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C-ALGEBRA AS A POSET

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ABSTARCT: *In this paper we define a partial ordering $\leq$ on a C-Algebra and prove that the centre B (A) of a C-algebra A with T becomes a Boolean algebra under this partial ordering. We derive a number of necessary and sufficient conditions for a C-algebra < A, $\vee$, $\wedge$ ´> to become a Boolean algebra.*

INTRODUCTION

Kleene [3] introduced the notion of a regular extension of classical logic. For our purpose a regular extension of an algebra A is obtained by adjoining a new element U (for "un Known", "undetermined" or undefined") to A and extending the operations on A to A $\cup$ {U}. If A is the two element Boolean algebra B = {T,F}, then A $\cup$ {U}: = {T,F,U} is denoted by C.

Guzman and squier [2] introduced the variety of C-algebras as the veriety generated by this 3-element algebra C = {T, F, U} which is the algebraic form of the three valued conditional logic. U.M. Swamy, G.C. Rao and R.V.G. Ravi Kumar [6] introduced the concept of the centre of a C-algebra A which is denoted by B(A) and proved that the centre of a C-algebra is a Boolean algebra. Many mathematicians like Berman. J, McCarthy. J, Kleene. S have worked on three valued logic.

In this paper we introduce a partial ordering $\leq$ on a C-Algebra and study the properties of C-algebra as a poset. We prove that the centre B (A) of a C-algebra A with T becomes a Boolean algebra under this partial ordering. We derive a number of necessary and sufficient conditions for a C-algebra < A, $\vee$, $\wedge$ ´> to become a Boolean algebra.

1. Preliminaries

In this section we recall the definition of C-algebra and some results from [2] and [6]. Also we prove some more results in a C-algebra which will be required later.

Definition 1.1 [2]: By a C-algebra we mean an algebra of type (1,2,2) with operations ´, $\wedge$ and $\vee$ satisfying the following identities.

(a) $x'' = x$

(b) $(x\wedge y)' = x' \vee y'$

(c) $(x\wedge y)\wedge z = x \wedge (y\wedge z)$

(d) $x\wedge (y\vee z) = (x\wedge y) \vee (x\wedge z)$

(e) $(x\vee y)\wedge z = (x\wedge z) \vee (x'\wedge y\wedge z)$

(f) $x\vee (x\wedge y) = x$

(g) $(x\wedge y) \vee (y\wedge x) = (y\wedge x)\vee (x\wedge y)$

Example 1.2 [2]: The two element algebra B = {T,F} is a C-algebra with operations ´ $\wedge$ and $\vee$ defined as in the following table.

X	X´
T	F
F	T

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	T
F	T	F

Example : 1.3 [2] : The 3 - element algebra C = {T,F,U} is a c - algebra with operations $\vee$, $\wedge$´, defined as in the following tables.

X	X´
T	F
F	T
U	U

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

Note 1.4 [2]:

(i) Identities 1.1a, 1.1b imply that the variety of C-algebras satisfies all the dual statements of 1.1b to 1.1g

(ii) $\wedge$ and $\vee$ are not commutative in C

(iii) The ordinary right distributive law of $\wedge$ over $\vee$ fail in C

(iv) Every Boolean algebra is a C-algebra

(v) Throughout this paper A represents a C-algebra unless otherwise mentioned.

Lemma: 1.5 [2]: Every C - algebra satisfies the following laws

(a) $x \wedge x = x$

(b) $x \wedge y = x \wedge (x' \vee y)$

(c) $x \vee (x' \wedge x) = x$

(d) $(x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y)$

(e) $(x \vee x') \wedge x = x$

Lemma 1.6 [6]: Every C - algebra satisfies the laws

(a) $x \vee x' = x' \vee x$

(b) $x \vee y \vee x = x \vee y$

(c) $x \wedge x' \wedge y = x \wedge x'$

Lemma 1.7 [6]: If $a \in A$ then

(i) a is left Identity for $\wedge$ if and only if a is right Identity for $\wedge$

(ii) a is left identity for $\vee$ if and only if a is right identity for $\vee$

Definition 1.8 [2]: A is said to have T (and F). if there is an Identity for $\wedge$. In this case we write T for the Identity for $\wedge$ and F in place of T′

Lemma 1.9 [2]: Let A be with T and $x, y \in A$.

