Comparative Study of IIR Notch Filters for Suppressing 60-Hz Interference in Electrocardiogram Signals

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Abstract: This paper gives the performance analysis of three well-known infinite impulse response (IIR) notch filters in two aspects. One is the amplitude property of the frequency response functions based on the measure of rectangular coefficient. The other is the phase property of the frequency response functions with respect to linearity. The analysis results suggest that the amplitude property of IIR notch filters seems satisfactory but the phase property may be unsatisfactory. The demonstration with a real ECG signal exhibits that an ECG signal may be considerably corrupted by nonlinear phase distortions.

Keywords: Electrocardiogram (ECG) signal processing, power line interference, interference suppression, IIR digital filters

1. INTRODUCTION

Electrocardiogram (ECG) plays a key role in quantitatively diagnosing cardiopathies. A high quality of ECG signals is obviously a basis of achieving high quality diagnostic analysis of these signals. However, since the electrical behavior of the human heart is measured by electrodes applied externally on the human body, the amplitude of an ECG signal is low, approximately 1 ~ 5 mV (Talmaon [1], Levkov et al. [2]). The low amplitude ECG signals are usually contaminated by various disturbances. Among them, the power line interference in 50 Hz (e.g., in China) or 60-Hz (e.g., in USA) is the most common one. This is because the strong 50 Hz or 60-Hz electromagnetic fields on the mains exist in modern hospitals and homes. Therefore, techniques for removing the power line interferences are desired.

Various types of digital filters and hardware circuits have been proposed in this regard, see e.g., Talmaon [1, pp. 151], Hamilton [3], Martens et al. [4], Ahlstrom and Tompkins [5], Pei and Tseng [6], Dotsinsky and Stoyanov [7], Ramos et al. [8], Ider et al. [9]. However, adopting notch filters in the standard sense remains a question (Breithardt et al. [10]). For this reason, this paper gives the performance analysis of notch filters discussed in [3,4,5,6,7]. The present analysis is for the 60-Hz interference notch filters but the analysis procedure is also usable for 50 Hz ones.

The present results suggest that the amplitude property of the notch filters in [3,4,6,7], which is measured by the rectangular coefficient, is satisfactory in the sense that their rectangular coefficients are approximately equal to 1 but their phases are nonlinear. A case study with a real ECG signal is used to demonstrate that the phase nonlinearity of the filters may seriously cause distortions of an ECG signal.

The remaining article is organized as follows. The preliminaries regarding digital filters are briefed in Section 2. The performance analysis of the notch filters in [3,4,5,6,7] is given in Section 3. Discussions are in Section 4, which is followed by conclusions.

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2. PRELIMINARIES OF DIGITAL FILTERS

The functionality of a digital filter is to transform an input sequence \( x(nT) \) of the filter to its output sequence \( y(nT) \), where \( n \) is the index of the sequence number and \( T \) the sampling interval. There are two categories of digital filters. One is infinite impulse response (IIR) filter and the other finite impulse response filters (FIR) (Harger [11], Lam [12], Mitra and Kaiser [13]).

One class of linear difference equations used to characterize a linear time-invarying digital filter is given by

\[
\sum_{k=0}^{N} a_k y(nT - kT) = \sum_{r=0}^{M} b_r x(nT - rT),
\]

where \( a_0 = 1 \).

Rewritng (1) by

\[
y(nT) = \sum_{r=0}^{M} b_r x(nT - rT) - \sum_{k=0}^{N} a_k y(nT - kT).
\]

Then, the transfer function of (2) is expressed by

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r z^{-r}}{\sum_{k=0}^{N} a_k z^{-k}}.
\]

By factoring of a polynomial (Korn and Kong [14, §1.7-2]), one has

\[
\sum_{r=0}^{M} b_r z^{-r} = b_0 \prod_{r=0}^{M} (z - z_r), \tag{4}
\]

\[
\sum_{k=0}^{N} a_k z^{-k} = a_0 \prod_{k=0}^{N} (z - z_k). \tag{5}
\]

Eqs. (4) – (5) imply that there are \( (M + 1) \) zeros \( z_r \) \( (r = 0, 1, \ldots, M) \) and \( (N + 1) \) poles \( z_k \) \( (k = 0, 1, \ldots, N) \) in \( H(z) \) expressed by (3) (Vegte [15]).

