FUZZY JOIN PRIME SEMI L-IDEAL

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The concept of fuzzy join prime semi L-ideals in fuzzy join subsemilattices is introduced. The properties of fuzzy join prime semi L-ideals are discussed.

Keywords: Fuzzy join prime semi L-ideals, f-invariant fuzzy join prime semi L-ideals, fuzzy join prime semi L-ideal homomorphism.

1. INTRODUCTION


Definition 1: A fuzzy join semi L-ideal $S(\mu)$ of a fuzzy join semilattice $A$ is said to be a fuzzy join prime semi L-ideal of $A$ if

(i) $S(\mu)$ is not a constant function and

(ii) For any two fuzzy join semi L-ideals $S(\sigma)$ and $S(\theta)$ in $A$ if $S(\sigma) \lor S(\theta) \subseteq S(\mu)$, then either

$S(\sigma) \subseteq S(\mu)$ or $S(\theta) \subseteq S(\mu)$.

Example 1: Let $A = \{0, a, b, 1\}$ be a fuzzy join semilattice.

Consider $S(\mu)$ is a fuzzy join prime semi L-ideal of $A$.

Then $S[\mu(0)] = 0.6$, $S[\mu(a)] = 0.5$, $S[\mu(b)] = 0.4$, $S[\mu(1)] = 0.7$

Let $S(\sigma)$ and $S(\theta)$ be any fuzzy join prime semi L-ideals of $A$.

Then $S[\sigma(0)] = 0.4$, $S[\sigma(a)] = 0.2$, $S[\sigma(b)] = 0.3$, $S[\sigma(1)] = 0.8$

and $S[\theta(0)] = 0.5$, $S[\theta(a)] = 0.3$, $S[\theta(b)] = 0.4$, $S[\theta(1)] = 0.7$

Here $S(\sigma) \lor S(\theta) = S(\mu)$, $S(\sigma) \not\subseteq S(\mu)$ or $S(\theta) \not\subseteq S(\mu)$.

Hence $S(\mu)$ is a fuzzy join prime semi L-ideal of $A$.

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Note 1: \(S(\sigma) \subseteq S(\mu)\) means \(S[\sigma(x)] \leq S[\mu(x)]\) for all \(x \in A\).

Definition 2: A fuzzy join prime semi L-ideal \(S(\mu)\) of a fuzzy join semilattice \(A\) is called fuzzy level join semi L-prime if the fuzzy level join semi L-ideal \(S(\mu_t)\), where \(t = S[\mu(0)]\) is a prime semi L-ideal of \(A\).

Proposition 1: Let \(S(\mu)\) be any fuzzy join prime semi L-ideal of a fuzzy join semilattice \(A\) such that each fuzzy level join prime semi L-ideal \(S(\mu_t)\), \(t \in \text{Im} S(\mu)\) is prime. If \(S[\mu(x)] < S[\mu(y)]\) for some \(x, y \in A\) then \(S[\mu(x \vee y)] = S[\mu(y)]\).

Proof: Let \(S[\mu(x)] = t\); \(S[\mu(y)] = t'\); and \(S[\mu(x \vee y)] = s\).

Given \(S[\mu(x)] < S[\mu(y)]\) (i.e) \(t < t'\).

Now, \(s = S[\mu(x \vee y)] \geq \max \{S[\mu(x)], S[\mu(y)]\}\)

\(= \max \{t, t'\}\)

\(= t'\)

Therefore \(t < t' \leq s\).

Suppose that \(t' < s\).

If \(x \vee y \in S(\mu)\) then either \(x \in S(\mu)\) or \(y \in S(\mu)\). Since \(S(\mu_s)\) is a fuzzy level join prime semi L-ideal of \(A\).

Now, \(x \in S(\mu) \Rightarrow S[\mu(x)] = s\) or \(y \in S(\mu) \Rightarrow S[\mu(y)] = s\).

Hence, \(t = S[\mu(x)] \geq s\) or \(t' = S[\mu(y)] \geq s\), which is not possible.