(a) $x \vee y = F$ if and only if $x = y = F$

(b) if $x \vee y = T$ then $x \vee x' = T$

Lemma: 1.10 [2]: Every C-algebra with T satisfies the law $x \vee T = x \vee x'$

Theorem 1.11[6]: The following are equivalent in A.

(a) A is a Boolean algebra

(b) $x \vee (y \wedge x) = x$ for all $x, y \in A$

(c) $x \wedge y = y \wedge x$ for all $x, y \in A$

(d) $(x \vee y) \wedge y = y$ for all $x, y \in A$

(e) $x \vee x'$ is Identity for $\wedge$ for all $x \in A$

(f) $x \vee x' = y \vee y'$ for all $x, y \in A$

(g) A has a right zero for $\wedge$

(h) For any $x, y \in A$, there exists $a \in A$ such that $x \wedge a = y \wedge a = a$.

(i) For any $x, y \in A$ if $x \vee y = y$ then $y \wedge x = x$

Definition 1.12 [6]: Let A be with T. An element $x \in A$ is called a central element of A if $x \vee x' = T$. The set $\{x \in A \mid x \vee x' = T\}$ of all central elements of A is called the centre of A and is denoted by B (A).

Theorem: 1.13: let A be with T. Then B (A) is a Boolean algebra with induced operations $\wedge$, $\vee$ and $'$

Note: 1.14: (i). a is the identity for $\wedge$ if and only if a' is Identity for $\vee$

since $a \wedge x = x = x \wedge a$ for all $x \in A \Leftrightarrow a' \vee x = x = x \vee a'$ for all $x \in A$.

(ii) observe that if Identity for $\wedge$ exists then it is unique and it is denoted by T, then T', denoted by F, is the unique Identity for $\vee$

Theorem: 1.15: Let $a \in A$. Then the following are equivalent.

(1) a is left Identity for $\wedge$

(2) a is right Identity for $\wedge$

(3) a is Identity for $\wedge$

(4) a' is left Identity for $\vee$

(5) a' is right Identity for $\vee$

(6) a' is Identity for $\vee$

2. Some Properties of C-Algebra

Guzman & Squier [2] proved that if $a \in A$ and $a' = a$ then a is left zero for both $\vee$ and $\wedge$. Now we prove the converse.

Theorem 2.1: Let $a \in A$.Then $a' = a$ if andonly if a is leftzero for both $\vee$ and $\wedge$

Proof: Suppose that a is left zero for both $\vee$ and $\wedge$

$a \wedge x = a = a \vee x$ for all $x \in A$ …(1)

$a' = a' \vee (a \wedge a')$ (by Lemma 1.5c)

$= a' \vee a$ (by (1))

$= a \vee a'$ (by Lemma 1.6 (a)

$= a$ by (1)

Hence $a' = a$

In a C-algebra A with T, if $q = p'$ then $p \vee q = T$ and $p \wedge q = F$ need not hold, for example in C, $U \vee U' = U \wedge U' = U$. But we prove the converse in the following.

Theorem 2.2: Let A be a C - algebra with T and $p, q \in A$.

If $p \wedge q = F$, $p \vee q = T$ then $q = p'$

Proof. Let $p \wedge q = F$, $p \vee q = T$

Now $p \vee q = T \Rightarrow p \vee p' = T$ (By 1.9) …(1)

Now $q = T \wedge q = (p \vee p') \wedge q$ (By (1))

$= (p \wedge q) \vee (p' \wedge q)$ (1.5d)

$= F \vee (p' \wedge q)$

$= (p' \wedge p) \vee (p' \wedge q)$ (By (1))

$= p' \wedge (p \vee q) = p' \wedge T = p'$

Hence $q = p'$

By 1.5 b, we know that in a C-algebra, $x \wedge y = x \wedge (x' \vee y)$. Now we prove the following.

Theorem 2.3 Every C-algebra satisfies the laws.