Doing the inverse of \( Z \) transform yeilds the impulse function \( h(n) \) written by

\[
h(n) = Z^{-1}[H(z)] = Z^{-1} \left[ \sum_{r=0}^{M} b_r z^{-r} \right] \left[ \sum_{k=0}^{N} a_k z^{-k} \right],
\]

where \( Z^{-1} \) represents the inverse of \( Z \) transform. Note that

\[
\frac{1}{\sum_{k=0}^{N} a_k z^{-k}} = \left( 1 + \sum_{k=1}^{N} a_k z^{-k} \right)^{-1}. \tag{7}
\]
Then, one immediately sees that \( h(n) \) is a function that is non-zero over an infinite length of time. Hence, IIR. The frequency response function of a digital filter is written by

\[
H(e^{j2\pi f}) = H(z)|_{z=e^{j2\pi f}}. \tag{8}
\]

The amplitude of \( H(e^{j2\pi f}) \) is denoted by \( A(f) = |H(e^{j2\pi f})| \) and the phase of \( H(e^{j2\pi f}) \) by \( \phi(f) \)

\[
\phi(f) = \tan^{-1}\left( \frac{\text{Im}[H(e^{j2\pi f})]}{\text{Re}[H(e^{j2\pi f})]} \right).
\]

An IIR filter is stable in the sense of bounded input and bounded output if all poles of the transfer function have absolute values smaller than one. In other words, all poles must be located within a unit circle in the \( z \)-plane.

In contrast to IIR filters, finite impulse response filters (FIR) have fixed-duration impulse responses. Based on FIR filters, a linear difference equation to characterize a linear time-invariant digital filter is expressed by

\[
y(n) = \sum_{r=0}^{M} b_r x(n-r). \tag{9}
\]

The transfer function corresponding to (9) is expressed by

\[
H(z) = \sum_{r=0}^{M} b_r z^{-r}. \tag{10}
\]

An FIR filter is always stable as its system function has no poles. Its impulse function is given by

\[
h(r) = \begin{cases} 
  b_r, & 0 \leq r \leq M \\
  0, & \text{elsewhere}.
\end{cases} \tag{11}
\]

Hence, FIR.

Usually, IIR filters do not have linear phase property but the phases of FIR ones are always linear. IIR filters are recursive but FIR ones are not. Due to the recursive property of IIR, in general, an FIR filter is time consuming and expensive in comparison with an IIR one. Therefore, IIR filters are often preferred in the design of digital filters in engineering though their phases are nonlinear (Lam [12, §12-2-3]). Three types of notch filters analyzed in this paper are all IIR.

3. PERFORMANCE ANALYSIS

The ideal amplitude property of a notch filter in the sense of low pass to remove the 60-Hz power line interference is expressed by

\[
A(f) = |H(e^{j2\pi f})| = u(f) - u(f - (60 - \varepsilon)) + u(f - (60 + \varepsilon)), \tag{12}
\]

where \( \varepsilon > 0 \) is an infinitesimal. Unfortunately, the above is physically unrealizable according to the Paley-Wiener criterion (Papoulis [16]). In practical terms, the above can only be approximated. Therefore, there is a side effect of a practical notch filter. More precisely, it may suppress useful frequency components in a way while eliminating the 60-Hz interference. The side effect is expected to be minimized. That is, the functionality of a filter is to suppress the 60-Hz interference as much as possible while keep the information in the other frequency components as much as possible. In fact, medical scientists are paying efforts on
seeking for useful information from electrocardiograms, see e.g., Rijnbeek [17], Zywietz et al. [18], Farrell and Rowlandson [19]. Thus, a notch filter is expected to approximate to (12) as much as possible.

