FUZZY EDGE COLORING OF FUZZY GRAPHS

R. Govindarajan¹ and S. Lavanya²

Abstract: In this paper the concept fuzzy edge coloring of fuzzy graph is defined as a family of fuzzy sets satisfying some conditions. It has been proved that the edge coloring of a fuzzy graph G is equivalent to the vertex coloring of its line graph and explained this through an example. The edge chromatic number of various graphs is found.

Keywords: Fuzzy coloring, Fuzzy graph, Fuzzy chromatic number.

INTRODUCTION

Coloring in fuzzy graphs was introduced by Mounox et al in [5]. They considered fuzzy graphs with crisp vertices and fuzzy edges. They defined fuzzy chromatic number as a fuzzy number defined through the α-cuts of the given fuzzy graph. The fuzzy vertex coloring of a fuzzy graph was defined by the authors Eslahchi and Onagh [2] as family of fuzzy sets satisfying some conditions. This definition was slightly modified and redefined by Lavanya and Sattanathan[3]. They found the fuzzy chromatic number for complete graphs, trees and complement of fuzzy graphs. In crisp graphs k-edge coloring of a graph G is a mapping c : E(G) → {1, 2, ..., k} such that incident edges receive different colors. Given a graph G, the edge chromatic number or chromatic index χ'(G) is the least k for which G is k-edge-colorable. In this paper we extend this concept of edge coloring to fuzzy graphs and define k-fuzzy edge coloring. The famous Vizing’s theorem on edge coloring states that any graph with a maximum vertex degree of Δ can be edge colored using at most Δ+1 colors. We prove that this theorem holds for fuzzy edge coloring also and find the fuzzy edge chromatic number for various fuzzy graphs and for complement of fuzzy graphs.

The set of edges that share a common vertex v are said to be incident if Max {μ(vv) / vv are set of edges from v} ≤ σ(v).

By definition of incidence in fuzzy graphs we can say that all the edges that share a common vertex are incident edges similar to incidence in crisp case.

Maximum number of edges incident from a vertex of a fuzzy graph G is denoted by (G).

Definition 1: A family Γ = {γ₁, ..., γₖ} of fuzzy sets on E(V × V) is called a k-fuzzy edge coloring of G = (σ, μ) if

¹, ² D.G. Vaishnav College, University of Madras, Chennai, Tamilnadu, India
²E-mail: lavanyaprasad1@gmail.com
(a) Max $\gamma_i(\text{uv}) = \mu(\text{uv})$ for all edge $\text{uv} \in E$.

(b) $\gamma_i \land \gamma_j = 0$.

(c) For every incident edges $\text{Min}\{\gamma_i(v_j) \mid v_j \in S_i\}$ are set of incident edges from the vertex $v_j$, $j = 1, 2, ..., |V| = 0$. $i = 1, 2, ..., k$.

Fuzzy edge chromatic number (fuzzy chromatic index) is the least value of $k$ for which the fuzzy graph $G$ has $k$-fuzzy edge coloring and is denoted by $\chi^k_{ef}(G)$.

Any edge coloring problem on a crisp graph $G$ can be converted to the problem of finding a vertex coloring on the line graph $L(G)$ where the line graph is defined as follows.

Definition 2: Given a graph $G$, its line graph $L(G)$ is a graph such that

- each vertex of $L(G)$ represents an edge of $G$; and
- two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent") in $G$.

That is, it is the intersection graph of the edges of $G$, representing each edge by the set of its two endpoints.

Same way we consider the fuzzy line graphs [10] and verify whether the fuzzy edge coloring of a fuzzy graph can be converted to fuzzy vertex coloring on its fuzzy line graph.

Fuzzy Line Graphs

Fuzzy line graph is the fuzzy intersection graph defined on the edge set $E$. Now we give the definitions of fuzzy intersection graph and fuzzy line graph as in [10].

Let $G = (V, E)$ be a graph where $V = \{v_1, v_2, ..., v_n\}$. Let $S_i = \{v_i, x_{i1}, x_{i2}, ..., x_{im}\}$ where $x_{ij} \in E$ and $x_{ij}$ has $v_i$ as a vertex, $j = 1, 2, ..., m; i = 1, 2, ..., n$.

Let $S = \{S_1, S_2, ..., S_n\}$ and $T = \{(S_i, S_j) \mid S_i, S_j \in S, S_i \cap S_j \neq \emptyset\}$. Then $I(S) = (S, T)$ is an intersection graph. Any partial fuzzy subgraph $(\tau, \nu)$ of $I(S)$ with supp($\nu$) = $T$ is called a fuzzy intersection graph.

If $G = (\sigma, \mu)$ is a fuzzy graph and $I(S, \tau, \nu)$ is the fuzzy intersection graph then the fuzzy subsets $\tau$ and $\nu$ of $S$ and $T$ are defined as follows

$$\tau(S_i) = \sigma(v) \quad \forall S_i \in S$$

$$\nu(S_i, S_j) = \mu(v_i, v_j) \quad \forall S_i, S_j \in T$$

For the graph $G = (V, E)$, Line graph $L(G)$ is the intersection graph defined on the edge set $E$. i.e., $I(E)$.

