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A GENERALIZATION OF TOTALLY POSITIVE FUNCTIONS

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Abstract: *A real valued function defined on a closed bounded interval is said to be generalized totally positive if all of its generalized Lagrange interpolants are positive. In this paper, we show that a real valued function which is generalized totally positive is infinitely differentiable.*

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1. INTRODUCTION

In [2] T. Popoviciu introduced generalized divided differences based on any complete Tchebycheff system. Interpolation of functions in an extended complete Tchebycheff Space (ECT space) can be done by using generalized divided differences. If f is any function defined on a closed interval $[a, b]$, then an explicit expression for the generalized polynomial interpolating to f at given points can be derived in a way similar to the Newton form for Lagrange interpolating polynomials.

Alan. L. Horwitz and Lee. A. Rubel in [1] introduced totally positive functions on [–1,1]. We generalize totally positive functions by using generalized Lagrange interpolants.

Definitions and results from theory of Tchebycheff spaces and generalized divided differences are discussed in the next section. In section 3, we introduce generalized totally positive funcions and prove our main result there. Throughout this paper, we denote the closed bounded interval $[a, b]$ by I.

2. PRELIMINARIES

We refer to chapters 2 and 9 of [3] for the definitions and results in this section. The results given in this section are those needed to prove our main theorem in the next section.

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Extended completeTchebycheff spaces.

Definition 2.1. Let $u_1, u_2, \ldots, u_m$ be real valued functions defined on $I = [a, b]$ and let $x_1 \le x_2 \le \cdots \le x_m$ be points in I. The collocation matrix associated with $\{u_i\}_1^m$ *and* $\{x_i\}_1^m$ is denoted by $M\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, u_m \end{pmatrix}$ and is defined by

$$M\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, u_m \end{pmatrix} = [D^{d_i} u_j(x_i)]_{i,j=1}^{m}$$

where $d_i = \max\{j : x_i = x_{i-j}\}$, $i = 1, 2 \cdots, m$, provided the d_i^{th} derivative of u_j exists at the points x_i, $i, j = 1, 2, \cdots, m$.

Remark 1. In the above definition, if the points $x_1, x_2, \cdots, x_m$ are all distinct, then the collocationmatrix becomes

$$M\begin{pmatrix} x_1 \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, u_m \end{pmatrix} = \begin{bmatrix} u_1(x_1) & \cdots & u_{m-1}(x_1) u_m(x_1) \\ u_1(x_2) & \cdots & u_{m-1}(x_2) u_m(x_2) \\ \cdots & \cdots & \cdots \\ u_1(x_m) & \cdots & u_{m-1}(x_m) u_m(x_m) \end{bmatrix}$$

The Determinant associated with the matrix $\mathrm{M}\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, u_m \end{pmatrix}$ is denoted by $\mathrm{D}\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, u_m \end{pmatrix}$.

Definition 2.2. Let $U_m = \{u_i\}_1^m$ be any collection of functions in $C^{m-1}(I)$. U_m is called an extended Tchebycheff system (ET-system) on I if the determinants associated with the collocation matrix $\mathrm{M}\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, u_m \end{pmatrix}$ is positive for all $x_1 \le x_2 \le \cdots \le x_m$ in I.

Definition 2.3. Let $\{u_1, u_2, \cdots\}$ be any finite or infinite sequence of functions in I. If for each k, $\{u_1, u_2, \ldots, u_k\}$ forms an ET-system on I, then $\{u_1, u_2, \cdots\}$ is called an extended complete Tchebycheff system (ECT-system) on I.

Remark 2. The determinant of the collocation matrix arising from an ECT-system $U_m = \{u_1, u_2, \cdots u_m\}$ is denoted by $D_{U_m}(x_1, x_2, \cdots, x_m)$. That is,

$$D_{U_m}(x_1, x_2, \cdots, x_m) = \mathrm{D}\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, u_m \end{pmatrix}$$

Definition 2.4. A subspace of $C(I)$, finite or infinite dimensional, is called an Extended complete Tchebycheff space (ECT-space) if it has an ordered basis which is an ECT-system.

Elements of an ECT-space are called generalized polynomials.

The following theorem gives an estimate for the derivatives of generalized polynomials.

Theorem 2.1. (cf.[3], p. 370) (Markov inequality) Let $\mathcal{U}_m$ be an ECT-space on I. Then there exists a constant c_m (depending only on $\mathcal{U}_m$) such that for each $u \in \mathcal{U}_m$ and $j = 1, 2, \cdots m - 1$,

$$\| D^j u \|_\infty \leq c_m \, \mathrm{h}^{-j-1} \| u \|_\infty$$

where h is the length of I.

