Qualitative Analysis and Control of an Eleven-Term Novel 4-D Hyperchaotic System with Two Quadratic Nonlinearities

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ABSTRACT
First, this paper announces an eleven-term novel 4-D hyperchaotic system and discusses its qualitative properties. The proposed 4-D system is an eleven-term novel polynomial hyperchaotic system with only two quadratic nonlinearities. The novel hyperchaotic system has a saddle-point equilibrium at the origin, which is unstable. The Lyapunov exponents of the novel hyperchaotic system are obtained as $L_1 = 1.39805$, $L_2 = 0.23933$, $L_3 = 0$ and $L_4 = -17.65085$. The maximal Lyapunov exponent (MLE) for the novel hyperchaotic system is obtained as $L_1 = 1.39805$ and Lyapunov dimension as $D_L = 3.09277$. Next, we derive a new result for the adaptive controller to globally stabilize the novel hyperchaotic system with unknown parameters. The adaptive control result has been established using adaptive control theory and Lyapunov stability theory. Numerical simulations with MATLAB have been shown to illustrate the phase portraits of the novel 4-D hyperchaotic system and the adaptive control results for the hyperchaotic system.

Keywords: Chaos, hyperchaos, hyperchaotic systems, adaptive control, Lyapunov stability theory.

1. INTRODUCTION

A chaotic system is commonly defined as a nonlinear dissipative dynamical system that is highly sensitive to even small perturbations in its initial conditions [1]. A chaotic system is also defined as a dynamical system having at least one positive Lyapunov exponent.

In the last four decades, many chaotic systems have been found such as Lorenz system [2], Rössler system [3], Shimizu-Morioka system [4], Shaw system [5], Chen system [6], Lü system [7], Chen-Lee system [8], Cai system [9], Tigan system [10], Li system [11], Zhou system [12], Sundarapandian system [13], Sundarapandian-Pehlivan system [14], Vaidyanathan systems [15-19], Vaidyanathan-Madhavan system [20], Pehlivan-Moroz-Vaidyanathan system [20], etc.

A hyperchaotic system is a chaotic system having more than one Lyapunov exponent. For continuous-time dynamical systems, the minimal dimension for a hyperchaotic system is four. The first hyperchaotic system was found by Rössler [21]. This was followed by the finding of many hyperchaotic systems such as hyperchaotic Lorenz system [22], hyperchaotic Lü system [23], hyperchaotic Chen system [24], hyperchaotic Wang system [25], hyperchaotic Vaidyanathan system [26], etc.

Hyperchaotic systems have attractive features like high security, high capacity and high efficiency and they find miscellaneous applications in several areas like neural networks [27-29], oscillators [30-31], circuits [32-35], secure communication [36-37], encryption [38], synchronization [39-56], etc.

The problem of control of a chaotic system is to find a state feedback control law to stabilize a chaotic system around its unstable equilibrium [57-63].

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In this paper, we have announced an eleven-term novel 4-D hyperchaotic system with two quadratic nonlinearities. We establish that the novel hyperchaotic system has a saddle-point equilibrium at the origin, which is unstable. The Lyapunov exponents of the novel hyperchaotic system are obtained as \( L_1 = 1.39805 \), \( L_2 = 0.23933 \), \( L_3 = 0 \) and \( L_4 = -17.65085 \). The maximal Lyapunov exponent (MLE) for the novel hyperchaotic system is obtained as \( L_1 = 1.39805 \) and Lyapunov dimension as \( D_L = 3.09277 \). Next, we derive a new result for the adaptive controller to globally stabilize the novel hyperchaotic system with unknown parameters. The adaptive control result has been established using adaptive control theory and Lyapunov stability theory.

The rest of this paper is organized as follows. Section 2 contains the description of the eleven-term novel 4-D hyperchaotic system proposed in this paper. Section 3 contains the qualitative properties of the novel hyperchaotic system. Section 4 contains the adaptive control results for the novel hyperchaotic system with unknown parameters. MATLAB simulations have been provided to illustrate the phase portraits and the adaptive control results obtained in this paper.

2. ANELEVEN-TERM NOVEL 4-DHYPERCHAOTIC SYSTEM

In this section, we describe an eleven-term novel 4-D hyperchaotic system with only two quadratic nonlinearities.

The novel 4-D hyperchaotic system is modeled by

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_3 + x_4 \\
\dot{x}_2 &= cx_1 - x_1x_3 + x_4 \\
\dot{x}_3 &= -bx_3 + x_1x_2 \\
\dot{x}_4 &= -d(x_1 + x_2)
\end{align*}
\]

where \( x_1, x_2, x_3, x_4 \) are the state variables and \( a, b, c, d \) are constant, positive, parameters of the system.

