$B_p^{(\xi,\zeta,\eta)}(c(x_1,y_1)+d(x_2,y_2)) = \int_0^1 t^{cx_1+dx_2-1}(1-t)^{cy_1+dy_2-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})dt$$

$$=\int_0^1 t^{cx_1+dx_2-(c+d)}(1-t)^{cy_1+dy_2-(c+d)}\left[{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})\right]^{c+d}dt$$

$$=\int_0^1 t^{c(x_1-1)}t^{dx_2-1}(1-t)^{c(y_1-1)}(1-t)^{d(y_2-1)} \times\left[{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})\right]^c\left[{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})\right]^d dt$$

$$\int_0^1\left[t^{x_1-1}(1-t)^{y_1-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})\right]^c\left[t^{x_2-1}t^{y_2-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})\right]^d dt$$

Setting $\theta_1=\frac{1}{c}$ and $\theta_2=\frac{1}{d}$ and using the Holder inequality (2.2 give)

$$B_p^{(\xi,\zeta,\eta)}(c(x_1,y_1)+d(x_2,y_2)) \le \left[\int_0^1 t^{x_1-1}(1-t)^{y_1-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})dt\right]^c$$

$$\times\left[\int_0^1 t^{x_2-1}t^{y_2-1}{}_1F_1(\xi,\zeta,-\frac{p}{t^\eta(1-t)^\eta})dt\right]^d$$

$$=\left[B_p^{(\xi,\zeta,\eta)}(x_1,y_1)\right]^c\left[B_p^{(\xi,\zeta,\eta)}(x_2,y_2)\right]^d.$$

Accordingly, the function $B_p^{(\xi,\zeta,\eta)}(x,y)$ is logarithmically convex on $(0,\infty)^2$.

When $c = d = \frac{1}{2}$, the above inequality reduce to (3.7).

Remark 1: Letting $x, y > 0$ such that $\min_{a\in\mathbb{R}}(x+a, x-a) > 0$ and taking $x_1 = x+a$, $x_2 = x-a$, $y_1 = y+b$, $y_2 = y-b$ in (3.7) result in

$$\left[B_p^{(\xi,\zeta,\eta)}(x,y)\right]^2 \le B_p^{(\xi,\zeta,\eta)}(x+a, y+b)B_p^{(\xi,\zeta,\eta)}(x-a, y-b)$$

For all $\Re(p) \ge 0$ and $\min\{\Re(x), \Re(y), \Re(\xi), \Re(\zeta), \Re(\eta)\} > 0$.

4. INEQUALITIES FOR EXTENDED CONFLUENT HYPERGEOMETRIC FUNCTIONS

Now we find the logarithmic convexity and the monotonicity related to the extended confluent hyper geometric function $\Phi_p^{\xi,\zeta,\eta}(x,y;t)$ defined in (1.5).

Theorem 3: Let $x \ge 0$ and $y, z > 0$.

(1) For $y \ge z$, the function $t \mapsto \dfrac{\Phi_p^{\xi,\zeta,\eta}(x,y;t)}{\Phi_p^{\xi,\zeta,\eta}(x,z;t)}$ is increasing on $(0,\infty)$.

(2) For $y \ge z$,

$$z\Phi_p^{\xi,\zeta,\eta}(x+1, y+1;t)\Phi_p^{\xi,\zeta,\eta}(x,z;t) \ge y\Phi_p^{\xi,\zeta,\eta}(x,y;t)\Phi_p^{\xi,\zeta,\eta}(x+1, z+1;t). \quad (4.1)$$

(3) The function $t \mapsto \Phi_p^{\xi,\zeta,\eta}(x,y;t)$ is logarithmically convex on $\mathbb{R}$.

(4) For $\sigma, y, t > 0$, the function

$$x \mapsto \frac{B(x,y)\Phi_p^{\xi,\zeta,\eta}(x+\sigma, y;t)}{B(x+\sigma, y)\Phi_p^{\xi,\zeta,\eta}(x,y;t)}$$

Is decreasing on $(0,\infty)$.