Definition 2.5. Let $U_m = \{u_1, u_2, \cdots u_m\}$ be an ECT-system on I, and let f be a sufficiently differentiable function defined on I. Associated with the points $x_1 \leq x_2 \leq \cdots \leq x_m$ in I we define a function on I as follows:

$$\mathrm{D}\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \,; x \\ u_1, \cdots, u_{m-1}, u_m, f \end{pmatrix} = \det \begin{bmatrix} u_1^{(d_1)}(x_1) & \cdots & u_m^{(d_1)}(x_1) & f^{(d_1)}(x_1) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ u_1^{(d_i)}(x_i) & \cdots & u_m^{(d_i)}(x_i) & f^{(d_i)}(x_i) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ u_1^{(d_m)}(x_m) & \cdots & u_m^{(d_m)}(x_m) & f^{(d_m)}(x_m) \\ u_1(x) & \cdots & u_m(x) & f(x) \end{bmatrix}$$

where $d_i = \max\{j : x_i = x_{i-j}\}$, $i = 1, 2, \cdots, m$.

Remark 3. If $U_{m+1} = \{u_1, u_2, \cdots u_m, u_{m+1}\}$ is an ECT-system on $I = [a, b]$, then the function in Definition 2.5 with f replaced by u_{m+1} is denoted by $D_{U_{m+1}}(x_1, x_2, \cdots, x_m; x)$.

Remark 4. A well-known example of an infinite ECT-system on any interval $I = [a, b]$ is $V = \{1, x, x^2, \cdots\}$. In this case, V forms an ordered basis for $\mathcal{P}$, the space of polynomials on I. For each n, $V_n = \{1, x, x^2, \cdots, x^{n-1}\}$ is an ECT-system forming a basis for $\mathcal{P}_n$, the space of all polynomials of degree atmost $n - 1$.

Remark 5. (cf.[3], p.30) For the ECT-system $V_n = \{1, x, x^2, \cdots, x^{n-1}\}$ on I, the determinant $D_{V_n}(x_1, x_2, \cdots, x_n)$ is the well-known Vandermonde determinant. When $x_1, x_2, \cdots, x_n$ are distinct points in I, this determinant has the following value.

$$D_{V_n}(x_1, x_2, \cdots, x_n) = \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j)$$

The following theorem establishes the connection between the determinants $D_{V_n}(x_1, x_2, \cdots, x_n)$ and $D_{U_n}(x_1, x_2, \cdots, x_n)$, where U_n is any ECT-system on I.

Theorem 2.2 (cf [3], p. 367) Let U_n be any ECT-system on $I = [a, b]$. There exists constants $c_{1,n}$ and $c_{2,n}$ such that for all choices of points $x_1 < x_2 < \cdots < x_m$ in I

$$c_{1,n} D_{V_n}(x_1, x_2, \cdots, x_n) \leq D_{U_n}(x_1, x_2, \cdots, x_n) \leq c_{2,n} D_{V_n}(x_1, x_2, \cdots, x_n)$$

and for any x in I,

$$c_{1,n}|D_{V_n}(x_1, x_2, \cdots, x_{n-1}; x)| \leq |D_{U_n}(x_1, x_2, \cdots, x_{n-1}; x)| \leq c_{2,n}|D_{V_n}(x_1, x_2, \cdots, x_{n-1}; x)|$$

where

$$D_{V_n}(x_1, x_2, \cdots, x_n) = \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j)$$

and for all $x \in I$,

$$D_{V_n}(x_1, x_2, \cdots, x_{n-1}; x) = \left\{ \prod_{\substack{i,j=1 \\ i>j}}^{n-1} (x_i - x_j) \right\} \left\{ \prod_{i=1}^{n-1} (x - x_i) \right\}.$$

Generalized Divided Differences

Definition 2.6: Suppose $U_m = \{u_1, u_2, \cdots u_m\}$ is an ECT-system in I. Given any function f defined on I, its $(m-1)$th order divided difference with respect to U_m at m distinct points $x_1, x_2, \cdots, x_m$ in I is defined as

$$[x_1, x_2, \cdots, x_m]_{U_m} f = \frac{\mathrm{D}\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, f \end{pmatrix}}{\mathrm{D}\begin{pmatrix} x_1, \cdots, x_{m-1}, x_m \\ u_1, \cdots, u_{m-1}, u_m \end{pmatrix}}$$

Remark 6. For the ECT-system $V_n = \{1, x, x^2, \cdots, x^{n-1}\}$ on $I = [a, b]$, $[x_1, x_2, \cdots, x_n]_{V_n} f$ is the classical divided difference.

For the classical divided difference, we have the following result.