In this paper, we use the rectangular coefficient to measure the amplitude property of a frequency response function. Without the generality losing, we use the normalized frequency response function in what follows. By normalized frequency response function, we mean that

\[ |H(e^{j2\pi f})|_{max} = 1. \]  \tag{13}

Denote \( B_{0.7} \) the 3-dB bandwidth. Then,

\[ |H(e^{j2\pi f})|_{f=B_{0.7}} = 0.707. \]  \tag{14}

Denote \( B_{0.1} \) the bandwidth for

\[ |H(e^{j2\pi f})|_{f=B_{0.1}} = 0.1. \]  \tag{15}

Then, the rectangular coefficient is defined by

\[ \text{Rec} = \frac{B_{0.7}}{B_{0.1}}. \]  \tag{16}

According to (16), one sees that \( \text{Rec} \to 1 \) for \( \varepsilon \to 0 \) for the ideal filter expressed by (12). Fig. 1 shows the plot of the amplitude of the ideal notch filter for removing the 60-Hz interference.

As a matter of fact, \( \text{Rec} \) is a parameter to measure how much similar of the amplitude of a frequency response function to an ideal notch filter in the form of low pass. For example, suppose there are two filters. Filter 1 has the rectangular coefficient \( \text{Rec}_1 \) while filter 2 has \( \text{Rec}_2 \). Then, if \( \text{Rec}_1 < \text{Rec}_2 \), we say that filter 2 has better amplitude response than that of filter 1.

![Figure 1: The Amplitude of Ideal Notch Filter for Removing the 60-Hz Interference](image)

4. COMPARATIVE STUDY OF NOTCH FILTERS IN ECG SYSTEMS

4.1 Notch Filter \( H_1(z) \) in [3,4]

The transfer function of the 60-Hz notch filter (IIR) discussed by Hamilton [3] and Martens et al. [4] is given by

\[ H_1(z) = \frac{1 - 2 \cos(2\pi \cdot 60 \cdot T) z^{-1} + z^{-2}}{1 - 2r \cos(2\pi \cdot 60 \cdot T) z^{-1} + r^2 z^{-2}}, \]  \tag{17}
where $T$ is the sampling interval. The parameter $r$ is used to adjust the bandwidth of the notch filter. The bandwidth of the filter becomes narrow and the transient response time tends to be larger when the value of $r$ increases. It corresponds to the difference equation given by

$$y(t) = rN y(t - nT) - r^2 y(t - 2nT) + x(t) - N(x) (t - nT) + x(t)(t - 2nT), \tag{18}$$

where $N = 2 \cos(2\pi 60T)$.

The amplitude of the frequency response is

$$A_1(f) = |H_1(e^{j2\pi f})| = \left| \frac{a - jb}{c - jd} \right|, \tag{19}$$

where

$$a = 1 - 2 \cos (2\pi \cdot 60 \cdot T) \cos (2\pi fT) + \cos (2 \cdot 2\pi fT), \tag{20}$$
$$b = \sin (2 \cdot 2\pi f T) - 2 \cos (2\pi \cdot 60 \cdot T) \sin (2\pi f T), \tag{21}$$
$$c = 1 - 2r \cos (2\pi \cdot 60 \cdot T) \cos (2\pi fT) + r^2 \cos (2 \cdot 2\pi fT), \tag{22}$$
$$d = r^2 \sin (2 \cdot 2\pi f T) - 2r \cos (2 \pi \cdot 60 \cdot T) \sin (2\pi fT). \tag{23}$$

Denote the phase of $H_1(e^{j2\pi f})$ by

$$\phi_1(f) = \tan^{-1} \frac{\text{Im}[H_1(e^{j2\pi f})]}{\text{Re}[H_1(e^{j2\pi f})]}. \tag{24}$$

In the cases of $r = 0.98$ and $T = 0.004 \text{ s}$, $A_1(f)$ and $\phi_1(f)$ are indicated in Fig. 2 (a) and Fig. 2 (b), respectively.

In Fig. 2 (a) that $A_1(f)$ has a high Rec. In fact,

$$A_1(59.175) = 0.707, \tag{25}$$
and

$$A_1 (59.914) = 0.1. \tag{26}$$

Thus, one has

$$\text{Rec1} = \frac{B_{0.7}}{B_{0.1}} = \frac{59.175}{59.914} = 0.988. \tag{27}$$
Note 1. This filter has satisfactory amplitude property since $\text{Rec} 1 \approx 1$ but its phase is nonlinear, see Fig. 2 (b).

