STRONG / WEAK EDGE VERTEX MIXED DOMINATION NUMBER OF A GRAPH

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Abstract: For any $v \in V$, the set $N(v) = \{u \in V \mid uv \in E\}$ is the open neighbourhood of the vertex $v$; while the set $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of $v$. Similarly for any edge $x = uv$, $N(x) = \{y \in E \mid y$ is adjacent to $x\}$ and $N[x] = N(x) \cup \{y\}$. Further for any edge $x = uv$, $V_x = V_x(N[x]) = \{w \in V \mid uw \in E \text{ or } vw \in E\}$. An edge $x$, $m$-dominates a vertex $v$ if $v \in V_x$. An edge $x$ strongly (weakly) $m$-dominates a vertex $v$ if $v \in V_x$ and $\deg(x) \geq \deg(y)$ ($\deg(x) \in \deg(y)$) for every $y$ which $m$-dominates the vertex $v$. A set $L \subseteq E$ is an Edge Vertex Dominating set (EVD-set) if every vertex in $G$ is $m$-dominated by an edge in $L$. The edge vertex domination number $\gamma_{ev}(G)$ is the minimum cardinality of an EVD-set. A set $L \subseteq E$ is a Strong Edge Vertex Dominating Set (SEVD-set) (Weak Edge Vertex Dominating Set (WEVD-set)) if every vertex in $G$ is strongly (weakly) $m$-dominated by an edge in $L$. The strong (weak) edge vertex domination number $\gamma_{sev}(G)$ ($\gamma_{wev}(G)$) is the minimum cardinality of a SEVD-set (WEVD-set). Besides finding the relationship between the existing graph parameters, we prove a Gallai’s type result for edges. A new parameter called Edge Vertex degree of an edge is defined and a bound in terms of the maximum and minimum EV degree is established.

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1. INTRODUCTION

By a graph we mean a simple undirected isolate free graph. For undefined terminologies refer [1]. Degree of a vertex $v$, $\deg(v)$ is the number of edges incident on $v$. Similarly the degree of an edge $x = uv$, $\deg(x)$ is the number of edges adjacent to the edge $x$. Equivalently $\deg(x) = \deg(u) + \deg(v) - 2$. Let $\Delta(G)$, $\delta(G)$, $\Delta_e(G)$, $\delta_e(G)$ denote the maximum degree, minimum degree, maximum edge degree and minimum edge degree of $G$ respectively. For any $v \in V$, the set $N(v) = \{u \in V \mid uv \in E\}$ is the open neighbourhood of the vertex $v$; while the set $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of $v$. Similarly for any edge $x = uv$, $N(x) = \{y \in E \mid y$ is adjacent to $x\}$ and $N[x] = N(x) \cup \{y\}$. Further for any edge $x = uv$, $V_x = V_x(N[x]) = \{w \in V \mid uw \in E \text{ or } vw \in E\}$. Also $\langle N[v]\rangle$ denotes the subgraph induced by the set $N[v]$.

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Domination is a well studied concept in graph theory. For details reader can refer [2, 3 and 16]. The strong weak domination was first introduced by Sampathkumar and Pushpalatha [13]. For any two adjacent vertices u and v in a graph \( G = (V, E) \), u strongly (weakly) dominates v if \( \deg(u) \geq \deg(v) \) \( \deg(u) \leq \deg(v) \). A set \( D \subseteq V \) is a dominating set (strong dominating set [SD-set], weak dominating set [WD-set] respectively) of \( G \) if every \( v \in V - D \) is dominated (strongly dominated, weakly dominated respectively) by some \( u \in D \). The domination number (strong domination number, weak domination number respectively) \( \gamma(G), \gamma_s(G), g_w(G) \) respectively) of \( G \) is the minimum cardinality of a dominating set (SD-set, WD-set respectively) of \( G \). Similarly its edge analogue Edge domination, strong (weak) edge domination numbers are defined. Sampathkumar and P. S. Neeralagi [11, 12] defined the neighbourhood sets and line neighbourhood sets as follows. A set \( S \subseteq V \) is a neighbourhood set (n-set) if \( G \) is a line neighbourhood set (ln-set) if \( G \). The neighbourhood number \( n_s = n_s(G) \) [line neighbourhood number \( n_{ln} = n_{ln}(G) \)] is the minimum cardinality of a n-set [ln set] of \( G \).