Theorem 2.3 (cf.[3], p. 47) If f is in $C^{n-1}[a, b]$ and if $a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b$ are n points, then

$$[x_1, x_2, \cdots, x_n]_{V_n} f = \frac{f^{n-1}(\xi)}{(n-1)!}$$

for some ξ, $x_1 < \xi < x_n$.

In this paper, we further make use of the following property of classical divided difference which can be obtained from the results in section 8 in chapter 2 of [3].

Theorem 2.4. Let $n > 1$. Let f be a bounded function defined in $[a, b]$. If there exists constants M_n such that for all $a < x_1 < x_2 < \cdots < x_n < b$,

$$|\,[x_1, x_2, \cdots, x_n]_{V_n} f\,| \leq M_n,$$

then f belongs to $C^{n-2}\,[a, b]$.

As an application of generalized divided differences, interpolation by functions in an ECT-space can be done. That is, if $\mathcal{U}_m$ is an ECT-space on $I = [a, b]$ and if $x_1 < x_2 < \cdots < x_m$ are prescribed points in I, then for any given function f defined on I, there corresponds a unique generalized polynomial in $\mathcal{U}_m$ which interpolates to f at the points $x_1 < x_2 < \cdots < x_m$ (cf.[3]). An explicit expression for this unique generalized polynomial p_m in $\mathcal{U}_m$ and also an expression for the error $f - p_m$ can be derived.

Theorem 2.5 (cf.[3]. p. 370) Let $\mathcal{U}_m$ be an ECT-space on $I = [a, b]$ and let x_1, x_2, $\cdots$, x_m be distinct points in I. Let f be any function defined on I, which is sufficiently differentiable.

(i) Then an explicit expression for the unique generalized polynomial p_m in $\mathcal{U}_m$ satisfying the condition

$$p_m(x_i) = f(x_i), \qquad i = 1,2, \cdots, m$$

is given by

$$p_m(x) = [x_1]_{U_1} f.\, D_{U_1}(x) + [x_1, x_2]_{U_2} f.\, \frac{D_{U_2}(x_1;\ x)}{D_{U_1}(x_1)} + \cdots + [x_1, x_2, \cdots, x_m]_{U_m} f.\, \frac{D_{U_m}(x_1, x_2, \cdots, x_{m-1}; x)}{D_{U_{m-1}}(x_1, \ \cdots, x_{m-2}, x_{m-1})}$$

where $U_m = \{u_1, u_2, \cdots, u_m\}$ is an ECT-system forming a basis for $\mathcal{U}_m$ and $U_k = \{u_1, u_2, \cdots, u_k\}$, $k = 1, 2, \cdots, m$.

(iii) The error is given by

$$f(x) - p_m(x) = [x_1, x_2, \cdots, x_m; x]_{U_{m+1}} f.\, \frac{D_{U_{m+1}}(x_1, \cdots, x_m; x)}{D_{U_m}(x_1, \cdots, x_m)}$$

where $U_{m+1} = \{u_1, u_2, \cdots u_m, u_{m+1}\}$ is an ECT-system on I containing U_m.

3. GENERALIZED TOTALLY POSITIVE FUNCTIONS

Throughout this section, $\mathcal{U}$ is an infinite dimensional ECT-space on $I = [a, b]$ with an ordered basis $U = \{u_1, u_2, \cdots\}$ which is an ECT-system on I. For each m, $\mathcal{U}_m$ is the ECT- space spanned by $U_m = \{u_1, u_2, \cdots, u_m\}$.

For a function f defined on I, we denote by $L(x_1, x_2, \cdots, x_m; f)$, the unique generalized polynomial in $\mathcal{U}_m$ interpolating to f at m distinct points $x_1, x_2, \cdots, x_m$ in I.

Definition 3.1. Let f be a real valued function defined on I. A generalized polynomial p is called a generalized Lagrange interpolant to f from $\mathcal{U}$ if

$$p = L(x_1, x_2, \cdots, x_m; f)$$

for some distinct points $x_1, x_2, \cdots, x_m$ in I.

The collection of all generalized Lagrange interpolants to f from $\mathcal{U}$ is denoted by $\mathcal{GL}_{\mathcal{U}}(f)$.

Definition 3.2. A function f defined on I is said to be generalized totally positive on I if $p > 0$ on I for all $p \in \mathcal{GL}_{\mathcal{U}}(f)$.

The collection of all generalized totally positive functions on I is denoted by $\mathcal{GTP}_{\mathcal{U}}$.

