CHARACTERIZATION OF FUZZY MEASURES VIA CONCAVITY AND RECURSIVITY

Om Parkash, P. K. Sharma & J. Kumar

ABSTRACT: *Two well-known measures of fuzzy entropy due to De Luca and Termini [3] and Kapur [7] have been characterized by using sum, concavity and recursivity properties. It is shown that these are the only measures which can be characterized in terms of sum property, concavity and recursivity.*

1. INTRODUCTION

The probabilistic measures of entropy due to Shannon [14] and Havrada-Charvat [5] are well known in the existing literature. There have been many characterizations of these measures provided by Shannon [14], Havrada-Charvat [5], Aczel and Daroczy [1], Fadeev [4], Mathai and Rathie [11] etc. These authors have not taken into consideration the concavity property of these measures which is very important because of its applications to maximum entropy principle. It is therefore necessary that concavity property of these measures should be considered as a property of primary importance and should be emphasized in characterization theorems.

One of the main problems in the description and modeling of complex systems is the impossibility of defining their meaningful parameters in an unambiguous and crisp manner. As a consequence, a sort of vagueness is present from the beginning and often every effort of eliminating it at a subsequent stage induces an oversimplification of the model and so a loss of information of the real system under study. A second problem is presented by the complexity of these systems which usually prevents the finding of solutions of the models without simplifications and forced approximations.

A possible way of avoiding these two drawbacks may be found by a suitable change of the descriptive language of the system. This new language should be compatible with the presence of elements of "fuzziness". A mathematical theory describing imprecise and vague notions is meaningful if the theory is also able to measure and control the indeterminacy which is introduced. A way of achieving this goal in the theory of fuzzy sets is provided by the theory of entropy measures

The measures of fuzzy entropy corresponding to Shannon's [14] and Havrada-Charvat's [5] probabilistic measures have been derived by De Luca and Termini

[3] and Kapur [7] respectively. Corresponding to Shannon's [14] measure of probabilistic entropy, De Luca and Termini [3] suggested the measure of fuzzy entropy as

$$H(A) = -\sum_{i=1}^{n}[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))] \tag{1.1}$$

Corresponding to Renyi's [13] measure of probabilistic entropy, Bhandari and Pal [2] suggested the measure of fuzzy entropy as

$$H_\alpha(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha]; \quad \alpha > 0, \alpha \neq 1 \tag{1.2}$$

Corresponding to Havrada-Charvat's [5] probabilistic measure of entropy, Kapur [7] suggested the following measure of fuzzy entropy:

$$H^\alpha(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha - 1]; \quad \alpha > 0, \alpha \neq 1 \tag{1.3}$$

Many other parametric and non-parametric measures of fuzzy entropy have been discussed and derived by Zadeh [15], Kapur [7], Kandel [6], Klir and Folger [8], Kosko [9], Loo [10], Parkash [12] etc. In the present paper, our aim is to characterize measures of fuzzy entropy in terms of concavity and recursivity properties.

In section 2, two well known fuzzy measures of entropy due to De Luca and Termini [3] and Kapur [7] have been characterized. In section 3, it is proved that these are the only measures which can be characterized in terms of sum property, concavity and recursivity.

2. CHARACTERIZATION THEOREMS FOR MEASURES OF FUZZY ENTROPY

(a) Derivation of De Luca and Termini's Fuzzy Entropy

Theorem 1: The only function $H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))$ of the form $\sum_{i=1}^{n}[\phi(\mu_A(x_i)) + \phi(1-\mu_A(x_i))]$ where $\phi(x)$ is a continuous, differentiable, concave function defined on [0, 1] for which $\phi(0) = \phi(1) = 0$ and which satisfies the recursivity property is an arbitrary positive multiple of fuzzy measure corresponding to Shannon's [14] measure.

Proof: Let $H_n(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n))$

$$= \phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + ... + \phi(\mu_A(x_n)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) + ... + \phi(1 - \mu_A(x_n)) \tag{2.1}$$

where $\phi(x)$ is continuous, differentiable, concave function in (0, 1) for which $\phi(0) = \phi(1) = 0$ and satisfies recursivity property.

