DOMINATION IN DEGREE SPLITTING GRAPHS

B. BASAVANAGOUD\textsuperscript{1*}, PRASHANT V. PATIL\textsuperscript{2} AND SUNILKUMAR M. HOSAMANI\textsuperscript{3}

Abstract

Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup ..., \cup S_t \cup T$ where each $S_i$ is a set of vertices having at least two vertices and having the same degree and $T = V - \bigcup S_i$. The degree splitting graph of $G$ is denoted by $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, ..., w_t$ and joining $w_i$ to each vertex of $S_i$ $(1 \leq i \leq t)$. Let the vertices and the edges of a graph $G$ are called the elements of $G$. In this paper, we study the variation in domination from the graph $G$ to the degree splitting graph $DS(G)$. Also we establish many bounds on $\gamma(DS(G))$ in terms of elements of $G$ but not in terms of elements of $DS(G)$.

2000 Mathematics Subject Classification: 05C69.

Keywords: Degree splitting graph, domination number, domatic number.

1. INTRODUCTION

The graphs considered here are finite, undirected without loops or multiple edges. Let $G = (V, E)$ be a graph and the vertices and edges be called the elements of $G$. Any undefined term in this paper may be found in Harary [2].

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set of $G$ if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number ($\gamma(G)$) of $G$ is the minimum cardinality of a minimal dominating set of $G$. For an early survey on domination refer [1].

In [4], R. Ponaraj and S. Somasundaram have initiated a study of degree splitting graph $DS(G)$ of a $(p, q)$ graph which is stated as follows.

Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup ..., \cup S_t \cup T$ where each $S_i$ is a set of vertices having at least two vertices and having the same degree and $T = V - \bigcup S_i$.

The degree splitting graph of $G$ is denoted by $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, ..., w_t$ and joining $w_i$ to each vertex of $S_i$ $(1 \leq i \leq t)$.

\textsuperscript{1,2,3} Department of Mathematics, Karnatak University, Dharwad-580 003
\textsuperscript{*} Corresponding Author E-mail: b.basavanagoud@gmail.com
In this paper, we study the variation in domination from the graph $G$ to the degree splitting graph $DS(G)$. Also we establish, many bounds on $\gamma(DS(G))$ in terms of elements of $G$ but not in terms of elements of $DS(G)$.

In Figure 1, a graph $G$ and its degree splitting graph $DS(G)$ are shown.

![Figure 1:](image)

Here $S_1 = \{1, 7\}$, $S_2 = \{3, 4\}$, $S_3 = \{2, 5, 6\}$ and $T = \emptyset$

2. RESULTS

First we calculate the domination number of $DS(G)$ for some standard class of graphs.

**Proposition 1:** For any path $G = P_p$, $p \geq 3$ then $\gamma(DS(G)) = 2$.

**Proposition 2:** For any wheel $G = W_p$, $p \geq 5$ then $\gamma(DS(G)) = 2$.

**Proposition 3:** If $G$ is a regular graph, then $\gamma(DS(G)) = 1$. 
**Proof:** Let $G$ be any regular graph. Then $DS(G) = G + K_1$ and $\Delta (DS(G)) = p - 1$. Hence $\gamma(DS(G)) = 1$.

The next result gives the upper bound for $\gamma(DS(G))$.

**Theorem 2.1:** For any graph $G$, $\gamma(DS(G)) \leq |w_i \cup T|$, where $w_i; 1 \leq i \leq t$ and $T$ are as defined in the definition of $DS(G)$.

**Proof:** Let $G$ be any graph. By the definition of $DS(G)$, we have $V(DS(G)) = S_1 \cup S_2 \cup ... \cup S_t \cup T$ and $w_i$ be the set of vertices of all the corresponding sets of $S_i; 1 \leq i \leq t$ in $DS(G)$. To prove the above inequality we consider the following cases.

**Case 1:** Let $T = \emptyset$. Since each $w_i; 1 \leq i \leq t$ is independent in $DS(G)$. If $T = \emptyset$ then clearly $w_i$ will be a maximal independent set in $DS(G)$. Since every maximal independent set is a minimal dominating set. Therefore, $\gamma(DS(G)) = |w_i|$.