Theorem 3.1. Let $f \in \mathcal{GTP}_{\mathcal{U}}$. Then

(i) for each n, there exists a constant M_n such that for all choices of n points $a \le x_1 < x_2 < \cdots < x_n \le b$,

$$|\, [x_1, x_2, \cdots, x_n]_{U_n} f \,| \le M_n \tag{3.1}$$

(ii) for each n, there exists a constant K_n such that for all $p \in \mathcal{U}_n \cap \mathcal{GL}_{\mathcal{U}}(f)$

$$\| p \|_\infty \le K_n \tag{3.2}$$

Proof. Let x_1 be any point in I. Then

$$p = \frac{f(x_1)}{u_1(x_1)}.u_1$$

belongs to $\mathcal{GL}_{\mathcal{U}}(f)$ and hence $p > 0$ on I. Also $u_1 > 0$ on I. Hence $f(x_1) > 0$. Since x_1 is an arbitrary point in I,

$$f > 0 \text{ on } I. \tag{3.3}$$

We now show that the bound in (3.1) holds for $n = 1$. For, suppose (3.1) does not hold for $n = 1$. Then there exists a sequence (x^i) in I such that

$$\left|[x^i]_{U_1} f\right| \to \infty \qquad as \;\; i \to \infty.$$

That is,

$$\left|\frac{f(x^i)}{u_1(x^i)}\right| = \frac{f(x^i)}{u_1(x^i)} \text{as} i$$

Since u_1 is positive and bounded away from zero on I, $f(x^i) \to \infty$ as $i \to \infty$. Since I is compact, we may assume that (x^i) converges to some point x^0 in I. Choose a point $\alpha \in (a, b) \setminus \{x^0\}$. For each i, let $p_2^i = L(\alpha, x^i; f)$. Then by (i) of Theorem 2.5, for all $x \in$ I,

$$p_2^i(x) = \frac{f(\alpha)}{u_1(\alpha)}.\, u_1(x) + [\alpha, x^i]_{U_2} f.\ \frac{D_{U_2}(\alpha;\ x)}{D_{U_1}(\ \alpha\ )}$$

Also

$$[\alpha, x^i]_{U_2} f = \frac{D\begin{pmatrix}\alpha & x^i\\ u_1 & f\end{pmatrix}}{D_{U_2}(\alpha,\ x^i)} = \frac{u_1(\alpha) f(x^i) - u_1(x^i) f(\alpha)}{D_{U_2}(\alpha,\ x^i)}$$

Since $f(x^i) \to \infty$,

$$u_1(\alpha)\, f(x^i) - u_1(x^i)\, f(\alpha) \to \infty \ as\ i \to \infty.$$

Also, $D_{U_2}(\alpha, x^i) \to D_{U_2}(\alpha, x^0)$, which is positive if $\alpha < x^0$ and negative if $\alpha > x^0$. Therefore,

$$[\alpha, x^i]_{U_2} f \to \begin{cases} +\infty, & if \quad \alpha < x^0 \\ -\infty, & if \quad \alpha > x^0 \end{cases}$$

Choose a point β in I such that $\beta \in [a, \alpha)$ if $\alpha < x^0$ and $\beta \in (\alpha, b]$ if $\alpha > x^0$. Then

$$[\alpha, x^i]_{U_2} f . \frac{D_{U_2}(\alpha, \beta)}{D_{U_\perp}(\alpha)} \to -\infty \ as\ i \to \infty.$$

Therefore, $p_2^i(\beta) \to -\infty$ as $i \to \infty$, which is a contradiction, since $p_2^i > 0$ on I. This proves that there exists a constant M_1 such that for all $x_1 \in I$

$$|\, [x_1]_{U_1} f \,| \le M_1.$$

Therefore (3.1) holds for $n = 1$.

Now if p is any element of $\mathcal{U}_1 \cap \mathcal{GL}_{\mathcal{U}}(f)$, then for some x_1 in I, $p = L(x_1; f)$. Then

$$p(x) = [x_1]_{U_1} f \,.\, u_1(x)$$

Since (3.1) holds for $n = 1$,

$$|\, p(x) \,| \le M_1 \|u_1\|_\infty = K_1 \quad (say)$$

Therefore, for all $p \in \mathcal{U}_1 \cap \mathcal{GL}_{\mathcal{U}}(f)$, $\|p\|_\infty \le K_1$. Thus (3.2) also holds for $n = 1$.