Mixed domination was first studied by R.Laskar and Ken Peters [8]. A vertex \( v \) and an edge \( uw \) weakly dominate each other if (i) \( v = u \) or \( v = w \), or (ii) \( vu \) or \( vw \) is an edge in \( G \). Further a vertex \( v \) and an edge \( uw \) strongly dominate each other if (i) \( v = u \) or \( v = w \), or (ii) \( vu \) and \( vw \) both are edges in \( G \). A set \( S \subseteq V \) is said to be a vertex edge weak (strong) dominating set if every edge in \( G \) is weakly (strongly) dominated by a vertex in \( S \). A set \( F \subseteq E \) is an edge vertex weak (strong) dominating set if every vertex in \( G \) is weakly (strongly) dominated by an edge in \( F \). The vertex edge weak (strong) domination number \( \gamma_{01}(G) \) (\( \gamma_{10}(G) \)) is the minimum cardinality of a vertex edge weak (strong) dominating set of \( G \). Like wise the edge vertex weak (strong) domination number \( \gamma_{10}(G) \) (\( \gamma_{10}(G) \)) is defined. How ever our notion of strong weak mixed domination depends on the degree of an edge and hence entirely different from the concept of strong weak domination defined by R. Laskar and Ken Peters. In 1992, Sampathkumar and S. S. Kamath [10] independently studied the same concept of mixed domination with a different but simpler approach. An edge \( x \), \( m \)-dominates a vertex \( v \) if \( v \in V_x \). A set \( L \subseteq E \) is an Edge Vertex Dominating set (EVD-set) if every vertex in \( G \) is \( m \)-dominated by an edge in \( L \). The edge vertex domination number \( \gamma_{ev}(G) \) is the minimum cardinality of an EVD-set. We now define strong (weak) Edge Vertex Mixed Domination. These concepts are further studied in [4, 5, 6 and 7].

### 2. STRONG (WEAK) EDGE VERTEX MIXED DOMINATION

An edge \( x \) strongly (weakly) \( m \)-dominates a vertex \( v \) if \( v \in V_x \) and \( \deg(x) \geq \deg(y) \) \( \deg(x) \leq \deg(y) \) for every \( y \) which \( m \)-dominates the vertex \( v \). A set \( L \subseteq E \) is a Strong Edge Vertex Dominating Set (SEVD-set) [Weak Edge Vertex Dominating Set (WEVD-set)] if every vertex...
in $G$ is strongly (weakly) $m$-dominated by an edge in $L$. The strong (weak) edge vertex domination number $\gamma_{sev}(G)$ ($\gamma_{wev}(G)$) is the minimum cardinality of a SEVD-set (WEVD-set).

In [10] it is observed that $\gamma_{10} = \gamma_{ev} \leq \gamma_{10}'$ and $\gamma_{ev}' \leq n_o'$. In [12] it is proved that $n'_o \leq n_o$ and $n'_o \leq g'$. Sampathkumar and Pushpalatha [13] defined full number and $s$-full number. A set $D \subset V$ is full ($s$-full, $w$-full respectively) if every $u \in D$ dominates (strongly dominates, weakly dominates respectively) some $v \in V - D$. The full number ($s$-full number, $w$-full number, respectively) $f = f(G)$ ($f_s = f_s(G)$, $f_w = f_w(G)$, respectively) of a graph $G$ is the maximum cardinality of a full set ($s$-full set, $w$-full set respectively) of $G$.

2.1 Full Edge Vertex Dominating Set

Motivated by above definition we now define Full Edge Vertex Dominating (FEVD) sets, Full Strong Edge Vertex Dominating (FSEVD) sets and Full Weak Edge Vertex Dominating (FWEVD) sets. In what follows by $V(L)$ we mean the set $V(L) = \{ v \in V \mid v \text{ is incident on some } x \in L \}$.

A set $L \subset E$ is said to be FEVD set if for every $v \in V(L)$ there exists a $x \in E - L$ such that $x, m$-dominates the vertex $v$. A set $L \subset E$ is said to be FSEVD (FWEVD) set if for every $v \in V(L)$ there exists a $x \in E - L$ such that $x$, strongly (weakly) $m$-dominates $v$. The FEVD number $f_{ev}(G)$ (FSEVD number $f_{sev}(G)$ and FWEVD number $f_{wev}(G)$ respectively) is the maximum cardinality of a FEVD set (FSEVD set, WFEVD set respectively) of $G$.