The system (1) exhibits a strange hyperchaotic attractor when the constant parameter values are chosen as

\[ a = 12, \quad b = 4, \quad c = 100, \quad d = 5 \]

For numerical simulations, we take the initial values as

\[ x_1(0) = 1.5, \quad x_2(0) = 0.6, \quad x_3(0) = 1.8, \quad x_4(0) = 2.5 \]

Figures 1-4 give the 3-D view of the strange hyperchaotic attractor in \((x_1, x_2, x_3)\), \((x_1, x_2, x_4)\), \((x_1, x_3, x_4)\) and \((x_2, x_3, x_4)\) spaces, respectively.

3. PROPERTIES OF THE NOVELHYPERCHAOTIC SYSTEM

(A) Invariance

The \( x_3 \) – axis \((x_1 = 0, x_2 = 0, x_4 = 0)\) is invariant for the system (1). Hence, all orbits of the system (1) starting on the \( x_3 \) – axis stay in the \( x_3 \) – axis for all values of time.

(B) Dissipativity

We write the system (1) in vector notation as
Figure 1. 3-D View of the Novel Chaotic System in \((x_1, x_2, x_3)\) Space

Figure 2. 3-D View of the Novel Chaotic System in \((x_1, x_2, x_4)\) Space
Figure 3. 3-D View of the Novel Chaotic System in $(x_1, x_3, x_4)$ Space

Figure 4. 3-D View of the Novel Chaotic System in $(x_2, x_3, x_4)$ Space
\[ \dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{bmatrix} \]  

where

\[ f_1(x) = a(x_2 - x_1) + x_3 + x_4 \]
\[ f_2(x) = cx_1 - x_1x_3 + x_4 \]
\[ f_3(x) = -bx_3 + x_1x_2 \]
\[ f_4(x) = -d(x_1 + x_2) \]  

The divergence of the vector field \( f \) on \( \mathbb{R}^4 \) is obtained as

\[ \text{div } f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4} = -a - b = -\mu, \]  

where

\[ \mu = a + b > 0 \]

because \( a, b, c \) and \( d \) are assumed to be positive constants.

Let \( \Omega \) be any region in \( \mathbb{R}^4 \) having a smooth boundary.

Let \( \Omega(t) = \Phi_t(\Omega) \), where \( \Phi_t \) is the flow of \( f \).

Let \( V(t) \) denote the hypervolume of \( \Omega(t) \).

By Liouville’s theorem, it follows that

\[ \frac{dV(t)}{dt} = \int_{\Omega(t)} \text{div } f \, dx_1 dx_2 dx_3 dx_4 = -\mu \int_{\Omega(t)} dx_1 dx_2 dx_3 dx_4 = -\mu V(t) \]  

Integrating the linear differential equation (10), we get the solution as

\[ V(t) = V(0) \exp(-\mu t) \]  

From Eq. (9), it follows that the volume \( V(t) \) shrinks to zero exponentially as \( t \to \infty \).

Thus, the novel hyperchaotic system (1) is dissipative. Hence, the asymptotic motion of the system (1) settles exponentially onto a set of measure zero, i.e. a strange attractor.

(C) Equilibrium Points

The equilibrium points of the novel hyperchaotic system (1) are obtained by solving the nonlinear equations

\[ f_1(x) = a(x_2 - x_1) + x_3 + x_4 = 0 \]
\[ f_2(x) = cx_1 - x_1x_3 + x_4 = 0 \]
\[ f_3(x) = -bx_3 + x_1x_2 = 0 \]
\[ f_4(x) = -d(x_1 + x_2) = 0 \]  

From the last equation in (10), since \( d > 0 \), we must have
Substituting (11) into (10), we obtain the simplified system of equations as

\[
2ax_2 + x_3 + x_4 = 0 \\
-cx_2 + x_3x + x_4 = 0 \\
bx_3 + x_2^2 = 0
\]

We suppose that the parameter values are taken as in the chaotic case, i.e.

\[
a = 12, \quad b = 4, \quad c = 100, \quad d = 5
\]

Then, it is easy to show that the system (12) has only the trivial solution \( x_2 = 0, \ x_3 = 0, \) and \( x_4 = 0. \)

Using (11), we conclude that the system (10) has only the trivial solution \( x = 0. \)

Thus, \( x = 0 \) is the only equilibrium point of the novel hyperchaotic system (1).

The Jacobian matrix of the novel hyperchaotic system (1) at \( x = 0 \) is obtained as

\[
J = \begin{bmatrix}
-a & a & 1 & 1 \\
c & 0 & 0 & 1 \\
0 & 0 & -b & 0 \\
-d & -d & 0 & 0
\end{bmatrix} = \begin{bmatrix}
-12 & 12 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -4 & 0 \\
-5 & -5 & 0 & 0
\end{bmatrix}
\]

The matrix \( J \) has the eigenvalues

\[
\lambda_1 = -41.2284, \quad \lambda_2 = -4, \quad \lambda_3 = 28.7045, \quad \lambda_4 = 0.5239
\]

This shows that the equilibrium \( x = 0 \) is a saddle-point, which is unstable.