Proof. By the definition in (1.5), we have

$$\frac{\Phi_p^{\xi,\zeta,\eta}(x,y;t)}{\Phi_p^{\xi,\zeta,\eta}(x,z;t)} = \frac{\sum_{n=0}^{\infty} a_n(c)t^n}{\sum_{n=0}^{\infty} a_n(d)t^n},$$

Where

$$a_n(z) = \frac{B_p^{\xi,\zeta,\eta}(x+n, z-x)}{B_p^{\xi,\zeta,\eta}(x, z-x)}$$

If denoting $f_n = \frac{a_n(c)}{a_n(d)}$, then

$$f_n - f_{n+1} = \frac{a_n(c)}{a_n(d)} - \frac{a_{n+1}(c)}{a_{n+1}(d)}$$

$$= \frac{B(x, z-x)}{B(x, y-x)} \left[\frac{B_p^{\xi,\zeta,\eta}(x+n, y-x)}{B_p^{\xi,\zeta,\eta}(x+n, z-x)} - \frac{B_p^{\xi,\zeta,\eta}(x+n+1, y-x)}{B_p^{\xi,\zeta,\eta}(x+n+1, z-x)} \right]$$

By taking $p = x+n$, $q = z-x$ $r = x+n+1$, and $s = y-x$ and using (3.1), since $(p-r)(q-s) = y-z \geq 0$, it follows from Theorem 1 that

$$\frac{B_p^{\xi,\zeta,\eta}(x+n, y-x)}{B_p^{\xi,\zeta,\eta}(x+n, z-x)} \leq \frac{B_p^{\xi,\zeta,\eta}(x+n+1, y-x)}{B_p^{\xi,\zeta,\eta}(x+n+1, z-x)}$$

Which is equivalent to say that $\{f_n\}_{n\geq 0}$ is an increasing sequence. Hence, with the aid of Lemma 3, we conclude that $t \mapsto \frac{\Phi_p^{\xi,\zeta,\eta}(x, y; t)}{\Phi_p^{\xi,\zeta,\eta}(x, y; t)}$ is increasing on $(0, \infty)$.

Also from [13] that

$$\frac{d^n}{dx^n} \Phi_p^{\xi,\zeta,\eta}(x, y; t) = \frac{(x)_n}{(y)_n} \Phi_p^{\xi,\zeta,\eta}(x+n, y+n; t) \tag{4.2}$$

Since the increasing property of $t \mapsto \frac{\Phi_p^{\xi,\zeta,\eta}(x, y; t)}{\Phi_p^{\xi,\zeta,\eta}(x, y; t)}$ is equivalent to

$$\frac{d}{dx} \left[\frac{\Phi_p^{\xi,\zeta,\eta}(x, y; t)}{\Phi_p^{\xi,\zeta,\eta}(x, z; t)} \right] \geq 0$$

Together with (4.2), we further obtain

$$\Phi_p^{\xi,\zeta,\eta}(x,y;t)\Phi_p^{\xi,\zeta,\eta}(x,z;t)-\Phi_p^{\xi,\zeta,\eta}(x,y;t)\Phi_p^{\xi,\zeta,\eta}(x,z;t)$$

$$=\frac{x}{y}\Phi_p^{\xi,\zeta,\eta}(x+1,y+1;t)\Phi_p^{\xi,\zeta,\eta}(x,z;t)-\frac{x}{z}\Phi_p^{\xi,\zeta,\eta}(x,y;t)\Phi_p^{\xi,\zeta,\eta}(x+1,z+1;t)\geq 0$$

This implies the inequality (4.1).

