FINANCIAL DECISION MAKING WITH THE FUZZY GENERALIZED PROBABILISTIC WEIGHTED AVERAGING OPERATOR

MONTSERRAT CASANOVAS* & JOSÉ M. MERIGÓ**
Department of Business Administration, University of Barcelona, Av. Diagonal 690, 08034, Barcelona, Spain, E-mails: mcasanovas@ub.edu*, jmerigo@ub.edu**

ABSTRACT
We present a new financial decision making model by using the fuzzy generalized probabilistic weighted averaging (FGPWA) operator. The main advantage of this new approach is that it is able to deal with probabilities (objective information) and weighted averages (subjective information) in the same formulation. Moreover, it is also able to deal with an uncertain environment that can be assessed with fuzzy numbers. Furthermore, it uses generalized and quasi-arithmetic means providing a more robust formulation of the model. We study the applicability of the new approach on a financial decision making problem concerning the selection of financial strategies.

JEL Classification: C44, C49, D81, D89.

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1. INTRODUCTION
The weighted average (WA) is one of the most common aggregation operators found in the literature. Another interesting concept that can be used as an aggregation operator is the probability. These two concepts have been used in a lot of applications concerning statistics, economics and engineering. Probably, these two concepts are the most relevant in statistics. However, there are a lot of other aggregation operators such as the ordered weighted averaging (OWA) operator and others (Beliakov, 2005; Beliakov et al., 2007; Calvo et al., 2002; Fodor et al., 1995; Karayiannis, 2000; Kaufmann and Gil-Aluja, 1987; Merigó, 2008; 2009a; 2009b; 2009c; 2009d; Merigó and Casanovas, 2008; 2009a; 2009b; Merigó and Gil-Lafuente, 2009; Torra, 1997; Torra and Narukawa, 2007; Xu, 2007; Xu and Da, 2003; Yager, 1988; 1993; 2002; 2004; Yager and Kacprzyk, 1997). A very interesting type of generalization can be obtained by using generalized and quasi-arithmetic means. Then, we get the generalized weighted average (GWA) and the generalized probabilistic aggregation operator.
Usually, we assume that the available information are exact (or precise) numbers. However, this may not be the real situation found in the decision making problem. Sometimes, the available information is vague or imprecise and it is not possible to analyze it with exact numbers. Thus, it is necessary to use another approach that is able to assess the uncertainty such as the use of fuzzy numbers (FNs). With the use of FNs, we are able to analyze the best and worst possible scenario and the possibility that the internal values between them will occur. FNs (Chang and Zadeh, 1972; Dubois and Prade, 1980; Gil-Aluja, 1998; Gil-Lafuente, 2005; Kaufmann and Gupta, 1985; Kaufmann and Gil-Aluja, 1987; Merigó, 2008; Xu and Yager, 2008; Zadeh, 1975) appeared as an extension of the fuzzy sets (Zadeh, 1965). Since their introduction, they have been used in an astonishingly wide range of applications in a lot of sciences such as mathematics, statistics, economics and engineering.

Recently, Merigó has suggested a new model that unifies the weighted average with the probability (Merigó, 2009a). He called it the probabilistic weighted averaging (PWA) operator. The main advantage of the PWA is that it is able to unify the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation. Thus, each case can be seen as a particular case of this general framework. Moreover, he has presented a further generalization by using generalized and quasi-arithmetic means, the generalized PWA (GPWA) operator (Merigó, 2009b).

In this paper, we present a new approach that is able to deal with the GPWA in an uncertain environment where the information can not be represented with the usual exact numbers but it is possible to use FNs. For doing so, we introduce the fuzzy generalized PWA (FGPWA) operator. It is a new aggregation operator that unifies the weighted average and the probability in the same formulation and considering the degree of importance that each concept has in the aggregation. Moreover, it uses generalized means providing a more robust formulation that considers other situations rather than the arithmetic case. Furthermore, it also uses FNs providing a more complete representation of the aggregation process because we can consider the best and worst possible results and the possibility that the internal values will occur. One of the main advantages of this new model is that it includes a wide range of particular cases such as the GPWA, the fuzzy PWA (FPWA), the fuzzy geometric PWA (FPWGA) operator, the fuzzy quadratic PWA (FPWQA) operator, the fuzzy harmonic PWA (FPWHA) operator, the fuzzy generalized probabilistic aggregation and the fuzzy generalized weighted average (FGWA) operator.

We further generalize the FGPWA operator by using quasi-arithmetic means, obtaining the quasi-arithmetic FPWA (Quasi-FPWA) operator. This approach is much more general and thus more robust because it includes the FGPWA operator as a particular case and a lot of other cases.

We study the applicability of the FGPWA and we see that it is extremely broad because all the studies that use the WA or the probability can be revised and extended
with this new approach. For example, we could use it in statistics, in economics, in engineering and in decision theory. In this paper, we focus on a financial decision making problem regarding the selection of financial strategies. We analyze a company that wants to invest some money in a country and it is looking for the optimal investment. We analyze how to use different types of aggregation operators included in the FGPWA and we see that depending on the aggregation operator used, the results may lead to different decisions.

This paper is organized as follows. In Section 2, we briefly comment some basic concepts about the FNs, the weighted aggregation operators, the probabilistic aggregation operators and the generalized aggregation operators. Section 3 presents the FGPWA operator. Section 4 analyzes some of its main particular cases and Section 5 develops a further generalization by using quasi-arithmetic means. Section 6 presents an application in a financial decision making problem.