Therefore, \(t' = s\) (i.e) \(S[\mu(x \vee y)] = S[\mu(y)]\)

Corollary 1: If \(S(\mu)\) is any fuzzy join prime semi L-ideal of a fuzzy join semilattice \(A\) then \(S[\mu(x \vee y)] = \max \{S[\mu(x)], S[\mu(y)]\}\), for all \(x, y \in A\).

Proof: Let \(x, y \in A \Rightarrow x \vee y \in A\)

\(S(\mu)\) is a fuzzy join prime semi L-ideal.

\(\Rightarrow S[\mu(x \vee y)] \geq \max \{S[\mu(x)], S[\mu(y)]\}\).

\(\Rightarrow S[\mu(x)] < S[\mu(y)]\), then \(S[\mu(x \vee y)] = S[\mu(y)]\)

Similarly, if \(S[\mu(x)] > S[\mu(y)]\), then \(S[\mu(x \vee y)] = S[\mu(x)]\), by theorem 5.1.5.

Therefore, \(S[\mu(x \vee y)] = \max \{S[\mu(x)], S[\mu(y)]\}\).

Theorem 1: Let \(S(\mu)\) be a fuzzy join prime semi L-ideal of a fuzzy join semi lattice \(A\) then \(\text{Card \text{Im} } S(\mu) = 2\).

Proof: Since \(S(\mu)\) is non constant, \(\text{card \text{Im} } S(\mu) \geq 2\).
Suppose that \( \text{card} \, \text{Im} \, S(\mu) \geq 3 \).

Let \( S[\mu(0)] = s \) and \( k = \text{Sup} \{ S[\mu(x)]/x \in A \} \).

Then there exists \( t, m \in \text{Im} \, S(\mu) \) such that \( t < m < s \) and \( t \leq k \).

Let \( S(\sigma) \) and \( S(\theta) \) be two fuzzy join prime semi L-ideals of \( A \) such that

\[
S[\sigma(x)] = \frac{1}{2} (t + m), \quad \text{for all } x \in A \text{ and all } \mu \in \text{Im} \, S(\mu)
\]

\[
S[\theta(x)] = s, \quad \text{if } x \in S(\mu)
\]

Clearly, \( S(\sigma) \) is a fuzzy join prime semi L-ideal of \( A \).

To show that \( S(\theta) \) is a fuzzy join prime semi L-ideal of \( A \).

Let \( x, y \in A \).

**Case (i):** If \( x, y \in S(\mu) \), then \( S[\theta(x)] = s, S[\theta(y)] = s, x \lor y \in S(\mu) \)

Also,

\[
S[\theta(x \lor y)] = s = \text{max} \{ S[\theta(x)], S[\theta(y)] \}
\]

\[
\Rightarrow \quad S[\theta(x \lor y)] = s = \text{max} \{ S[\theta(x)], S[\theta(y)] \}
\]

Therefore, \( S(\theta) \) is a fuzzy join prime semi L-ideal of \( A \).

**Case (ii):** If \( x \in S(\mu) \) and \( y \notin S(\mu) \), then \( S[\theta(x)] = s, S[\theta(y)] = k, x \lor y \notin S(\mu) \)

Also,

\[
S[\theta(x \lor y)] = k = \text{max} \{ S[\theta(x)], S[\theta(y)] \}
\]

\[
\Rightarrow \quad S[\theta(x \lor y)] = k = \text{max} \{ S[\theta(x)], S[\theta(y)] \}
\]

Therefore, \( S(\theta) \) is a fuzzy join prime semi L-ideal of \( A \).

**Case (iii):** If \( x \notin S(\mu) \) and \( y \notin S(\mu) \), then \( S[\theta(x)] = S[\theta(y)] = k, x \lor y \notin S(\mu) \)

Also,

\[
S[\theta(x \lor y)] = k = \text{max} \{ S[\theta(x)], S[\theta(y)] \}
\]

\[
\Rightarrow \quad S[\theta(x \lor y)] = k = \text{max} \{ S[\theta(x)], S[\theta(y)] \}
\]

Therefore, \( S(\theta) \) is a fuzzy join prime semi L-ideal of \( A \).

**Claim:** \( S(\sigma) \lor S(\theta) \subseteq S(\mu) \)

Let \( x \in A \).