4.2. Notch Filter $H_2(z)$ in [5]

Ahlstrom and Tompkins [5] studied an IIR notch filter whose transfer function is expressed by

$$H_2(z) = \frac{1 - z^{-256}}{1 - z^{-1}}. \quad (28)$$

The difference equation for $H_2(z)$ is

$$y(nT) = y(nT - T) + x(nT) - x(nT - 256T) \quad (29)$$

The amplitude of the frequency response function is given by

$$A_2(f) = |H_2(e^{j2\pi f})| = \left| \frac{1 - \cos(2\pi f \cdot 256T) + j\sin(2\pi f \cdot 256T)}{1 - \cos(2\pi f T) + j\sin(2\pi f T)} \right|. \quad (30)$$

The phase of $H_2(e^{j2\pi f})$ is

$$\phi_2(f) = \tan^{-1} \frac{\text{Im}[H_2(e^{j2\pi f})]}{\text{Re}[H_2(e^{j2\pi f})]} \quad (31)$$

In the case of $T = 0.000065$, we have $A_2(f)$ and $\phi_2(f)$ as shown in Fig. 3.

![Figure 3: Frequency Response of (28) in the Case of $T = 0.000065$ s. (a) Amplitude. (b) Phase](image)

Since

$$A_2(26.6) = 0.707, \quad (32)$$

and

$$A_2(54.55) = 0.1, \quad (33)$$

we have

$$\text{Rec} 2 = \frac{B_{0.7}}{B_{0.1}} = \frac{26.6}{54.55} = 0.488. \quad (34)$$

Note 2: This filter has the functionality to eliminate dc offset because $A_2(0) = 0$.

Note 3: Due to the sudden change at $f = 30$ Hz as shown in Fig. 3 (b), the phase of this filter is never linear.

Note 4: In practice, $T = 0.000065$ s requires over high sampling rate. Thus, that is replaced by $Tm = 0.000065$, where $m < 1$ is a constant. For example, if $T = 0.001$ s, one can set $m = 0.065$ [5].
4.3 Notch Filter $H_3(z)$ in [6,7]

In the reference papers by Pei and Tseng [6], Dotsinsky and Stoyanov [7], a notch filter was discussed. Its transfer function is written by

$$
H_3(z) = \frac{1}{2} \frac{(1 + a_2) - 2a_1 z^{-1} + (1 + a_2) z^{-2}}{1 - a_1 z^{-1} + a_2 z^{-2}}. \quad (35)
$$

The difference equation for is expressed by

$$
y(nT) = a_1 y(nT - T) - a_2 y(nT - 2T) \\
+ 0.5[(1 + a_2) x(nT) - 2a_1 x(nT - T) + (1 + a_2) x(nT - 2T)], \quad (36)
$$

where $y$ and $x$ are filtered and nonfiltered samples, respectively, and

$$
a_1 = \frac{2 \cos(\omega_0)}{1 + \tan(\Omega/2)}, \quad (37)
$$

$$
a_2 = \frac{1 - \tan(\Omega/2)}{1 + \tan(\Omega/2)}. \quad (38)
$$

The variables $\omega_0$ and $\Omega$ have to be computed for the interference frequency $f_0$ (Hz), the signal sampling rate $f_s$ (Hz), and the desired low and high cut-off frequencies $f_L$ and $f_H$ of the notch filter:

$$
\omega_0 = 2\pi \frac{f_0}{f_s}, \quad (39)
$$

$$
\Omega = 2\pi \frac{f_H - f_L}{f_s}. \quad (40)
$$

Let $f_0 = 60$ Hz, $f_s = 500$ Hz, $f_L = 59$ Hz, and $f_H = 61$ Hz. Then, $\Omega = 0.008\pi$, $\omega_0 = 0.24\pi$. Consequently,

$$
a_1 = \frac{2 \cos(\omega_0)}{1 + \tan(\Omega/2)} = 1.44, \quad a_2 = \frac{1 - \tan(\Omega/2)}{1 + \tan(\Omega/2)} = 0.975.
$$