Hence, the novel 4-D hyperchaotic system (1) has a unique equilibrium point at \( x = 0, \) which is unstable.

(D) Lyapunov Exponents

We take the parameter values of the system (1) as

\[
a = 12, \quad b = 4, \quad c = 100, \quad d = 5
\]

We take the initial state as

\[
x_1(0) = 1.5, \ x_3(0) = 0.6, \ x_4(0) = 1.8, \ x_4(0) = 2.5
\]

The Lyapunov exponents of the system (1) are numerically obtained with MATLAB as

\[
L_1 = 1.39805, \quad L_2 = 0.23933, \quad L_3 = 0, \quad L_4 = -17.65085
\]

Eq. (18) shows that the system (1) is hyperchaotic, since it has two positive Lyapunov exponents. Since the sum of the Lyapunov exponents is negative, the system (1) is a dissipative hyperchaotic system.

Also, the maximal Lyapunov exponent (MLE) of the system (1) is obtained as \( L_1 = 1.39805. \)

The dynamics of the Lyapunov exponents is depicted in Figure 5.
The Lyapunov dimension of the hyperchaotic system (1) is determined as

$$D_L = j + \frac{\sum L_j}{|L_{j+1}|} = 3 + \frac{L_4 + L_2 + L_3}{|L_4|} = 3.09277,$$

which is fractional. Thus, the ten-term 4-D system (1) is a dissipative hyperchaotic system with fractional Lyapunov dimension.

4. ADAPTIVE CONTROL OF THE NOVEL HYPERCHAOTIC SYSTEM

In this section, we derive new results for the adaptive controller to stabilize the unstable novel chaotic system with unknown parameters for all initial conditions.

Thus, we consider the controlled novel 4-D hyperchaotic system

$$\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + x_3 + x_4 + u_1 \\
\dot{x}_2 &= cx_1 - x_1 x_3 + x_4 + u_2 \\
\dot{x}_3 &= -bx_3 + x_1 x_2 + u_3 \\
\dot{x}_4 &= -d(x_1 + x_2) + u_4
\end{align*}$$

where $x_1, x_2, x_3, x_4$ are state variables, $a, b, c, d$ are constant, unknown, parameters of the system and $u_1, u_2, u_3, u_4$ are adaptive controls to be designed.
We aim to solve the adaptive control problem by considering the adaptive feedback control law
\[
\begin{align*}
    u_1 & = -A(t)(x_2 - x_1) - x_3 - x_4 - k_1 x_1 \\
    u_2 & = -C(t)x_1 + x_3 x_4 - k_2 x_2 \\
    u_3 & = B(t)x_3 - x_1 x_2 - k_3 x_3 \\
    u_4 & = D(t)(x_1 + x_2) - k_4 x_4
\end{align*}
\]
(21)
where \(A(t), B(t), C(t), D(t)\) are estimates for the unknown parameters \(a, b, c, d\), respectively, and \(k_1, k_2, k_3\) are positive gain constants.

The closed-loop system is obtained by substituting (21) into (20) as
\[
\begin{align*}
    \dot{x}_1 & = (a - A(t))(x_2 - x_1) - k_1 x_1 \\
    \dot{x}_2 & = (c - C(t))x_1 - k_2 x_2 \\
    \dot{x}_3 & = -(b - B(t))x_3 - k_3 x_3 \\
    \dot{x}_4 & = -(d - D(t))(x_1 + x_2) - k_4 x_4
\end{align*}
\]
(22)
To simplify (22), we define the parameter estimation error as
\[
\begin{align*}
    e_a(t) & = a - A(t) \\
    e_b(t) & = b - B(t) \\
    e_c(t) & = c - C(t) \\
    e_d(t) & = d - D(t)
\end{align*}
\]
(23)
Substituting (23) into (22), we obtain
\[
\begin{align*}
    \dot{x}_1 & = e_a(x_2 - x_1) - k_1 x_1 \\
    \dot{x}_2 & = e_c x_1 - k_2 x_2 \\
    \dot{x}_3 & = -e_b x_3 - k_3 x_3 \\
    \dot{x}_4 & = -e_d(x_1 + x_2) - k_4 x_4
\end{align*}
\]
(24)
Differentiating the parameter estimation error (23) with respect to \(t\), we get
\[
\begin{align*}
    \dot{e}_a(t) & = -\dot{A}(t) \\
    \dot{e}_b(t) & = -\dot{B}(t) \\
    \dot{e}_c(t) & = -\dot{C}(t) \\
    \dot{e}_d(t) & = -\dot{D}(t)
\end{align*}
\]
(25)
Next, we find an update law for parameter estimates using Lyapunov stability theory.