Next assume that (3.1) and (3.2) hold for some positive integer m. Suppose (3.1) does not hold for $n = m + 1$. Then there exists a sequence (x^i) in I^{m+1}, $x^i = (x_1^i, \cdots, x_{m+1}^i)$ with distinct co-ordinates such that

$$|\left[x_1^i,\, x_2^i, \cdots, x_{m+1}^i\right]_{U_{m+1}} f\,| \to \infty \text{ as } i \to \infty. \tag{3.4}$$

Now (x^i) is a sequence in I^{m+1} and since I^{m+1} is compact, we may assume that (x^i) converges to some $x^0 = \left(x_1^0,\, x_2^0, \cdots, x_{m+1}^0\right) \in I^{m+1}$. Now choose a point $\alpha \in (a,b) \setminus \{x_1^0,\, x_2^0, \cdots,\, x_{m+1}^0\}$. For each i, let $p_m^i = L\left(x_1^i, \cdots, x_m^i; f\right)$ and let $p_{m+1}^i = L(x_1^i, \cdots, x_{m+1}^i; f)$. Then by part(i) of Theorem 2.5

$$p_{m+1}^i(x) = p_m^i(x) + \left[x_1^i,\ x_2^i, \cdots, x_{m+1}^i\right]_{U_{m+1}} f \cdot \frac{D_{U_{m+1}}(x_1^i, \cdots,\ x_m^i\,; x)}{D_{U_m}(x_1^i, \cdots,\ x_m^i)} \tag{3.5}$$

Since $\mathcal{U}_m$ and $\mathcal{U}_{m+1}$ are ECT-spaces,

$$D_{U_m}\left(x_1^i, \cdots, x_m^i\right) \to D_{U_m}\left(x_1^0, \cdots, x_m^0\right) \neq 0$$

even if some of the points x_j^0 coincide and

$$D_{U_{m+1}}(x_1^i, \cdots,\ x_m^i\,; x) \to D_{U_{m+1}}(x_1^0, \cdots,\ x_m^0\,; x) \neq 0$$

for $x \neq x_1^0, \cdots,\ x_m^0$. In particular, this holds for $x = \alpha$. Therefore, by (3.4),

$$\left|\left[x_1^i, \cdots, x_{m+1}^i\right]_{U_{m+1}} f\right| \cdot \frac{|D_{U_{m+1}}(x_1^i, \cdots,\ x_m^i, \alpha)|}{|D_{U_m}(x_1^i, \cdots,\ x_m^i)|} \to \infty \text{ as } i \to \infty. \tag{3.6}$$

Since by assumption, (3.2) holds for $n = m$,

$$\left|p_m^i(\alpha)\right| \le K_m \tag{3.7}$$

From (3.5), (3.6) and (3.7)

$$|\ p_{m+1}^i(\alpha)| \to \infty \qquad \text{as } i \to \infty.$$

Therefore,

$$\left|\, f(\alpha) - p_{m+1}^i(\alpha)\right| \to \infty \text{ as } i \to \infty. \tag{3.8}$$

Now, by (ii) of Theorem 2.5

$$f(\alpha) - \ p_{m+1}^{i}(\alpha) = \left[x_1^i, \cdots, x_{m+1}^i, \alpha\right]_{U_{m+2}} f . \ \frac{D_{U_{m+2}}(x_1^i\,,\ \cdots,\ \ x_{m+1}^i\,,\ \alpha)}{D_{U_{m+1}}(x_1^i\,,\ \cdots,\ \ x_{m+1}^i)}$$

Therefore

$$\left[x_1^i, \cdots, x_{m+1}^i, \alpha\right]_{U_{m+2}} f \ = \left(f(\alpha) - \ p_{m+1}^i(\alpha)\right) . \frac{D_{U_{m+1}}(x_1^i, \cdots, \ x_{m+1}^i)}{D_{U_{m+2}}(x_1^i, \cdots, \ x_{m+1}^i, \alpha)} \quad (3.9)$$

Now

$$\frac{D_{U_{m+1}}(x_1^i, \cdots, \ x_{m+1}^i)}{D_{U_{m+2}}(x_1^i, \cdots, \ x_{m+1}^i, \alpha)} \ \to \ \frac{D_{U_{m+1}}(x_1^0, \cdots, \ x_{m+1}^0)}{D_{U_{m+2}}(x_1^0, \cdots, \ x_{m+1}^0, \alpha)} \ \neq 0 \qquad (3.10)$$

From (3.8), (3.9) and (3.10),

$$\left| \left[x_1^i, \cdots, x_{m+1}^i, \alpha\right]_{U_{m+2}} f \ \right| \to \ \infty \text{ as } i \to \infty. \qquad (3.11)$$

Then some subsequence of $\{\left[x_1^i, \cdots, x_{m+1}^i, \alpha\right]_{U_{m+2}} f\}$ tends to either + ∞ or -∞ as $i \to \infty$. For convenience, we may assume that the sequence itself tends to ± ∞. Then

$$\left[x_1^i, \cdots, x_m^i, \alpha, x_{m+1}^i\right]_{U_{m+2}} f \ = \ \left[x_1^i, \cdots, x_{m+1}^i, \alpha\right]_{U_{m+2}} f \ \to \ \pm\infty \text{ as } \quad i \to \infty.$$