2.2 Example

Here $\gamma_{ev}(G) = 2$, $\gamma_{sev}(G) = 5$ and $\gamma_{wev}(G) = 4$. The dotted edges in Fig. 1(a), Fig. 1(b) and Fig. 1(c) represent a $\gamma_{ev}$-set, a $\gamma_{sev}$-set and a $\gamma_{wev}$-set respectively. Further, $f_{ev}(G) = 18$, $f_{wev}(G) = 15$, $f_{sev}(G) = 16$. The heavy edges in Fig. 1(a), Fig. 1(b) and Fig. 1(c) form $f_{ev}$-set, $f_{sev}$-set and $f_{wev}$-set respectively.

**Fact 1:** Since every SEVD set and WEVD set is an EVD set we have $\gamma_{ev} \leq \gamma_{sev}$ and $\gamma_{ev} \leq \gamma_{wev}$. But $\gamma_{sev}$ and $\gamma_{wev}$ are not comparable. For example for the graph $G$, $\gamma_{sev}(G) = 5 > 4 = \gamma_{wev}(G)$ but for a path $P_n$ with $n$ vertices, $\gamma_{sev}(P_n) = \frac{n}{4} \leq \frac{n+2}{4} = \gamma_{wev}(P_n)$. 

Fact 2. An edge $x$ is said to be isolated if $\text{deg}(x) = 0$. A graph $G$ is said to be isolate edge free graph if $G$ has no isolated edges. It is easy to observe that every minimal EVD set of an isolated edge free graph is also a FEVD set of $G$. Further if $L$ is a minimal EVD set of a connected graph $G \neq K_2$, then $E - L$ is also an EVD set of $G$.

3. MAIN RESULTS

We observe that in the above example for the graph $G, f_{ev}$ set $\cup \gamma_{ev}$ set $= E$. It is not a coincidence!. We show that this happens for every graph.

Proposition 1: Let $G(V, E)$ be any graph. Then for any set $L \subseteq E$,

(i) $L$ is a FEVD set if, and only if, $E - L$ is an EVD set.

(ii) $L$ is a FSEVD (FWEVD) set if, and only if, $E - L$ is a WEVD (SEVD) set.

Proof: (i) Let $L \subseteq E$ be a FEVD set of $G$.

Case 1: $V(L) = V(G)$:

Since $L$ is a FEVD set, by definition for every $v \in V(L) = V(G)$, there exists a $x \in E - L$ such that $x, m$-dominates $v$. Hence $E - L$ is an EVD set of $G$.

Case 2: $V(L) = S \subseteq V(G)$:

Then again by definition for every $v \in S$, there exists an edge $x \in E - L$ such that $x, m$-dominates $v$. If $v \in V - S$, then $v$ is incident on some $x \in E - L$. Thus any $v \in V$ is $m$-dominated by some $x \in E - L$. Therefore $E - L$ is an EVD set of $G$. Converse is trivial. The result (ii) can be proved similarly and hence we omit the proof.

The above Proposition leads us to Gallai’s type results for edges.

Proposition 2. For any $(p, q)$ graph $G$,

$$\gamma_{ev} + f_{ev} = q \quad (1)$$

$$\gamma_{sev} + f_{wev} = q \quad (2)$$

$$\gamma_{wev} + f_{sev} = q \quad (3)$$

Proof: We prove (1) only. Results (2) and (3) can be proved with similar argument. Let $L$ be a $\gamma_{ev}$-set. Then by proposition 2, $E - L$ is a FEVD set.

Therefore $f_{ev} \geq |E - L| = q - \gamma_{ev} \quad \ldots(A)$. On the other hand, if $F$ is a $f_{ev}$ set, then by Proposition 2, $E - F$ is an EVD set. Hence $\gamma_{ev} \leq |E - F| = q - f_{ev} \quad \ldots(B)$

Then (1) follows from (A) and (B).
4. BOUNDS ON $\gamma_{ev}$, $\gamma_{sev}$ AND $\gamma_{wev}$

The following bound appears in [10].