Consider the quadratic Lyapunov function defined by
\[
V(x_1, x_2, x_3, x_4, e_a, e_b, e_c, e_d) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2),
\]
(26)
which is positive definite on \(R^8\).
Differentiating $V$ along the trajectories of (24) and (25), we obtain

$$
\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2 + e_a \left[ x_1 (x_2 - x_1) - \hat{A} \right] + e_b \left[ -x_3^2 - \hat{B} \right] + e_c \left[ x_1 x_2 - \hat{C} \right] + e_d \left[ -x_4 (x_1 + x_2) - \hat{D} \right]
$$

(27)

In view of (27), we define an update law for the parameter estimates as

$$
\dot{\hat{A}} = x_1 (x_2 - x_1) \\
\dot{\hat{B}} = -x_3^2 \\
\dot{\hat{C}} = x_1 x_2 \\
\dot{\hat{D}} = -x_4 (x_1 + x_2)
$$

(28)

**Theorem 1.** The novel 4-D hyperchaotic system (20) with unknown system parameters is globally and exponentially stabilized for all initial conditions by the adaptive control law (21) and the parameter update law (28), where $k_i$, $(i = 1, 2, 3, 4)$ are positive constants.

**Proof.** The result is proved using Lyapunov stability theory [64].

We consider the quadratic Lyapunov function $V$ defined by (26), which is a positive definite function on $R^8$.

Substituting the parameter update law (28) into (27), we obtain $\dot{V}$ as

$$
\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2
$$

(29)

which is a negative semi-definite function on $R^8$.

Thus, it can be concluded that the state vector $x(t)$ and the parameter estimation error are globally bounded, i.e.

$$
\begin{bmatrix}
x_1(t) & x_2(t) & x_3(t) & x_4(t) & e_a(t) & e_b(t) & e_c(t) & e_d(t) \end{bmatrix}^T \in L_\infty.
$$

(30)

We define $k = \min \{ k_1, k_2, k_3, k_4 \}$.

Then it follows from (29) that

$$
\dot{V} \leq -k \|x\|^2 \quad \text{or} \quad k \|x\|^2 \leq -\dot{V}.
$$

(31)

Integrating the inequality (31) from 0 to $t$, we get

$$
k \int_0^t \|x(\tau)\|^2 \, d\tau \leq -\int_0^t \dot{V}(\tau) \, d\tau = V(0) - V(t)
$$

(32)

From (32), it follows that $x(t) \in L_2$. Using (24), we can conclude that $x(t) \in L_\infty$.

Hence, using Barbalat’s lemma, we can conclude that $x(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $x(0) \in R^1$. This completes the proof.

**Numerical Results**

For the novel system (20), the parameter values are taken as in the hyperchaotic case, viz.
\[ a = 12, \ b = 4, \ c = 100, \ d = 5 \]  \hspace{1cm} (33)

We take the feedback gains as \( k_i = 5 \) for \( i = 1, 2, 3, 4 \).

The initial values of the chaotic system (27) are taken as

\[ x_i(0) = 4.8, \ x_2(0) = -2.9, \ x_3(0) = -4.7, \ x_4(0) = 9.2 \]  \hspace{1cm} (34)

The initial values of the parameter estimates are taken as

\[ A(0) = 21, \ B(0) = 12, \ C(0) = 5, \ D(0) = 22 \]  \hspace{1cm} (35)

Figure 6 depicts the time-history of the controlled novel hyperchaotic system.

![Figure 6: Time-History of the Controlled Novel Hyperchaotic System](image)

5. CONCLUSIONS

In this paper, we have announced an eleven-term novel 4-D hyperchaotic system with only two quadratic nonlinearities. We have given a detailed qualitative analysis of the proposed system in this paper. The novel hyperchaotic system has an unstable equilibrium at the origin, which is a saddle point. The novel hyperchaotic system has the Lyapunov exponents given by \( L_1 = 1.39805 \), \( L_2 = 0.23933 \), \( L_3 = 0 \) and \( L_4 = -17.65085 \). Since the sum of the Lyapunov exponents is negative, the novel hyperchaotic system is a dissipative system. The maximal Lyapunov exponent (MLE) for the novel hyperchaotic system is obtained as \( L = 1.39805 \) and Lyapunov dimension as \( D_L = 3.09277 \). In this paper, we have derived a new result for the adaptive controller to globally stabilize the novel hyperchaotic system with unknown parameters. MATLAB simulations have been shown to illustrate the phase portraits of the novel 4-D hyperchaotic system and the adaptive control results for the novel hyperchaotic system.
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