Now, since $\mathcal{U}_{m+1}$ and $\mathcal{U}_{m+2}$ are ECT-spaces,

$$\frac{D_{U_{m+2}}\left(x_1^i\,,\ \cdots,\ \ x_m^i,\ \alpha\,;x\right)}{D_{U_{m+1}}\left(x_1^i\,,\ \cdots,\ \ x_m^i\,,\ \alpha\right)} \ \to \ \frac{D_{U_{m+2}}(x_1^0\,,\ \cdots,\ \ x_m^0, \alpha\,;x)}{D_{U_{m+1}}(x_1^0\,,\ \cdots,\ \ x_m^0\,,\ \alpha)} \ \neq 0$$

for $x \neq x_1^0, \cdots, x_m^0, \alpha$. Choose $\beta \in [a,b] \setminus \{x_1^0, \cdots, x_m^0, \alpha\}$ such that

$$\text{sign of } \frac{D_{U_{m+2}}(x_1^i, \cdots, \ x_m^i, \ \alpha, \ \beta)}{D_{U_{m+1}}(x_1^i, \cdots, \ x_m^i, \alpha)} \ = \begin{cases} -1 & if \quad \lim_{i\to\infty}\left[x_1^i, \cdots, x_{m+1}^i, \alpha\right]_{U_{m+2}} f = \ +\infty \\ +1 & otherwise \end{cases}$$

Thus

$$\left[x_1^i, \cdots, x_m^i, \alpha, x_{m+1}^i\right]_{U_{m+2}} f \ . \ \frac{D_{U_{m+2}}(x_1^i, \cdots, \ x_m^i, \ \alpha, \ \beta)}{D_{U_{m+1}}(x_1^i, \cdots, \ x_m^i, \alpha)} \ \to \ -\infty \text{ as } i \to \infty. \quad (3.12)$$

Let $p_{m+2}^i = L\,(x_1^i\,,\ \cdots, x_m^i, \alpha,\ x_{m+1}^i\,;\ f)$. Again, applying (i) of Theorem 2.5

$$p_{m+2}^i(\beta) = \ p_m^i(\beta) + \ \left[x_1^i,, \cdots, x_m^i, \alpha\right]_{U_{m+1}} f . \frac{D_{U_{m+1}}(x_1^i\,,\ \cdots,\ \ x_m^i\,,\ \beta)}{D_{U_m}(x_1^i\,,\ \cdots,\ \ x_m^i)}$$

$$+ \ \left[x_1^i, \cdots, x_m^i, \alpha, x_{m+1}^i\right]_{U_{m+2}} f \ . \ \frac{D_{U_{m+2}}\left(x_1^i\,,\ \cdots,\ \ x_m^i,\ \ \alpha,\ \ \beta\right)}{D_{U_{m+1}}\left(x_1^i\,,\ \cdots,\ \ x_m^i\,,\ \alpha\right)}$$

Now, by assumption, (3.2) holds for $n = m$. Therefore, $\left|p_m^i(\beta)\right| \le K_m$. Also, by (ii) of Theorem 2.5,

$$f(\alpha) - p_m^i(\alpha) = [x_1^i, \cdots, x_m^i, \alpha]_{U_{m+1}} f . \frac{D_{U_{m+1}}(x_1^i, \cdots, x_m^i, \alpha)}{D_{U_m}(x_1^i, \cdots, x_m^i)}$$

Therefore

$$[x_1^i, \cdots, x_m^i, \alpha]_{U_{m+1}} f = \left(f(\alpha) - p_m^i(\alpha)\right) . \frac{D_{U_m}(x_1^i, \cdots, x_m^i)}{D_{U_{m+1}}(x_1^i, \cdots, x_m^i, \alpha)}$$

Now $\left|f(\alpha) - p_m^i(\alpha)\right| \le |f(\alpha)| + K_m$. Also

$$\frac{D_{U_{m+1}}(x_1^i, \cdots, x_m^i, \beta)}{D_{U_{m+1}}(x_1^i, \cdots, x_m^i, \alpha)} \to \frac{D_{U_{m+1}}(x_1^0, \cdots, x_m^0, \beta)}{D_{U_{m+1}}(x_1^0, \cdots, x_m^0, \alpha)} \neq 0$$

Therefore, for large i,

$$\left|\frac{D_{U_{m+1}}(x_1^i, \cdots, x_m^i, \beta)}{D_{U_{m+1}}(x_1^i, \cdots, x_m^i, \alpha)}\right| \le 2 \frac{D_{U_{m+1}}(x_1^0, \cdots, x_m^0, \beta)}{D_{U_{m+1}}(x_1^0, \cdots, x_m^0, \alpha)}$$