The frequency response function of is given by

$$
H_3(e^{j2\pi f}) = \begin{bmatrix}
1 - a_2 \cos(2\pi f) + (1 + a_2) \cos(4\pi f) \\
2 - a_1 \cos(2\pi f) + a_2 \cos(4\pi f) + j[a_1 \sin(2\pi f) - a_2 \sin(4\pi f)] \\
1 + 2a_1 \sin(2\pi f) - (1 + a_2) \sin(4\pi f)] \\
2 - a_1 \cos(2\pi f) + a_2 \cos(4\pi f) + j[a_1 \sin(2\pi f) - a_2 \sin(4\pi f)]
\end{bmatrix}. \quad (41)
$$

The amplitude of the frequency response function is given by

$$
A_3(f) = \begin{bmatrix}
1 - a_2 \cos(2\pi f) + (1 + a_2) \cos(4\pi f) \\
2 - a_1 \cos(2\pi f) + a_2 \cos(4\pi f) + j[a_1 \sin(2\pi f) - a_2 \sin(4\pi f)] \\
1 + 2a_1 \sin(2\pi f) - (1 + a_2) \sin(4\pi f)] \\
2 - a_1 \cos(2\pi f) + a_2 \cos(4\pi f) + j[a_1 \sin(2\pi f) - a_2 \sin(4\pi f)]
\end{bmatrix}. \quad (42)
$$
The phase of $H_3(e^{j2\pi f})$ is

$$\phi_3(f) = \tan^{-1} \frac{\text{Im}[H_3(e^{j2\pi f})]}{\text{Re}[H_3(e^{j2\pi f})]}.$$  \hspace{1cm} (43)$$

Fig. 4 (a) and Fig. 4 (b) show the plots of $A3(f)$ and $\phi_3(f)$, respectively.

Since

$$A3(59.006) = 0.707,$$ \hspace{1cm} (44)

and

$$A3(59.9) = 0.1,$$ \hspace{1cm} (45)

we have

$$\text{Rec}_3 = \frac{B_{0.7}}{B_{0.1}} = \frac{59.006}{59.9} = 0.985.$$ \hspace{1cm} (46)

**Note 5.** The amplitude property of the filter $H_3(z)$ is satisfactory since $\text{Rec} 1 \approx 1$ but its phase is nonlinear (Fig. 4 (b)).

### 4.4. Discussions

The previous discussions exhibit that $H_1(e^{j2\pi f})$ and $H_3(e^{j2\pi f})$ have the similar high rectangular coefficient but the rectangular coefficient of $H_2(e^{j2\pi f})$ is low. We summarize the analysis results for three filters discussed previously in Table 1.

<table>
<thead>
<tr>
<th>System function</th>
<th>Rectangular coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1(e^{j2\pi f})$</td>
<td>0.988 (high)</td>
</tr>
<tr>
<td>$H_3(e^{j2\pi f})$</td>
<td>0.488 (fair)</td>
</tr>
<tr>
<td>$H_4(e^{j2\pi f})$</td>
<td>0.985 (high)</td>
</tr>
</tbody>
</table>

It can be easily seen from the high rectangular coefficient that the amplitude of the frequency response function of the notch filters in [3,4,6,7] to suppress the 60-Hz interference is good enough. Nevertheless, their phases are nonlinear, which people in the field might be unsatisfactory, because the nonlinear phase of a notch filter may cause distortions of ECG signals.
We now use a real ECG signal \( x(t) \), which is with the file name 100 in the Records in the MIT-BIH Arrhythmia Database. It is available to download freely from the web site with the link [20]. Fig. 5 indicates that signal. Note that where \( FFT \) stands for the fast Fourier transform. Figs. 6 (a) and (b) show its normalized amplitude of the spectrum (b) its phase, respectively.

Figure 5: A real ECG Signal \( x(t) \)

We now purposely add a 60 Hz interference to \( x(t) \). Denote

\[
n(t) = 0.5 \cos (2\pi ft) \quad \text{for} \quad f = 60 \text{ Hz}.
\]  

(47)

Figs. 7 (a) and (b) indicate the plot of \( n(t) \) and its spectrum, i.e., \( N(f) = FFT[n(t)] \) respectively.