We now study the properties of $H_n(\mu_A(x_1), ..., \mu_A(x_n))$:

(i) Since sum of any number of continuous, differentiable, concave functions is itself a continuous, differentiable, concave function, $H_n(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n))$ is a continuous, differentiable and concave function.

(ii) $H_n(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n))$ is permutationally symmetric, that is, it does not change if $\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n))$ are permuted among themselves.

(iii) $H_{n+1}(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n), 0)$

$$= \phi(\mu_A(x_1) + \phi(\mu_A(x_2) + ..., + \phi(\mu_A(x_n) + \phi(0) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) + ... + \phi(1 - \mu_A(x_n)) + \phi(1 - 0)$$

$$= H_n(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n))$$

Thus our function satisfies the property of expansibility.

Now we assume that the recursivity or the branching property holds so that

$$H_n(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n))$$

$$= H_{n-1}(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), ..., \mu_A(x_n)) + (\mu_A(x_1) + \mu_A(x_2)) H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right) + H_{n-1}(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3) ...,$$

$$1 - \mu_A(x_n)) + (2 - \mu_A(x_1) - \mu_A(x_2))\, H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right) \tag{2.2}$$

This gives

$$\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) = \phi(\mu_A(x_1) + \mu_A(x_2))$$

$$+ (\mu_A(x_1) + \mu_A(x_2)) \left(\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right) + \phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right)$$

$$+ \phi(2 - \mu_A(x_1) - \mu_A(x_2)) + (2 - \mu_A(x_1) - \mu_A(x_2))$$

$$\left(\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right) \tag{2.3}$$

which is functional equation whose only solution is given by

$$\phi(x) = A\,x \log x \text{ (See Appendix I)}$$

Since $\phi(x)$ is to be concave, A has to be negative. Let $A = -K$, where K is positive.

Thus $\phi(x) = -K\,x \log x$, where K is an arbitrary positive constant, so that

$$H_n(\mu_A(x_1), \mu_A(x_2), \ldots., \mu_A(x_n)) = \sum_{i=1}^{n}\left[\phi(\mu_A(x_i)) + \phi(1-\mu_A(x_i))\right]$$

$$= -K\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]$$

which is arbitrary positive multiple of fuzzy entropy corresponding to Shannon's [14] probabilistic measure of entropy.

(b) Derivation of Kapur's Fuzzy Entropy

Theorem 2: The only function $H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))$ of the form $\sum_{i=1}^{n}\left[\phi(\mu_A(x_i)) + \phi(1-\mu_A(x_i))\right]$ where $\phi(x)$ is continuous, differentiable and concave function defined on [0, 1] for which $\phi(0) = \phi(1) = 0$ and which satisfies the recursivity property given by

$$H_n(\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))$$

$$= H_{n-1}(\mu_A(x_1) + \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_n))$$

$$+ (\mu_A(x_1) + \mu_A(x_2))^{\alpha}\ H_2\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}, \frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)$$

$$+ H_{n-1}(2 - \mu_A(x_1) - \mu_A(x_2), 1 - \mu_A(x_3), \ldots, 1 - \mu_A(x_n))$$

$$+ (2 - \mu_A(x_1) - \mu_A(x_2))^{\alpha}\ H_2\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}, \frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)$$

is a fuzzy measure corresponding to Havrada-Charvat's [5] measure.

Proof: Let $H_n(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n))$

$$= \phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + ... + \phi(\mu_A(x_n))$$

$$+ \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) + ... + \phi(1 - \mu_A(x_n))$$

$$= \sum_{i=1}^{n}[\phi(\mu_A(x_i)) + \phi(1 - \mu_A(x_i))]$$

where $\phi(x)$ is a continuous, differentiable and concave function defined in [0, 1] for which $\phi(0) = \phi(1) = 0$. Now by recursive property,

$$\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)$$

$$= \phi(\mu_A(x_1) + \mu_A(x_2)) + (\mu_A(x_1) + \mu_A(x_2))^\alpha$$

$$\left[\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right]$$

$$+ \phi(2 - \mu_A(x_1) - \mu_A(x_2)) + (2 - \mu_A(x_1) - \mu_A(x_2))^\alpha$$

$$\left[\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right] \tag{2.4}$$

which is a functional equation for which the only continuous, differentiable solution is