Thus

$$p_{m+2}^i(\beta) \le K_m + 2(|f(\alpha)| + K_m) . \frac{D_{U_{m+1}}(x_1^0, \cdots, x_m^0, \beta)}{D_{U_{m+1}}(x_1^0, \cdots, x_m^0, \alpha)}$$
$$+ [x_1^i, \cdots, \alpha, x_{m+1}^i]_{U_{m+2}} f . \frac{D_{U_{m+2}}(x_1^i, \cdots, x_m^i, \alpha, \beta)}{D_{U_{m+1}}(x_1^i, \cdots, x_m^i, \alpha)}$$

$$\to -\infty \text{ as } i \to \infty, \text{ by} \qquad (3.12).$$

This is a contradiction. Thus there exists some constant M_{m+1} such that

$$|[x_1, \cdots, x_{m+1}]_{U_{m+1}} f| \le M_{m+1}$$

for all choices of points $a \le x_1 < x_2 < \cdots < x_{m+1} \le b$. So (3.1) holds for $n = m + 1$ also. Now let $p \in \mathcal{U}_{m+1} \cap \mathcal{GL}_{\mathcal{U}}(f)$. If $p \in \mathcal{U}_m$, then by assumption, $\|p\|_\infty \le K_m$. If $p \in \mathcal{U}_{m+1} \setminus \mathcal{U}_m$, then $p = L(x_1, x_2, \cdots, x_{m+1}; f)$ for $m + 1$ distinct points $x_1 < x_2 < \cdots < x_{m+1}$ in I. Let $q = L(x_1, x_2, \cdots, x_m; f)$. Then for all $x \in I$,

$$p(x) = q(x) + [x_1, \cdots, x_{m+1}]_{U_{m+1}} f . \frac{D_{U_{m+1}}(x_1, \cdots, x_m; x)}{D_{U_m}(x_1, \cdots, x_m)}$$

$$|p(x)| \le K_m + M_{m+1} \frac{\left|D_{U_{m+1}}(x_1, \cdots, x_m; x)\right|}{D_{U_m}(x_1, \cdots, x_m)}$$

$$\leq K_m + M_{m+1} \cdot \frac{c_{2,m+1}}{c_{1,m}} \cdot \frac{|D_{V_{m+1}}(x_1,\cdots,x_m;x)|}{D_{V_m}(x_1,\cdots,x_m)}, \qquad \text{by Theorem 2.2}$$

$$\leq K_m + M_{m+1} \cdot \frac{c_{2,m+1}}{c_{1,m}} \cdot |(x-x_1)(x-x_2)\cdots(x-x_m)|$$

$$\leq K_m + M_{m+1} \cdot \frac{c_{2,m+1}}{c_{1,m}} \cdot (b-a)^m = K_{m+1} \text{ (say)}$$

Thus there exists a constant K_{m+1} such that for all $p \in \mathcal{U}_{m+1} \cap \mathcal{GL}_{\mathcal{U}}(f)$, $\|p\|_\infty \leq K_{m+1}$. Thus (3.2) also hold for $n = m + 1$. Using induction on n, we see that (3.1) and (3.2) hold for all n.

Remark 7. From Theorem 3.1 and its proof, it can be easily seen that if f is in $\mathcal{GTP}_{\mathcal{U}}$, then f is positive and bounded on I.

We are now in a position to prove our main theorem.

Theorem 3.2 Let $f \in \mathcal{GTP}_{\mathcal{U}}$. Then f is infinitely differentiable on I.

Proof. First we will prove that for each $n \geq 2$, there exists a constant R_n such that whenever $a \leq x_1 < x_2 < \cdots < x_n \leq b$,

$$|[x_1, x_2, \cdots, x_n]_{V_n} f| \leq R_n \tag{3.13}$$

where V_n is the ECT-system $\{1, x, x^2, \cdots, x^{n-1}\}$. Fix $n \geq 2$. Let $x_1 < x_2 < \cdots < x_n$ be n distinct points in I. Now $\mathcal{P}_{n-1}$ is the space spanned by V_{n-1}. Let p_{n-1} be the unique polynomial in $\mathcal{P}_{n-1}$ interpolating to f at $x_1, x_2, \cdots, x_{n-1}$ and let q_{n-1} be the unique generalized polynomial in $\mathcal{U}_{n-1}$ interpolating to f at these points. Then, by (ii) of Theorem 2.5

$$f(x_n) - p_{n-1}(x_n) = [x_1, \cdots, x_n]_{V_n} f \cdot \frac{D_{V_n}(x_1, x_2 \cdots, x_n)}{D_{V_{n-1}}(x_1, x_2 \cdots, x_{n-1})}$$

$$= [x_1, \cdots, x_n]_{V_n} f \cdot (x_n - x_1) \cdots (x_n - x_{n-1}) \tag{3.14}$$

and

$$|f(x_n) - q_{n-1}(x_n)| = [x_1, \cdots, x_n]_{U_n} f \cdot \frac{D_{U_n}(x_1, x_2 \cdots, x_n)}{D_{U_{n-1}}(x_1, x_2 \cdots, x_{n-1})}$$