(i) $x \wedge y = (x' \vee y) \wedge x$, (ii) $x \vee y = (x' \wedge y) \vee x$

Proof: $x \wedge y = x \wedge (x' \vee y)$

$= (x \wedge x') \vee (x \wedge y)$

$= (x' \wedge x) \vee (x \wedge y \wedge x)$ (by lemma 1.6 a,b)

$= (x' \vee y) \wedge x$ (by def 1.1)

similarly $x \vee y = (x' \wedge y) \vee x$

Corollary 2.4: For any $x, y \in A$, the elements x, $x'vy$ commute.

That is, (i) $x \wedge (x' \vee y) = (x' \vee y) \wedge x$

(ii) $x \vee (x' \wedge y) = (x' \wedge y) \vee x$

Corollary 2.5: Every C-algebra satisfies.

$$x \wedge (x' \vee x) = (x' \vee x) \wedge x = x \text{ v } (x' \wedge x) = (x' \wedge x) \vee x = x$$

Note 2.6: If A has an element a with the property $a' = a$ then it is unique and is denoted by U.

Theorem: 2.7. B is the only C-algebra of order two.

Proof: Let A be a C-algebra of order two. Let $A = \{x,y\}$

If $x' = x$ then $y' = y$ (by definition of C-algebra) which is not true (See 2.6)

Therefore $x' = y$, $A = \{x,x'\}$. Since $x \vee x' = x' \vee x$ (by 1.6 (a))

We get $x \vee y = y \vee x$ for all $x,y \in A$ Thus A is a Boolean algebra of order 2

Hence B is the only C-algebra of order two

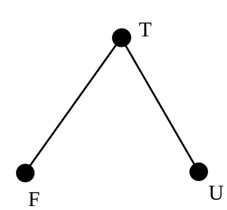
Theorem: 2.8. every C-algebra satisfies

$$(x \vee y) \wedge x = x \vee (y \wedge x)$$

3. C-ALGEBRA AS A POSET

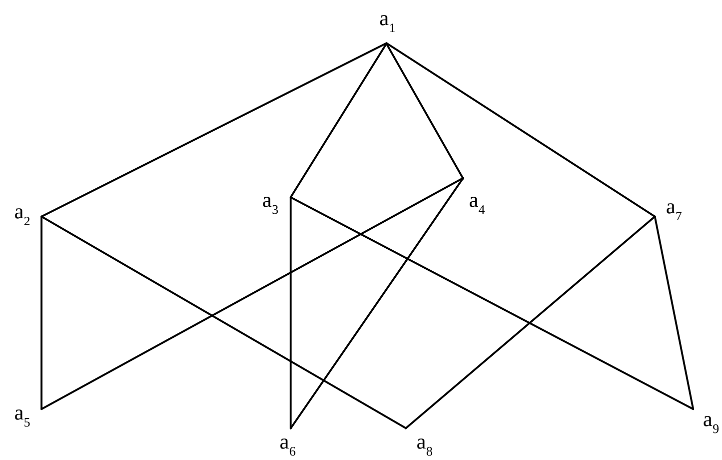
In this section we define a partial ordering "$\leq$" on C-algebra A and study the properties of the C-algebra as a poset. We prove that the centre B (A) of a C-algebra A becomes a Boolean algebra under this partial ordering. We derive a number of necessary and sufficient conditions for a C-algebra $< A, \vee, \wedge\, ' >$ to become a Boolean algebra.

Definition: 3.1 Let A be a C- algebra. Define '$\leq$' on A by $x \leq y$ if $y \wedge x = x$ then '$\leq$' is a partial ordering on C.



It is enough to verify that '$\leq$' is antisymmetric, remaining proof is a routine verification. Suppose that $x, y \in A$, $x \leq y$ and $y \leq x$

$$\{ \quad y \wedge x = x, \; x \wedge y = y \quad \}$$



Now $\quad x \vee y = x \vee (x \wedge y) = x$

$\quad y \vee x = y \vee (y \wedge x) = y$

So that $\quad x = y \wedge x = (y \vee x) \wedge (x \vee y) = (x \vee y) \wedge (y \vee x) = x \wedge y = y$

Hence '$\leq$' is antisymmetric

Note 3.2: If a C-algebra A has T. Then T is the greatest element of the poset $(A \leq)$ since $T \wedge x = x$ for all $x \in A$ (that is $x \leq T$ for all $x \in A$).