$\phi(x) = A(x^\alpha - x)$ (See Appendix I)

Since $\phi(x)$ has to be a concave function, to make it concave for all positive values of ? ? 1, we take $A = \frac{1}{1-\alpha}$, so that ?$(x) = \frac{x^\alpha - x}{1-\alpha}$.

Thus $$H_n(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n)) = \sum_{i=1}^{n}[\phi(\mu_A(x_i)) + \phi(1 - \mu_A(x_i))]$$

$$= \sum_{i=1}^{n}\left[\frac{\mu_A^\alpha(x_i) - \mu_A(x_i)}{1-\alpha} + \frac{(1-\mu_A(x_i))^\alpha - (1-\mu_A(x_i))}{1-\alpha}\right]$$

$$= \frac{1}{1-\alpha}\sum_{i=1}^{n}[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha - 1];\ \alpha > 0,\ \alpha \neq 1$$

which is fuzzy measure of entropy corresponding to Havrada and Charvat's [5] measure.

3. MEASURES WHICH CAN BE CHARACTERIZED IN TERMS OF SUM-FUNCTION PROPERTY, CONCAVITY AND RECURSIVITY

We now consider the more general recursivity relation

$$\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2))$$
$$= \phi\{\mu_A(x_1) + \mu_A(x_2)\} + g\{\mu_A(x_1) + \mu_A(x_2)\}$$
$$\left[\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right]$$
$$+ \phi(2 - \mu_A(x_1) - \mu_A(x_2)) + g(2 - \mu_A(x_1) - \mu_A(x_2))$$
$$\left[\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right] \tag{3.1}$$

This gives

$$\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(\mu_A(x_3)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) + \phi(1 - \mu_A(x_3))$$
$$= \{\phi(\mu_A(x_1)) + \mu_A(x_2)) + \phi(\mu_A(x_3))\} + g(\mu_A(x_1) + \mu_A(x_2))$$
$$\left[\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right]$$
$$+ \{\phi(2 - \mu_A(x_1) - \mu_A(x_2)) + \phi(1 - \mu_A(x_3))\} + g(2 - \mu_A(x_1) - \mu_A(x_2))$$
$$\left[\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right] \tag{3.2}$$

Using (3.1) in (3.2), we get

$$\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(\mu_A(x_3)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2)) + \phi(1 - \mu_A(x_3))$$
$$= \phi\{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)\} + g(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)\}$$
$$\left[\phi\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)+\phi\left(\frac{\mu_A(x_3)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)\right]$$
$$+ g(\mu_A(x_1) + \mu_A(x_2)) \left[\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right]$$

$$+ \phi(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)) + g(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3))$$

$$\left[\phi\left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)+\phi\left(\frac{1-\mu_A(x_3)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)\right]$$

$$+ g\{(2 - \mu_A(x_1) - \mu_A(x_2)\}\left[\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right] \quad (3.3)$$

Again using (3.1), we get

$$\phi\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)=\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)$$

$$+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)-g\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)$$

$$\left[\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right] \quad (3.4)$$

Similarly

$$\phi\left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)=\phi\left(\frac{1-\mu_A(x_1)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)$$

$$+\phi\left(\frac{1-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)-g\left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)$$

$$\left[\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right] \quad (3.5)$$