**Case 2:** Let $T \neq \emptyset$. Then there exist at least one vertex in $G$ which is not present in $S_i; 1 \leq i \leq t$. Since $G$ is an induced subgraph of $DS(G)$, to dominate all the vertices of $DS(G)$ we need at least $D \leq |w_i \cup T|$ vertices. Hence $\gamma(DS(G)) \leq |D| \leq |w_i \cup T|$.

**Theorem 2.2:** For any graph $G$, $\gamma(DS(G)) \leq \left\lceil \frac{p}{2} \right\rceil$.

**Proof:** Let $G$ be any graph of order $p$. To prove the result we consider the following cases.

**Case 1:** If $T = \emptyset$, $G$ has atmost $S_i \leq \frac{p}{2}$. Hence $w_i \leq \frac{p}{2}$. Therefore by Theorem 2.1, we get

$$\gamma(DS(G)) \leq |w_i| \leq \frac{p}{2} \leq \left\lceil \frac{p}{2} \right\rceil.$$

**Case 2:** If $T \neq \emptyset$, Then $G$ has atmost $S_i \leq \frac{p}{2} - T$. Hence $w_i \leq \frac{p}{2} - T$. Therefore by Theorem 2.1, we have

$$\gamma(DS(G)) \leq |w_i + T| \leq \left\lceil \frac{p}{2} - T + T \right\rceil = \frac{p}{2} \leq \left\lceil \frac{p}{2} \right\rceil.$$
In the next theorem, we characterize the graphs whose degree splitting graph have domination number equal to the half of the order of $G$.

**Theorem 2.3:** Let $G$ be any nontrivial connected graph $G$ of even order $p$.

Then $\gamma(\text{DS}(G)) = \frac{p}{2}$ if and only if $T = \emptyset$ and cardinality of each $S_i$; $1 \leq i \leq t$ is even.

**Proof:** Let $G$ be any nontrivial connected graph $G$ of even order $p$ with $T = \emptyset$ and each $|S_i|$; $1 \leq i \leq t$ is even. Then by using Theorem 2.1 and Theorem 2.2, we get

$$\gamma(\text{DS}(G)) = |w_i|$$

$$= \left\lfloor \frac{p}{2} \right\rfloor$$

$$= \frac{p}{2} (\because p \text{ is even})$$

Conversely, let $G$ be any nontrivial connected graph with $\gamma(\text{DS}(G)) = \frac{p}{2}$ and $T \neq \emptyset$. Then by Theorem 2.1, $\gamma(\text{DS}(G)) \leq |w_i \cup T|$ and by Theorem 2.2, we get

$$\gamma(\text{DS}(G)) \leq \left\lfloor \frac{p}{2} \right\rfloor.$$

This implies that $V(G)$ is odd and a contradiction to our assumption. Hence $G$ must be a graph of even order with $T = \emptyset$ and $|S|$ is even.

**Theorem 2.4:** Let $G$ be any nontrivial tree $T$. Then $\gamma(T) \leq \gamma(\text{DS}(T))$

**Proof:** Let $G$ be any nontrivial tree $T$. Let $D = \{v_1, v_2, ..., v_t\}; 1 \leq i \leq p$ be the minimum dominating set of $T$. Then $\gamma(T) = |D|$. Let $S_i$ be the set of all equal degree vertices in a tree $T$ and $H = V - \cup S_i$. Therefore $D' = w_i \cup (H \cap D)$ is a dominating set of $\text{DS}(G)$.

Hence

$$(\text{DS}(T)) \leq |D'|$$

$$= |w_i \cup (H \cap D)|$$

$$\leq |D|$$

$$= \gamma(T)$$

Next theorem gives the relation between independence number of $G$ and the domination number of $\text{DS}(G)$.

**Theorem 2.5:** For any graph $G$, $\gamma(\text{DS}(G)) \leq \beta_0(G)$. Further equality holds for $G = K_p$. 


**Proof:** For any graph $G$, we have $\beta_0(G) \leq p$. Also by Theorem 2.2, $\gamma(DS(G)) \leq \left\lceil \frac{p}{2} \right\rceil$. Therefore $\gamma(DS(G)) \leq \beta_0(G)$.

**Theorem A [3].** For any graph $G$, $p - q \leq \gamma(G)$ Furthermore, $\gamma(G) = p - q$ if and only if each component of $G$ is a star.

**Theorem 2.6:** Let $G = (p, q)$ be any nontrivial connected graph with maximum degree $\Delta$, then $p - q + 1 \leq \gamma(DS(G)) \leq p - \Delta(G) + |w_i|$. Further, the lower bound is attained if and only if each component of $G$ is a star and the upper bound is attained for $G = K_p$.