**Theorem A [10]:** For any graph $G$, $\frac{\gamma}{2} \leq \gamma_{ev} \leq \gamma$

In the next Proposition we get a simple but sharp upper bound for $\gamma_{ev}$

**Proposition 3:** For any $(p, q)$ graph $G$, $\gamma_{ev} \leq \min\left\{\left\lfloor \frac{p}{2} \right\rfloor, \left\lfloor \frac{q}{2} \right\rfloor \right\}$.

**Proof:** It is well known that $\gamma \leq \left\lfloor \frac{p}{2} \right\rfloor$. Then from Theorem A, we have $\gamma_{ev} \leq \gamma \leq \left\lfloor \frac{p}{2} \right\rfloor$. From the Fact 2, it follows that if $L$ is a minimal EVD set then $E - L$ is also an EVD set. Therefore $\gamma_{ev} \leq |L|$ and $\gamma_{ev} \leq |E - L| = q - |L|$. Adding these two inequalities we get $\gamma_{ev} \leq \left\lfloor \frac{q}{2} \right\rfloor$.

A spider is a tree on $2n + 1$ vertices obtained from a star $K_{1, n}$ by subdividing each edge of the star. The bound in the above Proposition is sharp for any spider $T$ as $\gamma_{ev}(T) = n = \left\lfloor \frac{2n + 1}{2} \right\rfloor$ and $\gamma_{ev}(T) = n = \left\lfloor \frac{2n}{2} \right\rfloor = \left\lfloor \frac{q}{2} \right\rfloor$.

**Corollary 3.1:** For any $(p, q)$ graph $G$, $f_{ev} \geq \max\left\{\frac{q}{2}, \frac{2q - p}{2} \right\}$.

We try to improve the above bound for certain class of graphs. A graph $G$ is Hamiltonian if there exists a spanning cycle in $G$. A graph $G$ is Semi Hamiltonian if there exists a spanning path in $G$. Every Hamiltonian graph is semi Hamiltonian but not conversely. For example Peterson’s graph is semi Hamiltonian but not Hamiltonian. It is well known that if $H$ is a spanning subgraph of $G$, then $\gamma(G) \leq \gamma(H)$. A similar result holds for EVD sets also.

**Proposition 4:** Let $G$, be a graph and $H$ be an isolate free spanning subgraph of $G$. Then $\gamma_{ev}(G) \leq \gamma_{ev}(H)$.

**Proof:** Let $F$ be a $\gamma_{ev}$-set of $H$. Since $H$ is a spanning subgraph of $G$, $F$ is an EVD-set of $G$. Hence $\gamma_{ev}(G) \leq |F| = \gamma_{ev}(H)$.

**Proposition 5:** Let $G(p, q)$ be a Semi Hamiltonian graph.

Then $\gamma_{ev}(G) \leq \frac{p}{4}$ and $\gamma(G) \leq \frac{p}{3}$.

**Proof:** It is easy to observe that for any path $P_n$ with $n$ vertices, $\gamma_{ev}(P_n) = \frac{p}{4}$ and $\gamma(P_n) = \frac{p}{3}$. Further since $G$ is Semi Hamiltonian, there exists a spanning path $P_{n'}$. Then from Proposition 4, $\gamma_{ev}(G) \leq \gamma_{ev}(P_{n'}) = \frac{p}{4}$. Similarly $\gamma(G) \leq \gamma(P_{n'}) = \frac{p}{3}$.

**Corollary 5.1:** Let $G(p, q)$ be a Semi Hamiltonian graph. Then $f_{ev}(G) \geq \frac{4q - p}{4}$ and $f(G) \geq \frac{3q - p}{3}$.
As every Hamiltonian graph is Semi Hamiltonian, the following corollary is immediate.

**Corollary 5.2:** If $G$ is a Hamiltonian graph, then $\gamma_{ev}(G) \leq \frac{p}{4}$ and $\gamma(G) \leq \frac{p}{3}$.

**Theorem B [9]:** If $G$ is a connected graph of order $p$ with minimum degree 2 and if $G$ is not isomorphic to any of the graphs in Fig. 3, then $\gamma(G) \leq \left\lfloor \frac{2p}{5} \right\rfloor$.