Using (3.4) and (3.5) in (3.3.), we get

$$\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(\mu_A(x_3)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2))$$

$$+ \phi(1 - \mu_A(x_3)) = \phi(\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)) + g\{\mu_A(x_1) + \mu_A(x_2) + \mu_A(x_3)\}$$

$$\left[\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)\right.$$

$$+\phi\left(\frac{\mu_A(x_3)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)-g\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)$$

$$\left.\left\{\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right\}\right] + g\{\mu_A(x_1) + \mu_A(x_2)\}$$

$$\left\{\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right\}$$

$$+ \phi(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3)) + g(3 - \mu_A(x_1) - \mu_A(x_2) - \mu_A(x_3))$$

$$\left[\phi\left(\frac{1-\mu_A(x_1)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)\right.$$

$$+\phi\left(\frac{1-\mu_A(x_3)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)-g\left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)$$

$$\left.\left\{\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right\}\right]$$

$$+ g(2 - \mu_A(x_1) - \mu_A(x_2))$$

$$\left[\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right] \tag{3.6}$$

Thus the recursivity property (3.1) can be extended to three fuzzy values if

$$g\left(\frac{\mu_A(x_1)+\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)}\right)=\frac{g\{\mu_A(x_1)+\mu_A(x_2)\}}{g\{\mu_A(x_1)+\mu_A(x_2)+\mu_A(x_3)\}} \tag{3.7}$$

and

$$g\left(\frac{2-\mu_A(x_1)-\mu_A(x_2)}{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)}\right)=\frac{g\{2-\mu_A(x_1)-\mu_A(x_2)\}}{g\{3-\mu_A(x_1)-\mu_A(x_2)-\mu_A(x_3)\}} \tag{3.8}$$

Equations (3.7) and (3.8) have the solution $g(x) = x^{\alpha}$ so that equation (3.1) becomes

$$\phi(\mu_A(x_1)) + \phi(\mu_A(x_2)) + \phi(1 - \mu_A(x_1)) + \phi(1 - \mu_A(x_2))$$

$$= \phi(\mu_A(x_1) + \mu_A(x_2)) + \mu_A(x_1)) + \mu_A(x_2))^{\alpha}$$

$$\left[\phi\left(\frac{\mu_A(x_1)}{\mu_A(x_1)+\mu_A(x_2)}\right)+\phi\left(\frac{\mu_A(x_2)}{\mu_A(x_1)+\mu_A(x_2)}\right)\right]$$

$$+ \phi(2 - \mu_A(x_1) + \mu_A(x_2)) + (2 - \mu_A(x_1) - \mu_A(x_2))^{\alpha}$$

$$\left[\phi\left(\frac{1-\mu_A(x_1)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)+\phi\left(\frac{1-\mu_A(x_2)}{2-\mu_A(x_1)-\mu_A(x_2)}\right)\right]$$

which is same as (2.4). Thus the only fuzzy measures which can be characterized in terms of sum-property, concavity and recursivity are those measures which correspond to Shannon's [14] and Havrada and Charvat's [5] measures.

Appendix

Consider the functional equation

$$\phi(x)+\phi(y)=\phi(x+y)+(x+y)^{\alpha}\left[\phi\left(\frac{x}{x+y}\right)+\phi\left(\frac{y}{x+y}\right)\right] \tag{A_1}$$

Putting $x = y$, we get

$$2\phi(x)=\phi(2x)+2^{\alpha}x^{\alpha}.\left[2\phi\left(\frac{1}{2}\right)\right] \tag{A_2}$$

Putting $\phi(x) = x^{\alpha}\psi(x)$, we get

$$2x^{\alpha}\psi(x)=2^{\alpha}x^{\alpha}\psi(2x)+2^{\alpha}.x^{\alpha}.2.\frac{1}{2^{\alpha}}\psi\left(\frac{1}{2}\right)$$

or
$$\psi(x) = 2^{\alpha-1}\psi(2x)+\psi\left(\frac{1}{2}\right) \text{ so that}$$

$$\psi' = 2^{\alpha}\,\psi'(2x)$$

Its solution is given by $\psi'(x) = \dfrac{A}{x^{\alpha}}$, where A is some constant

Integrating, we get

$$\phi(x) = \frac{Ax^{-\alpha+1}}{-\alpha+1} + B\text{, where } B \text{ is some other constant.}$$