Example 3.3: The Hasse diagram of the poset $(C, \leq)$ is

Example 3.4: We know that C x C is a C - algebra under point wise operations. The Hasse diagram of the poset $(C x C, \leq)$ is given below

C x C $\quad a_1 = (T_, T)$, $a_2 = (T,F)$, $a_3 = (T_, U)$, $a_4 = (F,T)$, $a_5 = (F_, F)$,
$a_6 = (F,U)$, $a_7 = (U,T)$ $a_8 = (U_, F)$, $a_9 = (U_, U)$

Note 3.5 : The above two examples suggest that, in general, if A is a C-algebra then $(A, \leq)$ is a join semi lattice. We do not know yet whether $(A, \leq)$ is a join semi lattice. But we have the following.

Theorem: 3.6 In the poset $(A, \leq)$, for any $x \in A$

Supremum of $\{x, x'\} = x \vee x'$, infimum of $\{x, x'\} = x \wedge x'$

Proof: $(x \vee x') \wedge x = x$ and $(x \vee x') \wedge x' = x'$ by cor. 2.5

Therefore $x \leq x \vee x'$ and $x' \leq x \vee x'$

Hence $x \vee x'$ is an upper bound of $\{x, x'\}$

Suppose t is an upper bound of $\{x, x'\}$.

We have $t \wedge x = x$. $t \wedge x' = x'$

Now $t \wedge (x \vee x') = (t \wedge x) \vee (t \wedge x') = x \vee x'$

Therefore $x \vee x'$ is least upper bound of $\{x, x'\}$, That is, supremum of $\{x, x'\} = x \vee x'$. Similarly, infimum of $\{x, x'\} = x \wedge x'$

Corollary: 3.7 In a C-algebra $x \vee x' = x' \vee x$ and $x \wedge x' = x' \wedge x$

In general for a C- algebra A with T. $x \vee y$ need not be the lub of $\{x,y\}$ in $(A, \leq)$ for example, in C, $U \vee F = U$ is not the lub of U and F, However we have the following results.

Theorem 3.8: Let A be a C - algebra with T and $x, y \in A$. If $x \in B(A)$ and $x \wedge y = y \wedge x$ then $\sup \{x,y\} = x \vee y$

Proof: Let $x \wedge y = y \wedge x$ and $x \vee x' = T$

$$(x \vee y) \wedge x = (x \wedge x) \vee (x' \wedge y \wedge x) = x \vee (x' \wedge x \wedge y)$$
$$= x \vee (x \wedge (x' \wedge y)) = x$$

Therefore $x \leq x \vee y$

$$(x \vee y) \wedge y = (x \wedge y) \vee (x' \wedge y \wedge y) = (x \wedge y) \vee (x' \wedge y) = (x \vee x') \wedge y$$
$$= T \wedge y = y$$

Therefore $y \leq x \vee y$.

Hence $x \vee y$ is an upper bound of $\{x, y\}$

Suppose u is an upper bound of $\{x, y\}$. Then $u \wedge x = x$ and $u \wedge y = y$

Now $u \wedge (x \vee y) = (u \wedge x) \vee (u \wedge y) = x \vee y$

Therefore $x \vee y \leq u$, Hence sup $\{x, y\} = x \vee y$.

Theorem 3.9: Let $x,y \in A$ and $x\wedge y = y\wedge x$ then inf $\{x,y) = x\wedge y$

Proof: Let $x, y \in A$ and $x \wedge y = y \wedge x$. Clearly, $x \wedge y \leq x$

$y \wedge (x \wedge y) = y \wedge (y\wedge x) = y\wedge x = x\wedge y$

Hence $x \wedge y \leq y$. Therefore $x\wedge y$ is a lower bound of $\{x,y\}$

Suppose l is a lower bound of $\{x,y\}$

$x \wedge l = l = y\wedge l$. Now $(x\wedge y) \wedge l = x \wedge (y\wedge l) = x\wedge l = l$

Therefore $l \leq x \wedge y$ and hence $x\wedge y$ is the greatest lower bound of $\{x,y\}$

Corollary 3.10: Let A be with T, $x \in B(A)$, $y\in A$ such that $x \wedge y = y\wedge x$ then $x \vee y$ is the lub of $\{x, y\}$ and $x \wedge y$ is the glb of $\{x,y\}$

Thus we have proved the following

Theorem 3.11: Let A be a C-algebra with T. Then the centre B (A) of A (def 1.12) is a Boolean algebra under the partial ordering '≤' induced from A.