Figure 2

Since $\gamma_{ev}(G) \leq \gamma(G)$ and $\gamma_{ev}(G) \leq \left\lfloor \frac{2p}{5} \right\rfloor$ for these seven graphs and therefore we have

**Proposition 6:** If $G$ is a connected graph of order $p$ with minimum degree 2, then $\gamma_{ev}(G) \leq \left\lfloor \frac{2p}{5} \right\rfloor$ and $f_{ev}(G) \geq \frac{5q - 2p}{5}$.

This bound is sharp is evident from the fact that the bound is attained for $C_3, C_4, C_5, K_4$ and the above seven graphs in Fig. 3.

### 5. DISTANCE $k$ DOMINATING SETS

The concept of distance $k$ domination is introduced by Henning et al., [3]. A set $S \subseteq V$ is a distance $k$ dominating set if every vertex in $V - S$ is at distance $\leq k$ from a vertex in $S$. The distance $k$ domination number $\gamma_k$ is the minimum cardinality of a distance $k$ dominating set of $G$. Shridharan et al., [15] studied in particular the distance 2 domination and found some bounds on $\gamma_2$. Extending the concept of strong weak domination to the $k$ domination we get strong (weak) $k$ domination. A vertex $v$ strongly (weakly) $k$ dominate a vertex $u$ if $d(u, v) \leq k$ and $\deg(u) \geq \deg(v)$ ($\deg(u) \leq \deg(v)$). The strong (weak) distance $k$ domination number $\gamma_{sk}(G)$ ($\gamma_{wk}(G)$) is the minimum cardinality of a distance $k$ dominating set of $G$. In particular $\gamma_{s2}(G)$ ($\gamma_{w2}(G)$) denote the distance 2 strong (weak) domination number of $G$.

**Proposition 7:** For any graph $G$,

$$
\gamma_2 \leq \gamma_{ev} \tag{4}
$$

$$
\gamma_{s2} \leq \gamma_{sev} \tag{5}
$$

$$
\gamma_{w2} \leq \gamma_{wev} \tag{6}
$$

**Proof:** Let $F = \{y_1, y_2, \ldots, y_t\}$ be a $\gamma_{ev}$ set so that $\gamma_{ev} = t$. Let $u_j$ be a vertex incident on the edge $y_j, 1 \leq j \leq t$. Then $S = \{u_1, u_2, \ldots, u_t\}$ of distinct vertices is a distance 2 dominating set. Therefore $\gamma_2 \leq |S| = t = \gamma_{ev}$. Thus result (4) follows. With the similar argument we can prove (5) and (6).
6. EV-DEGREE, SEV-DEGREE AND WEV-DEGREE
The *Edge Vertex degree* of an edge \( x \in E \), EV-deg \((x)\) is the number of vertices \( m \)-dominated by \( x \) or equivalently EV-deg \((x)\) is the number of vertices in \( N[x] \). For any edge \( x \in E \) the *Strong Edge Vertex degree* of \( x \), SEV-deg \((x)\) (Weak Edge Vertex degree WEV-deg \((x)\)) is the number of vertices strongly (weakly) \( m \)-dominated by \( x \). With respect to these degrees we get the following new graph parameters. \( \Delta_{ev}(G) = \max \{ \text{EV-deg}(x) \mid x \in E \} \) and \( \delta_{ev}(G) = \min \{ \text{EV-deg}(x) \mid x \in E \} \).

6.1 SEV Silent Number WEV Silent Number
An Edge \( x \) is called SEV-Silent, if SEV-deg \((x)\) = 0. Similarly \( x \) is called WEV-Silent if WEV-deg \((x)\) = 0. A set \( L \subset E \) is said to be SEV-Silent set if for every \( x \in L \), SEV-deg \((x)\) = 0. The *SEV-Silent number* \( \eta_{sev} = \eta_{sev}(G) \) is the maximum cardinality of a SEV-Silent set. Similarly the *WEV-Silent number* \( \eta_{wev} = \eta_{wev}(G) \) is defined.