Figure 7: Sixty Hz Interference. (a) Time Signal. (b) Spectrum

Denote

\[
g(t) = x(t) + n(t).
\]  

(48)

Then, the signal \( x(t) \) is seriously contaminated by \( n(t) \), as can be seen from Fig. 8.

Therefore, if we filter \( g(t) \) by \( H_1(e^{j2\pi f}) \) all frequency components of \( x(t) \) should pass through \( H_1(e^{j2\pi f}) \) with the 60 interference suppressing. Denote \( y(t) \) and \( Y(f) \) the time series and the spectrum of the output of \( H_1(e^{j2\pi f}) \), respectively. Then,
Figure 8: Signal $x(t)$ is Contaminated by a 60 Hz Interference (a) Time Signal (b) Spectrum

\[
Y(f) = H_1(e^{j2\pi f})G(f),
\]

(49)

\[
y(t) = IFFT[Y(f)],
\]

(50)

where $IFFT$ implies the inverse of $FFT$. Figs. 9 (a) and (b) show the amplitude of $Y(f)$ and its phase, respectively.

Figure 9: Spectrum of $y(t)$ (a) Amplitude (b) Phase

Fig. 9 (a) clearly indicates that the 60 Hz interference is satisfactorily eliminated by the notch filter $H_1(e^{j2\pi f})$. According to $y(t) = IFFT[Y(f)]$, we have a reconstructed signal in time as indicated in Fig. 10.

Figure 10: Time Signal $y(t)$

By comparing Fig. 5 to Fig. 10, we see that the $R$ waves have been completely reconstructed but others, such as $P$, $Q$, $T$ and $U$ waves, are seriously contaminated in a way, see Fig. 11. We shall explain the possible reason below.

Figure 11: Comparing $x(t)$ to $y(t)$. Solid line: $y(t)$. Dot line: $x(t)$
In appearance, the phase of \( y(t) \) is the same as that of \( x(t) \). However, there is the phase distortion caused by the filter \( H_1 (e^{j2\pi f}) \). Fig. 12 shows the comparison between the phase of \( X(f) \) and that of \( Y(f) \). From Fig. 12, we see that the phase of \( X(f) \) is completely overlapped with that of \( Y(f) \) except that there is an obvious phase distortion for \( f \) being near 60 Hz. This is natural because the filter \( H_1 (e^{j2\pi f}) \) has considerably nonlinear phase for \( f \) being near 60 Hz and its phase quite linear for other frequencies (see Fig. 2 (b)).

![Figure 12: Comparing the Phase of \( X(f) \) to that of \( Y(f) \). Solid Line: the Phase \( Y(f) \). Dot Line: the Phase of \( X(f) \)](image)

Note that the \( |X(f)| \) is totally overlapped with \( |Y(f)| \) (see Fig. 13), implying that \( H_1 (e^{j2\pi f}) \) has satisfactory amplitude response. Therefore, we conclude that the reason why there are distortions shown in Fig. 11 may be the nonlinearity of the phase of \( H_1 (e^{j2\pi f}) \) although its phase is only considerably nonlinear for \( f \) being near 60 Hz.

![Figure 13: Comparing \( |X(f)| \) to \( |Y(f)| \). Solid Line: \( |Y(f)| \). Dot Line: \( |X(f)| \)](image)

As the nonlinear phase is an intrinsic property of IIR filters while FIR notch filters have linear phases, we suppose that to find an FIR notch filter that has high rectangular coefficient may be a way to overcome the shortcoming of nonlinear phase of an IIR filter though an FIR filter may be time consuming.

**5. CONCLUSIONS**

We have analyzed three well-known IIR digital notch filters for suppressing 60-Hz interference in ECG signals. The present results exhibit that the filters in [3,4] and [6,7] have satisfactory rectangular coefficient but their phases are nonlinear. We have shown a case of applying the filter \( H_1 (e^{j2\pi f}) \) to remove 60 Hz interference with a real ECG signal. The analysis of this case implies that the nonlinearity of the phase of a filter may cause serious distortions of the ECG signal being analyzed. Consequently, to design FIR notch filters that have high rectangular coefficient may be a way to reduce the side effect of notch filters for removing 60-Hz interference.
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REFERENCES