Thus $\phi(x) = \dfrac{Ax}{1-\alpha} + Bx^{\alpha}$

Substituting in (A_2), we get

$$\frac{2Ax}{1-\alpha} + 2Bx^{\alpha} = \frac{2Ax}{1-\alpha} + B2^{\alpha}x^{\alpha} + 2^{\alpha+1}x^{\alpha}\left[\frac{1}{2}.\frac{A}{1-\alpha} + B\frac{1}{2^{\alpha}}\right]$$

or

$$B = \frac{A}{1-\alpha}$$

Thus $\phi(x) = A\dfrac{x^{\alpha}-x}{\alpha-1}$

Since $\phi(x)$ is to be a concave function, we choose

$A = -k$, so that $\phi(x) = k\dfrac{x^{\alpha}-x}{1-\alpha}$, $k > 0$

As α–1 $\phi(x) = -k\,x \log x$, which proves the result.

Concluding Remarks: One of the important applications of entropy measures is the study of maximum entropy principle. The importance arises due to the fact that the stationary value of a concave function, when it is obtained by Lagrange's method, will give the globally maximum value. Thus, when we get the stationary value, we have not to worry whether it is maximum or minimum and we have not to worry whether it is the largest maximum value. The problem of checking these can be complicated problem, since entropy is a function of *n* variables. However, because of the concavity of Shannon and Havrada-Charvat's measures, these problems are automatically taken care of and the problem of entropy maximization becomes a relatively simple matter. It is therefore emphasized that concavity property is of primary importance and consequently it should be satisfied in all characterization theorems. In this communication, we have developed only two measures of fuzzy entropy via concavity and recursivity. Similar developments can be extended for the other existing as well as new measures of fuzzy entropy.

REFRENCES

[1] Aczel, J. and Daroczy, Z. "On Measures of Information and their Characterizations", Academic Press, New York, (1975).

[2] Bhandari, D. and Pal, N. R. "Some New Information Measures for Fuzzy Sets", *Inform. Sci.*, **67**, (1993), 209–228.

[3] De Luca, A. and Termini, S. "A Definition of Non-probabilistic Entropy in Setting of Fuzzy Set theory", *Information and Control*, **20**, (1971), 301–312.

[4] Fadeev, D. K. "On the Concept of Entropy of Finite Probability Scheme (Russian) Uspeki Mat Naut (NS), **11**, (1956), 227–231.

[5] Havrada, J. H. and Charvat, F. "Quantification Methods of Classification Processes: Concept of Structural ?-entropy", *Kybernetika*, **3**, (1967), 30–35.

[6] Kandel, A. "Fuzzy Mathematical Techniques with Applications", Addison-Wesley (1986).

[7] Kapur, J. N. "Measures of Fuzzy Information", Mathematical Sciences Trust Society, New Delhi, (1997).

[8] Klir, G. J. and Folger, T. A. "Fuzzy Sets, Uncertainty and Indetermination", Prentice Hall, New York, (1988).

[9] Kosko, B. "Fuzzy Entropy and Conditioning", *Inform. Sci.*, **40**, (1986), 165–174.

[10] Loo, S. G. "Measures of Fuzziness", *Kybernetika*, **20**, (1977), 201–210.

[11] Mathai, A. M. and Rathie, P. N. "Basic Concepts of Information, (1975).

[12] Parkash, O. "A New Parametric Measure of Fuzzy Entropy", *Proc., 7th Int. Con. IPMU*' **98**, (1998), 1732–1737.

[13] Renyi, A. "On Measures of Entropy and Information", *Proc. 4th Ber. Symp. Math. Stat. and Prob.*, **1**, (1961), 547–561.

[14] Shannon, C. E. "A Mathematical Theory of Communication", *Bell. Sys. Tech. Jr.*, **27**, (1948), 379–423, 623–659.

[15] Zadeh, L. A. "Fuzzy Sets", *Inform. and Control*, **8**, (1965), 338–353.

Om Parkash
Department of Mathematics
Guru Nanak Dev University
Amritsar-143005, India

P. K. Sharma
Department of Mathematics
Hindu College
Amritsar-143001, India

J. Kumar
Departmtne of Mathematics
Guru Gobind Singh Khalsa College
Sarhali-143410, India