6.2 Illustrations

![Figure 3(a)](image1)

![Figure 3(b)](image2)

![Figure 3(c)](image3)

In \( G_1 \) of Fig. 3(a), for every pendant edge \( x \), EV-deg \((x)\) = WEV-deg \((x)\) = 4 and SEV-deg \((x)\) = 0. For every non pendant edge \( x \), EV-deg \((x)\) = SEV-deg \((x)\) = 7 and WEV-deg \((x)\) = 0. Further, \( \Delta_{ev}(G_1) = 7 \), \( \delta_{ev}(G_1) = 4 \). The \( \eta_{sev}(G_1) = 8 \) and \( \eta_{sev-set} \) is the set of all pedant edges in \( G_1 \) while the \( \eta_{wev} = 4 \) and \( \eta_{wev-set} \) is the set of all non pendant edges in \( G_1 \). In Fig. 3(b) and Fig. 3(c) the edge labels represent the SEV-degree and WEV-degree of each edge of \( G_2 \) respectively. \( \Delta_{ev}(G_2) = 6 \), \( \delta_{ev}(G_2) = 4 \). \( \eta_{sev}(G_2) = 6 \) and \( \eta_{sev-set} \) is the set of all edges with label ‘0’ in Fig. 3(b). \( \eta_{wev}(G_2) = 6 \) and \( \eta_{wev-set} \) is the set of all edges with label ‘0’ in Fig. 3(c).

The new parameters so defined yield a very sharp bounds for EVD, SEVD, WEVD numbers.

**Proposition 8:** For any \((p, q)\) graph \( G \),
Further these bounds are sharp.

**Proof:** Since any edge can \( m \)-dominate at most \( \Delta_{ev} \) vertices, and we have to exhaust all the \( p \) vertices, we need at least \( p \frac{p}{\Delta_{ev}} \) edges to \( m \)-dominate all the vertices. This implies the lower bound in (7). Let \( S \) be the \( \eta_{sev} \)-set of \( G \). By definition of \( \eta_{sev} \), all the edges in \( S \) are SEV-Silent. Hence no edge in \( S \) strongly \( m \)-dominate any vertex of \( G \). Hence \( E - S \) is a SEVD set of \( G \). Therefore \( \gamma_{sev} \leq |E - S| = q - \eta_{sev} \). The proof of (8) is similar and hence we omit the proof. The bounds are sharp as the graph \( G_3 \) in Fig. 3 attains the lower and upper bounds in (7) and (8), where as \( G_1 \) attains the upper bound in (7).

Using Proposition 2, in equations (7) and (8) the following bounds are immediate for FEVD, FSEVD and FWEVD numbers.

**Corollary 8.1:** For any \((p, q)\) graph \( G \),

\[
\eta_{sev} \leq f_{sev} \leq f_{es} \leq q - \frac{p}{\Delta_{ev}}
\]

\[
\eta_{wev} \leq f_{wev} \leq q - \frac{p}{\delta_{ev}}
\]

**Proposition 9:** For any \((p, q)\) graph \( G \),

\[
\gamma_{ev}(G) \leq p - \Delta_{ev} + 1
\]

\[
f_{ev}(G) \geq q - (p - \Delta_{ev} + 1)
\]

Further these bounds are sharp.

**Proof:** Let \( x \) be an edge of max EV degree \( \Delta_{ev} \). Let \( D \) be the set of vertices that are \( m \)-dominated by the edge \( x \), so that \( |D| = \Delta_{ev} \). Let \( V - D = \{u_1, u_2, \ldots, u_k\} \). Choose an edge \( y_i, 1 \leq i \leq k \) such that \( y_i \) is incident on the vertex \( u_i \). Then the set \( L = \{y_1, y_2, \ldots, y_k\} \) \( m \)-dominates all the vertices in \( V - D \). Hence \( L \cup \{x\} \) is an EVD set of \( G \). Therefore \( \gamma_{ev} \leq |L \cup \{x\}| = |L| + 1 = |V - D| + 1 = p - \Delta_{ev} + 1 \) as desired. Now (10) follows from proposition 2. The bounds in (9) and (10) are sharp and is attained for the graphs \( K_p, K_{m,n}, C_4 \) and \( C_5 \).

### 7. SOME BOUNDS ON TREES

We now find some bounds on \( \gamma_{ev}, \gamma_{sev}, \gamma_{wev} \) when \( G \) is a Tree. In a tree a support edge is an edge incident to a pendant edge. Let \( t, t_i \) and \( s \) be the number of pendant edges, independent
pendant edges and support edges in \( T \) respectively. An edge \( x \) is said to be strong if \( \text{deg}(x) \geq \text{deg}(y) \) for every \( y \in N(x) \). Let \( S \) be the set of all strong edges and \( m \) be maximum number of independent edges in \( S \).