Consider the following cases:

(i) \( \text{Let } x = 0. \)

Then

\[
[S(\sigma) \lor S(\theta)](x) = \text{max} \{ \text{max} \{ S[\sigma(y)], S[\theta(z)] \} \}
\]
\[ x = y \lor z \]
\[ \leq \frac{1}{2} (t + m) \]
\[ < s \]
\[ = S[\mu(0)] \]

(ii) Let \( x \neq 0, x \in S(\mu_{m}) \).

Then
\[ S[\mu(x)] \geq m \]

and
\[ [S(\sigma) \lor S(\theta)](x) = \max \{\max \{S[\sigma(y)], S[\theta(z)]\}\} \]
\[ x = y \lor z \]
\[ \leq \frac{1}{2} (t + m) \]
\[ < m \]
\[ = S[\mu(x)]. \]

Since
\[ \max \{S[\sigma(y)], S[\theta(z)]\} \leq S[\sigma(y)]. \]

(iii) Let \( x \neq 0, x \notin S(\mu_{m}). \)

Then for any \( y, z \in A \), such that \( x = y \lor z \), \( y \notin S(\mu_{m}) \) and \( z \notin S(\mu_{m}). \)

Thus, \( S[\theta(y)] = k \) and \( S[\theta(z)] = k \)

Hence, \([S(\sigma) \lor S(\theta)](x) = \max \{\max \{S[\sigma(y)], S[\theta(z)]\}\}\]
\[ x = y \lor z \]
\[ = \max \{\max (k, k)\} \]
\[ = k \]
\[ \leq S[\mu(x)]. \]

Thus in any case, \([S(\sigma) \lor S(\theta)](x) \leq S[\sigma(x)] \]

Hence \( S(\sigma) \lor S(\theta) \leq S(\mu) \)

Now, there exists \( y \in A \) such that \( S[\mu(y)] = t \)

Then \( S[\sigma(y)] = \frac{1}{2} (t + m) > S[\mu(y)] \)

\( \therefore \]
\[ S[\sigma(y)] > S[\mu(y)] \]

Hence, \( S(\sigma) \subseteq S(\mu) \)

Also there exists \( x \in A \) such that \( S[\mu(x)] = t. \)

Then, \( x \in S(\mu_{m}) \) and

thus \( S[\theta(x)] = s > m = S[\mu(x)] \)

\( \Rightarrow S[\theta(x)] > S[\mu(x)] \)
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Hence $S(\theta) \subseteq S(\mu)$

This shows that $S(\mu)$ is not a fuzzy join prime semi L-ideal of $A$, which is a contradiction to the hypothesis.

Hence, card $\text{Im} S(\mu) = 2$.

**Theorem 2:** Let $A$ be a fuzzy join semilattice and let $S(\mu)$ be a fuzzy join semi L-ideal of $A$ such that Card $\text{Im} S(\mu) = 2$, $S[\mu(0)] = 1$ and the set $S(\mu_0) = \{x \in A/S[\mu(x)] = S[\mu(0)]\}$ is a fuzzy level join prime semi L-ideal of $A$. Then $S(\mu)$ is a fuzzy join prime semi L-ideal of $A$.

**Proof:** Let $\text{Im} S(\mu) = \{t, 1\}$, $t < 1$.

Then $S[\mu(0)] = 1$.

Let $x, y \in A$.

**Case (i):** If $x, y \in S(\mu_0)$ then $x \lor y \in S(\mu_0)$ and $S[\mu(x \lor y)] = 1 = \max \{S[\mu(x)], S[\mu(y)]\}$.

**Case (ii):** If $x \in S(\mu_0)$ and $y \in S(\mu_0)$, then $x \lor y \not\in S(\mu_0)$ and $S[\mu(x \lor y)] = t = \max \{S[\mu(x)], S[\mu(y)]\}$.

**Case (iii):** If $x, y \in S(\mu_0)$ and $y \not\in S(\mu_0)$, then $S[\mu(x)] = S[\mu(y)] = t$.

Thus, $S[\mu(x \lor y)] \geq t = \max \{S[\mu(x)], S[\mu(y)]\}$.