**Proof:** First consider the lower bound.

Let $q = p - 1$, then $G$ is a tree. Therefore the lower bound follows from the fact that for any nontrivial tree $\gamma(DS(G)) \geq 2$.

For equality, suppose each component of a graph $G$ is a star then the result can be easily verified.

Conversely, let $p - q + 1 = \gamma(DS(G))$. Then $G$ has exactly $p - q$ components with domination number equal to one. Thus $G$ is a forest. Since $p - q + 1 = \gamma(DS(G))$, therefore by Theorem A, each component of a graph $G$ is a star.

Now consider the upper bound.

Let $v$ be a maximum degree vertex in $G$. Then $v = p - \Delta(G)$. Since $G$ is a subgraph of $DS(G)$ and each $w_i$ is independent in $DS(G)$. Therefore $\gamma(DS(G)) \leq |w_i \cup \{v\}| \leq p - \Delta(G) + |w_i|$.

Suppose $G = K_p$, then the equality can be easily verified.

**Proposition 4:** Let $G$ be any $(p, q)$ graph having a vertex of $\Delta(G) = p - 1$. Then $G = H + K_i$ and $DS(G)$ has the following properties.

(i) If $H$ is a complete graph, then $\gamma(DS(G))$ sets are precisely $\{w_1\}, \{v_1\}, \{v_2\}, \ldots, \{v_p\}$

(ii) If $H$ is a block (which is not complete) then there exist an unique $\gamma(DS(G))$ - set $\{v_i, w_i\}; 1 \leq i \leq t$ in $DS(G)$.

By observing all these results we raise one open problem which is stated as follows.

**Open Problem:** Characterize the graphs for which $\gamma(G) = \gamma(DS(G))$.

A partial solution to the above problem is as follows.

**Theorem 2.7:** For any graph $G$, $\gamma(G) = \gamma(DS(G))$ if $G$ is one of the following graphs $(i) G = K_p$ $(ii) G = K_{m,n}$ $(iii)$ bistar or tristar.

**Proof:** If $G$ is any one of the graph mentioned in the statement of the theorem, then one can easily verify the result.
3. DOMATIC NUMBER OF DS(G)

Let $G$ be a graph. A partition $\Delta$ of its vertex set $V(G)$ is called a domatic partition of $G$ if each class of $\Delta$ is a dominating set in $G$. The maximum number of classes of a domatic partition of $G$ is called the domatic number of $G$ and it is denoted by $d(G)$. The domatic number was introduced by Cockayne and Hedetniemi [1].

**Theorem 3.1:** For any graph $G$, $d(DS(G)) \leq \delta(G) + 2$.

**Proof:** For any graph $G$, $d(G) \leq \delta(G) + 1$, equality holds if and only if $G = K_1$. If $G = K_1$ then $DS(G) = K_{p+1}$. Hence the theorem.

**Theorem 3.2:** For any graph $G$ having $p$ vertices, $d(DS(G)) + d(DS(\overline{G})) \leq p + 3$.

**Proof:** Let $G$ be any graph having $p$ vertices. Then by Theorem 3.1,

$$d(DS(G)) \leq \delta(G) + 2 \text{ and } d(DS(\overline{G})) \leq \delta(\overline{G}) + 2 \leq \Delta(\overline{G}) + 2.$$  
Therefore,

$$d(DS(G)) + d(DS(\overline{G})) \leq \delta(G) + \Delta(\overline{G}) + 4 \leq p - 1 + 4 = p + 3.$$  

**Theorem 3.3:** Let $G$ be any graph having $p$ vertices. Then $d(DS(G)) + d(DS(\overline{G})) = p + 3$ if and only if $G = K_p$ or $\overline{K}_p$.

**Proof:** Let $d(DS(G)) + d(DS(\overline{G})) = p + 3$.

We proceed by induction on $p$. Suppose, contrary to the assertion that $G \neq K_p$ or $\overline{K}_p$ has $p$ vertices and $d(DS(G)) + d(DS(\overline{G})) = p + 3$. We consider the following cases.

**Case 1:** In $G$ if has an unique vertex of $\delta(G) = 0$ then $\delta(G) = \delta(DS(G))$ and $d(DS(G)) = 1$, and $(\overline{G})$ has a vertex of degree $p - 1$, which is unique, then $\Delta(\overline{G}) = p - 1$.