**Proposition 10:** Let \( T(V, E) \) be any tree with \( p \geq 4 \) vertices and \( T \neq K_{1, p-1} \).

Then,
\[
\begin{align*}
\gamma_{ev} & \leq \gamma_{sev} \leq q-t = \gamma_c-1 \leq p-(\Delta+1) \quad (11) \\
t_i & \leq \gamma_{wev} \leq q-(m+t-t_i) \quad (12)
\end{align*}
\]

**Proof:** In a tree there always exists a minimum EVD set containing all support edges. Hence \( s \leq \gamma_{ev} \). Since the set of non-pendant edges form a SEVD set we have \( \gamma_{sev} \leq q-t \). As \( \gamma_c = p-t \) (see [14]) and \( q = p-1 \) we have \( q-t = \gamma_c-1 \). Further, since \( t \geq \Delta \) we have \( q-t \leq p-(\Delta+1) \). We already have \( \gamma_{ev} \leq \gamma_{sev} \). Hence (11) follows. Let \( L \) be the set of all pendant edges and \( L_i \) be the set of all independent pendant edges of \( T \). Let \( S \) be the set of all strong edges and \( S_i \) be a maximum independent set of edges in \( S \). Then \( E - (S_i \cup (L-L_i)) \) is a WEVD set of \( T \). Hence \( \gamma_{wev} \leq |E - (S_i \cup (L-L_i))| = q-(m+t-t_i) \). Since there always exists a minimum WEVD set containing the set \( L_i \) we have \( t_i \leq \gamma_{wev} \). Thus (12) holds.

The bounds in the above proposition is sharp as the tree in the Fig. 3(a) attains the upper and lower bounds in (11) and lower bound in (12). \( P_4 \) and \( P_6 \) attain the upper bound in (12).

In what follows we use the following notations, \( \gamma_{ev} = \gamma_{ev}(T) \) and \( \Delta_e = \Delta_e(T) = \text{maximum edge degree of } T \) and \( \delta_e = \delta_e(T) = \text{minimum edge degree of } T \).

**Proposition 11:** Let \( T(V, E) \) be any tree with \( p \geq 4 \) vertices, \( T \neq K_{1, p-1} \).

Then \( \gamma_{ev} = \gamma_{sev} = 1 \) and \( \gamma_{wev} \leq 2 \).

**Proof:** If \( u \) and \( v \) are two pendant vertices adjacent to different supports, then for the edge \( x = uv \) we have \( N(u) \cup N(v) = V \) in \( T \). Hence \( \{x\} \) is the EVD set of \( T \). Further since \( u \) and \( v \) are pendant vertices in \( T \), \( \text{deg}(u) = \text{deg}(v) = p-2 \) in \( T \). Hence \( \text{deg}(x) = 2p-4 = \Delta_e(T) \). Hence \( \{x\} \) is a SEVD set of \( T \). Thus \( \gamma_{ev} = \gamma_{sev} = 1 \).

To show that \( \gamma_{wev} \leq 2 \). Let \( u \) be a maximum degree vertex and \( v \) be a next maximum degree vertex in \( T \). Then we have the following two possibilities.

**Case 1:** \( u \) and \( v \) are not adjacent in \( T \). Then \( x = uv \) is an edge in \( \overline{T} \) and since, \( u \) is a minimum degree vertex and \( v \) is next minimum degree vertex in \( \overline{T} \), the \( \text{deg}(x) = \delta_e \).

**Sub Case 1.1:** \( N(u) \cap N(v) = \emptyset \) in \( T \). Then \( N(u) \cup N(v) = V \) in and \( \text{deg}(x) = \delta_e \) together implies that set \( \{x\} \) is a WEVD set of and \( \overline{T} \) hence \( \gamma_{wev} = 1 \).