Then, by Theorem 2.2 and Theorem 3.1,

$$|f(x_n) - q_{n-1}(x_n)| \leq \left|[x_1, \cdots, x_n]_{U_n} f\right| \cdot \frac{c_{2,n}}{c_{1,n}} \cdot (x_n - x_1) \cdots (x_n - x_{n-1})$$

$$\leq M_n \cdot \frac{c_{2,n}}{c_{1,n}} \cdot (x_n - x_1) \cdots (x_n - x_{n-1}) \tag{3.15}$$

Now p_{n-1} is also the unique polynomial in $\mathcal{P}_{n-1}$ interpolating to q_{n-1} at $x_1, x_2, \cdots, x_{n-1}$.

Therefore, by (ii) of Theorem 2.5

$$q_{n-1}(x_n) - p_{n-1}(x_n) = [x_1, \cdots, x_n]_{V_n} q_{n-1} \cdot \frac{D_{V_n}(x_1, \cdots, x_n)}{D_{V_{n-1}}(x_1, \cdots, x_{n-1})}$$

Using Theorem 2.3 there exists a point ξ, $x_1 < \xi < x_n$ such that

$$[x_1, x_2, \cdots, x_n]_{V_n} q_{n-1} = \frac{q_{n-1}^{(n-1)}(\xi)}{(n-1)!}$$

Therefore

$$q_{n-1}(x_n) - p_{n-1}(x_n) = \frac{q_{n-1}^{(n-1)}(\xi)}{(n-1)!} \cdot \frac{D_{V_n}(x_1, \cdots, x_n)}{D_{V_{n-1}}(x_1, \cdots, x_{n-1})}$$

$$= \frac{q_{n-1}^{(n-1)}(\xi)}{(n-1)!} \cdot (x_n - x_1) \cdots (x_n - x_{n-1}) \quad (3.16)$$

using the value of the Vandermonde determinants D_{V_n} and $D_{V_{n-1}}$. Now applying Theorem 2.1 (Markov inequality),

$$\| q_{n-1}^{(n-1)} \|_\infty \le \frac{c_{n-1}}{(b-a)^{n-1}} \| q_{n-1} \|_\infty \quad (3.17)$$

Since $q_{n-1} \in \mathcal{U}_{n-1} \cap \mathcal{GL}_{\mathcal{U}}(f)$, by (ii) of Theorem 3.1

$$\| q_{n-1} \|_\infty \le K_{n-1} \quad (3.18)$$

From (3.16), (3.17) and (3.18),

$$|q_{n-1}(x_n) - p_{n-1}(x_n)| \le \frac{c_{n-1}}{(b-a)^{n-1}} \cdot \frac{K_{n-1}}{(n-1)!} \cdot (x_n - x_1) \cdots (x_n - x_{n-1}) \quad (3.19)$$

From (3.14), (3.15) and (3.19)

$$|[x_1, \cdots, x_n]_{V_n} f| = \frac{|f(x_n) - p_{n-1}(x_n)|}{(x_n - x_1) \cdots (x_n - x_{n-1})}$$

$$\le M_n \cdot \frac{c_{2,n}}{c_{1,n}} + \frac{c_{n-1}}{(b-a)^{n-1}} \cdot \frac{K_{n-1}}{(n-1)!} = R_n \qquad (say)$$

That is,

$$|[x_1, \cdots, x_n]_{V_n} f| \le R_n$$

This proves (3.13). Since $f \in \mathcal{GTP}_{\mathcal{U}}$, f is bounded on I. Thus, by Theorem 2.4, we see that $f \in C^{n-2}[a, b]$. Thus $f \in C^{n-2}[a, b]$ for all $n \ge 2$. Therefore, f is infinitely differentiable on $[a, b]$.

References

[1] Horwitz, A.L., and L.A.Rubel, Totally Positive Functions and Totally Bounded Functions on [-1, 1]., *J.Approx. Theory* 52 (1988), 204-216.

[2] Popoviciu, T., Sur le reste dans certaines formules lineaures d' approximation de l'analyse, Mathematica (Cluj), (1959), 95 -142.

[3] Schumaker, L., L., *Spline Functions : Basic Theory,* John Wiley & Sons, New York, 1981.