The line graph $L(G) = (Z, W)$ together with the fuzzy sets $\lambda, \omega$ is the fuzzy line graph $L_{\lambda}(G) = (\lambda, \omega)$ where $Z = \{S_1, S_2, ..., S_n\}, S_i = \{x_i \cup \{u, v\}/x_i E, x_i = \{u, v\}, u, v \in V\} i = 1, 2, ..., |E|$ and $W = \{(S_i, S_j) \mid S_i \cap S_j \neq \emptyset\}$. The fuzzy sets $\lambda, \omega$ of $Z$ and $W$ are defined as follows

$$\lambda(S_i) = \mu(x) \quad \forall S_i \in Z$$
The necessary and sufficient condition for a fuzzy graph to be a fuzzy line graph of some fuzzy graph is given in [10] which is as follows

**Theorem 1:** $L_F(G) = (\lambda, \omega)$ is a fuzzy line graph of $G = (\sigma, \mu)$ if and only if $(\text{supp} (\lambda), \text{supp} (\omega))$ is a line graph and $\forall (u, v) \in \text{supp}(\omega), \omega(u, v) = \lambda(u) \wedge \lambda(v)$.

**Example 1:** Consider the fuzzy graph $G = (\sigma, \mu)$ with vertex set $V = \{1, 2, 3, 4\}$ and edge set $E = \{12, 13, 14, 23, 34\}$

where

$$
\begin{align*}
\sigma(v_i) &= \begin{cases} 
0.5 & i = 3, 4 \\
0.8 & i = 2 \\
0.2 & i = 1 
\end{cases} \\
\mu(x_i) &= \begin{cases} 
0.12 & i = 12, 13, 14 \\
0.375 & i = 23, 24 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
$$

Figure 1: Fuzzy Graph $G(\sigma, \mu)$ for Example 1

The fuzzy line graph $L_F(G) = (\lambda, \omega)$ for $G = (\sigma, \mu)$ is constructed as follows

$$
Z = \{S_1, S_2, \ldots, S_5\}
$$

where
\[ S_1 = \{ x_{12} \cup \{ v_1, v_2 \} \}, S_2 = \{ x_{23} \cup \{ v_2, v_3 \} \}, S_3 = \{ x_{34} \cup \{ v_3, v_4 \} \}, S_4 = \{ x_{13} \cup \{ v_1, v_3 \} \}, S_5 = \{ x_{14} \cup \{ v_1, v_4 \} \} \text{ and } W = \{ (S_1, S_2), (S_1, S_4), (S_1, S_5), (S_2, S_3), (S_2, S_5), (S_3, S_4), (S_3, S_2), (S_3, S_3) \}. \]

Figure 2: Fuzzy Line Graph \( L_F(G) \) of Fuzzy Graph \( G \) in Example 1

Now we show that the fuzzy edge coloring of a fuzzy graph can be converted to fuzzy vertex coloring on its fuzzy line graph.

By theorem [1] we have \( \omega(uv) = \lambda(u)^\alpha \lambda(v)^\beta \) for \( uv \in W \). So for all \( uv \in W \) with \( \omega(u, v) > 0 \) the vertices \( u \) and \( v \) are strongly adjacent. Thus the definition of fuzzy vertex coloring can be applied to the fuzzy line graph to obtain the fuzzy chromatic number which is nothing but the fuzzy edge chromatic number of the fuzzy graph \( G \).

BOUND FOR FUZZY EDGE CHROMATIC NUMBER

The famous Vizing’s theorem on edge coloring in crisp graphs states that any graph with a maximum vertex degree of \( \Delta \) can be edge colored using at most \( \Delta + 1 \) colors i.e., the edge chromatic number \( \chi'(G) \leq \Delta (G) + 1 \).

For fuzzy graphs same bound can be given to the fuzzy edge chromatic number.
Theorem 2: For a fuzzy graph $G(\sigma, \mu)$, the fuzzy edge chromatic number $\chi'_e(G) \leq \Delta(G) + 1$ where $\Delta(G)$ is the maximum number of edges incident to a vertex of $G$.

FUZZY EDGE CHROMATIC NUMBER FOR VARIOUS FUZZY GRAPHS

- The edge chromatic number of complete fuzzy graph on $n$ vertices is $n$ if $n$ is odd and $n-1$ if $n$ is even whatever may be the membership functions.
- The edge chromatic number of fuzzy trees is $\Delta$ (we consider only the unique maximum spanning tree of the fuzzy tree) where $\Delta$ is the maximum number of incident edges from a vertex.
- If $\mu(xy) = \frac{1}{2} (\sigma(x) \wedge \sigma(y))$ for all $x, y \in V$ then every pair of vertices are strongly adjacent. Then $G$ is self complementary and $G \cong \overline{G}$. So $\chi'_e(G) = \overline{\chi'_e}(G)$.

REFERENCES