**Sub Case 1.2:** \( N(u) \cap N(v) \neq \emptyset \) in \( T \). Let \( N(u) \cap N(v) = S \).
Claim: $|S| = 1$. Suppose $|S| > 1$ and let $S = \{a, b\}$. Then $u$ and $v$ are adjacent to both $a$ and $b$. But then $(a, u, v, b, u)$ is a cycle in $T$ a contradiction and hence our claim. Therefore let $S = \{a\}$. Then $N(u) \cup N(v) = V - \{a\}$. Select an edge $y$, of $\tilde{T}$, such that $y$, weakly $m$-dominate the vertex $a$. Hence $\{x, y\}$ is a $\gamma_{vev}$ set of $\tilde{T}$ and $\gamma_{wev} = 2$.

Case 2: $u$ and $v$ are adjacent in $T$. Then $uv$ is not an edge in $\tilde{T}$. Hence $N(u) \cup N(v) = V - \{u, v\}$. In this case choose two vertices $a$ and $b$ such that $x = ua$ and $y = vb$ are edges in $\tilde{T}$ and $\deg(u) + \deg(a)$ is maximum and $\deg(v) + \deg(b)$ is next maximum in $T$, so that as in Sub case 1.2, we have $\{x, y\}$ is a $\gamma_{wev}$ set of $\tilde{T}$ and $\gamma_{wev} = 2$.

Thus in any case $\gamma_{wev} \leq 2$ holds.

We now characterize the trees for which $\gamma_{wev} = 1$ and $\gamma_{wev} = 2$.

Proposition 12: Let $T(V, E)$ be any tree with $p \geq 4$ vertices, $T \neq K_1, p - 1$. Let $u$ be a maximum degree vertex and $v$ be a next maximum degree vertex in $T$. Then, $\gamma_{wev} = 1$ if and only if $N[u] \cap N[v] = \phi$.

Proof: If $N[u] \cap N[v] = \phi$, then $u$ and $v$ are not adjacent and $N(u) \cap N(v) = \phi$. Then by Sub case 1.1 of Proposition 11, $\gamma_{wev} = 1$. Conversely, let $\gamma_{wev} = 1$. Suppose $N[u] \cap N[v] \neq \phi$. Then either $u$ and $v$ are adjacent or $N(u) \cap N(v) \neq \phi$. Then by sub Case 1.2 and Case 2 of Proposition 11, $\gamma_{wev} = 2$, a contradiction.

Corollary 12.1: $\gamma_{wev} = 2$ if and only if $N[u] \cap N[v] \neq \phi$.

We close this section by giving a Nordhaus Guddum type results for Trees.

Proposition 13: Let $T(V, E)$ be any tree with $p \geq 4$ vertices, $T \neq K_1, p - 1$. Then

\[
2 \leq \gamma_{ev} + \gamma_{ev} \leq \gamma_{sev} + \gamma_{sev} \leq p - 2 \tag{13}
\]

\[
3 \leq \gamma_{wev} + \gamma_{wev} \leq p \tag{14}
\]

\[
(\gamma_{ev})(\gamma_{ev}) \leq * \tag{15}
\]

\[
(\gamma_{wev})(\gamma_{wev}) \leq 2p - 4 \tag{16}
\]

Further these bounds are sharp.

Proof: Since any tree $T$ has at least one support edge and $\gamma_{ev} = 1$ the lower bound in (13) follows from (11). Again since any tree has at least two pendant edges, $q(T) = p - 1$ and $\gamma_{sev} = 1$ the upper bound in (13) follows from (11). Further any tree with $p \geq 4$ has at least two independent pendant edges, $\gamma_{wev}$ is at least 1 the lower bound in (14) follows from (12). Again since $m \geq 1$, we have $\gamma_{wev} \leq p - 2$ from (12) and $\gamma_{wev} \leq 2$ implies the upper bound in (14) and (16). From Proposition 3, $\gamma_{ev} \leq \frac{p}{2}$ and $\gamma_{ev} = 1$. Hence (15) follows. $P_4$ attains the lower bound in (13) and upper bound in (14) and (16). $P_5$ attains the upper bound in (13).
Let $K_{1,r}$ and $K_{1,s}$ be any two star graphs. Join a pendant vertex of $K_{1,r}$ with a pendant vertex of $K_{1,s}$. For the new tree $T$ so obtained $\gamma_{tw} = 2$ and $\gamma_{tw} = 1$. Hence such a tree attains the lower bound in (14). Any spider satisfies the upper bound in (15). Hence the bounds in the Proposition are sharp.

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