Now we give the following equivalent conditions for "$x \leq y$"

Theorem: 3.12: In a C-algebra

(i) $x \leq y \Leftrightarrow y \wedge (y' \vee x) = x \Leftrightarrow (y' \vee x) \wedge y = x$

(ii) $x \leq y \Rightarrow y \vee x = y$

Proof: (i) is clear from 2.3 and 2.4

(ii) $x \leq y \Rightarrow y \wedge x = x \Rightarrow y \vee (y\wedge x) = y\vee x \Rightarrow y = y\vee x$

The converse of 3.12 (ii) is not true, for example, in 'C', $U_{\vee}F = U$ but $F \nleq U$

Theorem 3.13. In the poset $(A, \leq)$

1. $x \vee y \leq x \vee x'$
2. if $x \leq y$ then, for any $z\in A$, (i) $z\wedge x \leq z\wedge y$ (ii) $z\vee x \leq z\vee y$
3. $x\wedge y \leq x$

4. if A is a C-algebra with T and $x \in B(A)$ then $y \leq x \vee y$

Proof: 1. $(x \vee x') \wedge (x \vee y) = (x \wedge (x \vee y)) \vee (x' \wedge (x \vee y))$

$= x \vee (x' \wedge y) = x \vee y$ therefore $x \vee y \leq x \vee x'$

2. Suppose $x \leq y$, and $z \in A$.

(i) $(z \wedge y) \wedge (z \wedge x) = (z \wedge y \wedge z) \wedge x = z \wedge y \wedge x$ (By 1.6)

$= z \wedge x$ (since $y \wedge x = x$)

Therefore $z \wedge x \leq z \wedge y$

(ii) is a direct verification

3. $x \wedge (x \wedge y) = (x \wedge x) \wedge y = x \wedge y$. Therefore $x \wedge y \leq x$.

4. Suppose A is C-algebra with T and $x \in B(A)$. Then

$(x \vee y) \wedge y = (x \wedge y) \vee (x' \wedge y \wedge y) = (x \wedge y) \vee (x' \wedge y)$

$= (x \vee x') \wedge y = T \wedge y = y$

Therefore $y \leq x \vee y$.

Note 3.14

(i) In the above, $x \vee y \leq y \vee y'$ not hold; for example $T \vee U \nleq U \vee U'$ in C.

(ii) In the above, $x \leq x \vee y$ not hold; for example $F \nleq F_{V}U$ in C.

Now we prove modular type results in the following

Lemma: 3.15: In the poset $(A, \leq)$

$$x \leq y \Rightarrow x \vee (y \wedge z) = y \wedge (x \vee z)$$

Proof: Suppose $x \leq y \Rightarrow y \wedge x = x$

$y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = x \vee (y \wedge z)$.

Theorem 3.16: For any $x, y, z, \in A$

$x \leq y$, $x \vee z = y \vee z$ and $x \wedge z = y \wedge z$ then $x = y$

Proof: $x = x \vee (x \wedge z) = x \vee (y \wedge z)$

$= y \wedge (x \vee z)$ (By above Lemma 3.15)

$= y \wedge (y \vee z) = y$

If $x \in B(A)$ then we have proved (3.12) that $y \leq x \vee y$ and if $x \in B(A)$, $x \wedge y = y \wedge x$ then $x \vee y$ is sup $\{x,y\}$ (3.8). In general $x \vee y$ need not be an upper bound of $\{x,y\}$ in the poset $(A, \leq)$. If $x \vee y$ is an upper bound of $\{x,y\}$ in $(A, \leq)$ then A becomes a Boolean algebra. Also it can be observed that, for $x,y \in A$ $x \wedge y \leq x$ is true But $x \wedge y \leq y$ need not hold in A. If $x \wedge y \leq y$ for all $x,y \in A$ then A becomes a Boolean algebra. We conclude this paper with the following.