Therefore,

$$d(DS(G)) + d(DS(\overline{G})) = \delta(G) + 2 + \delta(\overline{G}) + 2 = 2 + (p - 2) + 2 \leq p + 2 < p + 3.$$
Case 2: \[ 0 < \delta(G) < \frac{p}{2} + 1 \] (3.1)

By assumption, \( d(DS(G)) + d(DS(\overline{G})) = p + 3 \).

Therefore, \( d(DS(\overline{G})) = p + 3 - d(DS(G)) \)

\[ = p + 3 - (\delta(G) + 2) \text{ (By Theorem 3.1)} \]
\[ = p - \delta(G) + 1 \] (3.2)

If all dominating sets in a maximum \( D \)-partition of \( DS(\overline{G}) \) have at least two vertices.

Then \[ p \geq 2d(DS(\overline{G})) \]
\[ \geq 2(p - \delta(G) + 1) \text{ (using 3.2)} \]
\[ = p + 3, \text{ a contradiction.} \]

Hence some vertex \( v \) dominates \( \overline{G} \) and \( \{v \cup w_i\} \) dominates \( DS(\overline{G}) \),

Therefore \( deg_{DS(G)}(v) = p - |w_i|, \text{ } deg_{\overline{G}}(v) = p - 1 \text{ and } deg_{G}(v) = 0. \)

Hence \( \delta(G) = 0 = \delta(DS(G)), \text{ contradiction to (3.1)}. \)

If \( \delta(DS(G)) = \delta(G) = 0, \) apply Case 1, to \( DS(G) \) otherwise apply Case 2, to \( DS(G). \)

**Theorem 3.4:** For any graph \( G \) with \( p \) vertices, \( d(DS(G)) + \gamma(DS(G)) \leq p + 2 \), equality holds if and only if \( G = K_p \).

**Proof:** If \( G = K_p \), trivially \( d(DS(G)) + \gamma(DS(G)) \leq p + 2 \). Therefore, let \( G \) have \( p \) vertices and \( G \neq K_p \). From Theorem 2.6, we have \( \gamma(D_s(G)) \leq p - \Delta(G) + |w_i| \), hence \( \gamma(DS(G)) \leq p - \delta(G) + |w_i| \).

We claim that this inequality, \( d(DS(G)) \leq \delta(G) + 2 \), is strict. Suppose not then,

\[ \gamma(DS(G)) = p - \delta(G) \]
\[ = p - \delta(DS(G)) + 2 \]

Using the fact that \( d(DS(G)) \leq \frac{p + 1}{\gamma(DS(G))} \),

\[ d(DS(G)) \leq \frac{p + 1}{p - d(DS(G)) + 2} \]

i.e., \( d(DS(G)) \cdot (p - d(DS(G)) + 2) \leq p + 1 \) from which on solving this equation we get, \( d(DS(G)) = p + 1 \) and \( d(DS(G)) = 1 \).
In the previous case \( G = K_p \) while \( d(G) = 1 \) implies \( G \) has an isolated vertex \( v \). Since \( G - v \neq K_{p-1} \), \( \gamma(DS(G)) > 1 \).

In this case \( (G) = 0 \), \( w_i = p - 1 \) and \( T = \{ v \} \).

Hence the inequality, \( \gamma(DS(G)) \leq p - \delta(G) + |w_i| \) is strict.

Thus one of the inequalities \( \gamma(DS(G)) \leq p - \delta(G) + |w_i| \) or \( d(DS(G)) \leq d(G) + 2 \) is strict.

Hence \( d(DS(G)) + \gamma(DS(G)) < p + 2 \) as asserted.

Nordhaus-Gaddum type results for \( DS(G) \)

**Theorem 3.5:** Let \( G \) be any graph having \( p \) vertices, then

\[
(i) \quad \gamma(DS(G)) + \gamma(DS(\overline{G})) \leq \left\lfloor \frac{p}{2} \right\rfloor
\]

\[
(ii) \quad \gamma(DS(G)) \cdot \gamma(DS(\overline{G})) \leq \left\lfloor \frac{p}{2} \right\rfloor^2
\]

**ACKNOWLEDGEMENT**

This research was supported by UGC-SAP DRS-II New Delhi, India: for 2010-2015.

**REFERENCES**