The logarithmic convexity of $t\mapsto\Phi_p^{\xi,\zeta,\eta}(x,y;t)$ can be proved by using the integral representation (1.5) and by applying the Holder inequality (2.2) as follows:

$$\Phi_p^{\xi,\zeta,\eta}(x,y;al+(1-a)m)$$
$$=\frac{1}{B(x,y-x)}\int_0^1 t^{x-1}(1-t)^{y-x-1}\exp(alt+(1-a)mt)\,{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})dt$$

$$=\frac{1}{B(x,y-x)}\int_0^1\left(t^{x-1}(1-t)^{y-x-1}\exp(lt)\,{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})\right)^a$$

$$\times\left(t^{x-1}(1-t)^{y-x-1}\exp(mt)\,{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})\right)^{1-\xi}dt$$

$$\leq\left[\frac{1}{B(x,y-x)}\int_0^1 t^{x-1}(1-t)^{y-x-1}\exp(lt)\,{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})dt\right]^a$$

$$\times\left[\frac{1}{B(x,y-x)}\int_0^1 t^{x-1}(1-t)^{y-x-1}\exp(mt)\,{}_1F_1(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}})dt\right]^{1-a}$$
$$=\left[\Phi_p^{\xi,\zeta,\eta}\left(x,y;l\right)\right]^a\left[\Phi_p^{\xi,\zeta,\eta}\left(x,y;m\right)\right]^{1-a}$$

For $l,m>0$ and $\xi\in[0,1]$.

For the case l<0, the assertion follows immediately from the identity

$$\Phi_p^{\xi,\zeta,\eta}\left(x,y;l\right)=e^l\Phi_p^{\xi,\zeta,\eta}\left(y-x,y;-l\right)$$

In [13]

Let $x'\geq x$ and

$$h(t)=t^{x'-1}(1-t)^{y-x'-1}\exp(lt)\,{}_1F_1\left(\xi,\zeta,-\frac{p}{t^{\eta}(1-t)^{\eta}}\right)$$

$$f(t)=\left(\frac{t}{1-t}\right)^{x-x'},\quad g(t)=\left(\frac{t}{1-t}\right)^{\sigma}$$

Using the integral representation (1.5), we have

$$\frac{B(x,y)\Phi_p^{\xi,\zeta,\eta}(x+\sigma,y;l)}{B(x+\sigma,l)\Phi_p^{\xi,\zeta,\eta}(x,y;l)}-\frac{B(x',y)\Phi_p^{\xi,\zeta,\eta}(x'+\sigma,y;l)}{B(x'+\sigma,y)\Phi_p^{\xi,\zeta,\eta}(x',y;l)} \tag{4.3}$$

$$=\frac{\int_0^1 f(t)g(t)h(t)dt}{\int_0^1 f(t)h(t)dt}-\frac{\int_0^1 g(t)h(t)dt}{\int_0^1 h(t)dt}$$

When $\sigma\geq 0$ and the function *f* is decreasing and the function g is increasing. Since h is a non-negative function for $t\in[0,1]$, by Chebyshev's inequality (1), it follows that

$$\int_0^1 f(t)h(t)dt\int_0^1 g(t)h(t)dt\leq\int_0^1 h(t)dt\int_0^1 f(t)g(t)h(t)dt$$

Combining this with (4.3) yields

$$\frac{B(x,y)\Phi_p^{\xi,\zeta,\eta}(x+\sigma,y;l)}{B(x+\sigma,l)\Phi_p^{\xi,\zeta,\eta}(x,y;l)}-\frac{B(x',y)\Phi_p^{\xi,\zeta,\eta}(x'+\sigma,y;l)}{B(x'+\sigma,y)\Phi_p^{\xi,\zeta,\eta}(x',y;l)}\geq 0$$

Which is equivalent to say that the function

$$x\mapsto\frac{B(x,y)\Phi_p^{\xi,\zeta,\eta}(x+\sigma,y;l)}{B(x+\sigma,y)\Phi_p^{\xi,\zeta,\eta}(x,y;l)}$$

Is decreasing on $(0,\infty)$. The proof of the Theorem 3 is complete.

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