Theorem: 3.17: The following are equivalent in a C-algebra A

(1) $< A, \vee, \wedge, ' >$ is a Boolean algebra

(2) $y \leq x \vee y$ in $(A, \leq)$ for all $x, y \in A$

(3) $x \leq x \vee y$ in $(A, \leq)$ for all $x, y \in A$

(4) $x \vee y$ is an upper bound of $\{x,y\}$ in $(A, \leq)$ for all $x, y \in A$

(5) Sup $\{x,y\} = x \vee y$ in $(A, \leq)$ for all $x, y \in A$

(6) $x \wedge y \leq y$ in $(A, \leq)$ for all $x, y \in A$

(7) $x \wedge y$ is a lower bound of x,y for all $x, y \in A$

(8) inf $\{x,y\} = x \wedge y$ in $(A, \leq)$ for all $x, y \in A$

(9) The poset $(A, \leq)$ is directed below

(10) $x \vee x'$ is the greatest element in the poset $(A, \leq)$ for all $x \in A$

(11) For any $x, y \in A$, if $y' \leq x'$ then $x \leq y$

Proof: (1) $\Rightarrow$ (2) is clear

(2) $\Rightarrow$ (3): Assume (2), Therefore $(x \vee y) \wedge y = y$ for all $x, y \in A$

$(x \vee y) \wedge (y \vee x) = (x \vee (y \vee x)) \wedge (y \vee x)$ (By 1.6 b) $= y \vee x$

Similarly $(y \vee x) \wedge (x \vee y) = x \vee y$

Therefore $x \vee y = y \vee x$, for all $x, y \in A$

Hence by (2) $x \leq y \vee x = x \vee y$ for all $x, y \in A$

(3) $\Rightarrow$ (4): Assume, (3) we have $(x \vee y) \wedge x = x$, for all $x, y \in A$

$\Rightarrow x \vee (y \wedge x) = x$ for all $x, y \in A$. By (2.8)

$(x \wedge y) \vee (y \wedge x) = (x \wedge y) \vee (y \wedge (x \wedge y))$ (by 1.6b)

$= x \wedge y$

Similarly, $(y \wedge x) \vee (x \wedge y) = y \wedge x$

Therefore $x \wedge y = y \wedge x$ for all $x, y \in A$

Therefore $x \vee y = y \vee x$ for all $x, y \in A$

Now $(x \vee y) \wedge y = (y \vee x) \wedge y = y$, that is $y \leq x \vee y$.

$x \vee y$ is upper bound of $\{x,y\}$ for all x,y

(4) $\Rightarrow$ (5): Assume (4). Suppose u is upper bound of $\{x,y\}$,

$u \wedge x = x$, $u \wedge y = y$ now $u \wedge (x \vee y) = (u \wedge x) \vee (u \wedge y) = x \vee y$

Therefore, sup $\{x,y\} = x \vee y$ for all $x,y \in A$

(5) $\Rightarrow$ (6): Assume (5). Then, $x \vee y = y \vee x$ for all $x, y \in A$
so that $x \wedge y = y \wedge x$ for all $x,y \in A$

Now $y \wedge (x \wedge y) = y \wedge (y \wedge x) = y \wedge x = x \wedge y$ Therefore $x \wedge y \leq y$

(6) $\Rightarrow$ (7): Assume (6). we know that $x \wedge y \leq x$.

Therefore $x \wedge y$ is lower bound of $\{x,y\}$, for all $x, y \in A$

(7) $\Rightarrow$ (8): Assume (7). suppose t is lower bound of $\{x,y\}$

$x \wedge t = t = y \wedge t$, Now $(x \wedge y) \wedge t = x \wedge (y \wedge t) = x \wedge t = t$

Therefore $x \wedge y$ is greatest lower bound of $\{x,y\}$

(8) $\Rightarrow$ (1): Assume (8). Therefore $x \wedge y = y \wedge x$ for all $x, y \in A$

Hence, by thereon 1.11, A is Boolean algebra

Therefore (1) to (8) are equivalent

Clearly (9), (10), (11) are equivalent to (1) by theorem 1.11.

Hence the theorem.

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