Hence, $S[\mu(x \lor y)] \geq \max \{S[\mu(x)], S[\mu(y)]\}$, for all $x, y \in A$.

Now, if $x \in S(\mu_0)$, then $x \lor y, y \lor x \in S(\mu(0))$.

Therefore, $S[\mu(x \lor y)] = S[\mu(y \lor x)] = 1 = S[\mu(x)]$.

If $x \not\in S(\mu_0)$, then $S[\mu(x \lor y)] \geq t = S[\mu(x)]$ and $S[\mu(y \lor x)] \geq t = S[\mu(x)]$.

Hence, $S(\sigma)$ is a fuzzy join prime semi L-ideal of $A$.

Let $S(\sigma)$ and $S(\theta)$ be fuzzy join prime semi L-ideals of $A$ such that $S(\sigma) \lor S(\theta) \subseteq S(\mu)$.

Suppose that $S(\sigma) \subseteq S(\mu)$ and $S(\theta) \subseteq S(\mu)$.

Then there exists $x, y \in A$ such that $S[\sigma(x)] > S[\mu(x)]$ and $S[\theta(x)] > S[\mu(x)]$.

Since for all $a \in S(\mu_0)$, $S[\mu(a)] = 1 = S[\mu(0)]$, $x \not\in S(\mu_0)$ and $y \not\in S(\mu_0)$.

Now, since $S(\mu_0)$ is a fuzzy join prime semi L-ideal of $A$, there exists $z \in A$ such that $x \lor z \lor y \not\in S(\mu_0)$.

Let $a = x \lor z \lor y$.

Then, $S[\mu(a)] = S[\mu(x)] = S[\mu(y)] = t$.

Now, $S[\sigma(a)] = \max \{S[\sigma(u)] \lor S[\sigma(v)]\}$

$x = u \lor v$

$\geq \max \{S[\sigma(x)], S[\sigma(z \lor y)]\} > t = S[\mu(a)]$. 
Since $S(\sigma(x)) > S(\sigma(y)) = t$ and $S[\theta(x \lor y)] > S[\theta(y)] > S[\mu(y)] = t$
That is, $S(\sigma) \lor S(\theta) \subseteq S(\mu)$.

This contradicts the assumption that $S(\sigma) \lor S(\theta) \subseteq S(\mu)$

Thus, $S(\mu)$ is a fuzzy join prime semi L-ideal of $A$.

**Theorem 3:** If $f$ is a fuzzy join prime semi L-ideal homomorphism from a fuzzy join semi L-ideal of $A$ onto a fuzzy join semi L-ideal of $A'$ and $S(\mu')$ is any fuzzy join prime semi L-ideal of $A'$, then $f^{-1}[S(\mu')]$ is a fuzzy join prime semi L-ideal of $A$.

**Proof:** Let $S(\mu)$ and $S(\sigma)$ be any two fuzzy join prime semi L-ideals of $A$ such that $S(\mu) \lor S(\sigma) \subseteq f^{-1}[S(\mu')]$.

$\Rightarrow \ f[S(\mu) \lor S(\sigma)] \subseteq ff^{-1}[S(\mu)] = S(\mu')$

$\Rightarrow \ f[S(\mu)] \lor f[S(\sigma)] \subseteq S(\mu').$

Since $f$ is a fuzzy join prime semi L-ideal homomorphism.

$\Rightarrow$ Either $f[S(\mu)] \subseteq S(\mu')$ or $f[S(\sigma)] \subseteq S(\mu'),$

Since $S(\mu')$ is fuzzy join prime semi L-ideal of $A'$.

$\Rightarrow$ Either $f^{-1}[S(\mu)] \subseteq f^{-1}[S(\mu')]$ or $f^{-1}[S(\sigma)] \subseteq f^{-1}[S(\mu')]$.

$\Rightarrow$ Either $S(\mu) \subseteq f^{-1}[S(\mu')]$ or $S(\sigma) \subseteq f^{-1}[S(\mu')]$

Hence $f^{-1}[S(\mu')]$ is a fuzzy join prime semi L-ideal of $A$.

**REFERENCES**