2. PRELIMINARIES

In this Section, we briefly describe the FNs, the weighted aggregation functions, the probabilistic aggregation functions, the generalized aggregation functions and the FPWA operator.

2.1 Fuzzy Numbers

The FN was introduced by Chang and Zadeh (1972) and Zadeh (1975). Since its creation, it has been studied and applied by a lot of authors (Dubois and Prade, 1980; Kaufmann and Gupta, 1985; Kaufman and Gil-Aluja, 1987; Merigó, 2008).

A FN $A$ is defined as a fuzzy subset of a universe of discourse that is both convex (i.e., $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$; for $\forall x_1, x_2 \in R$ and $\lambda \in [0, 1]$) and normal (i.e., sup$_{x \in R} \mu_A(x) = 1$).

In the literature, we find a wide range of FNs (Merigó, 2008) including the usual triangular FNs (TFNs), trapezoidal FNs (TpFNs), L-R FNs, interval-valued FNs (IVFNs), interval-valued intuitionistic FNs (IVIFNs) and interval-valued intuitionistic generalized FNs (IVIGFNs).

For example, a TpFN $A$ of a universe of discourse $R$ can be characterized by a trapezoidal membership function ($\alpha$-cut representation) $A = (\tilde{a}, \tilde{a})$ such that

$$\tilde{a}(\alpha) = a_1 + \alpha(a_2 - a_1),$$

$$\tilde{a}(\alpha) = a_4 - \alpha(a_4 - a_3).$$

(1)

where $\alpha \in [0, 1]$ and parameterized by $(a_1, a_2, a_3, a_4)$ where $a_1 \leq a_2 \leq a_3 \leq a_4$, are real values. Note that if $a_1 = a_2 = a_3 = a_4$, then the FN is a crisp value and if $a_2 = a_3$, the FN is represented by a TFN. Note that the TFN can be parameterized by $(a_1, a_2, a_4)$. 

The TpFN can also be represented in the following way:

\[
m(t) = \begin{cases} 
1 & \text{if } t \in [a_2, a_3], \\
\frac{t - a_1}{a_2 - a_1} & \text{if } t \in [a_1, a_2], \\
\frac{a_4 - t}{a_4 - a_3} & \text{if } t \in [a_3, a_4], \\
0 & \text{otherwise},
\end{cases}
\]  

(2)

where \(a_1, a_2, a_3, a_4 \in R\) and \(a_1 \leq a_2 \leq a_3 \leq a_4\). Note that in this paper, especially when developing the illustrative example, we denote the TpFN as \((a_1, a_2, a_3, a_4)\). Furthermore, we denote all the FNs in a general way as \(\tilde{a}\). Thus, by providing this abbreviation we are able to represent all the FNs in the same formulation.

In the following, we are going to review some basic FN arithmetic operations as follows. Let \(A\) and \(B\) be two TFNs, where \(A = (a_1, a_2, a_3)\) and \(B = (b_1, b_2, b_3)\). Then:

1. \(A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)\).
2. \(A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)\).
3. \(A \times k = (k \times a_1, k \times a_2, k \times a_3); \text{ for } k > 0\).

Among the wide range of methods existing in the literature for ranking FNs, for simplicity, we recommend the use of the methods commented by Merigó (2008) such as the use of the value found in the highest membership level (\(a = 1\)) and if it is an interval, the average of the interval.

Note that other operations and ranking methods could be studied but in this paper we focus on these ones. For a more complete overview about FNs, see for example (Dubois and Prade, 1980; Kaufmann and Gupta, 1985; Kaufmann and Gil-Aluja, 1987; Merigó, 2008).

2.2 Weighted Aggregation Functions

Weighted aggregation functions are those functions that weight the aggregation process by using the weighted average (WA). The weighted average can be defined as follows.

**Definition 1**: A WA operator of dimension \(n\) is a mapping \(WA: R^n \rightarrow R\) that has an associated weighting vector \(W\), with \(w_i \in [0, 1]\) and \(\sum_{i=1}^{n} w_i = 1\), such that

\[
WA (a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i
\]  

(3)

where \(a_i\) represents the \(i^{th}\) argument variable.

Other extensions of the weighted average are those that use it with the OWA operator such as the WOWA operator (Torra, 1997; Torra and Narukawa, 2007) and
the hybrid averaging (HA) operator (Xu and Da, 2003). Recently, Merigó (2009c)
suggested another approach called the OWA weighted average (OWAWA) operator.
Its main advantage against the WOWA and the HA is that it includes the OWA and
the WA considering the degree of importance that each concept have in the
aggregation. It can be defined as follows.

**Definition 2:** An OWAWA operator of dimension \( n \) is a mapping \( \text{OWAWA} : \mathbb{R}^n \rightarrow \mathbb{R} \)
that has an associated weighting vector \( W \) of dimension \( n \) such that \( w_j \in [0, 1] \) and
\( \sum_{j=1}^{n} w_j = 1 \), according to the following formula:

\[
\text{OWAWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j
\]

where \( b_j \) is the \( j^{th} \) largest of the \( a_i \), each argument \( a_i \) has an associated weight (WA) \( v_i \)
with \( \sum_{i=1}^{n} v_i = 1 \) and \( v_i \in [0, 1] \), \( \hat{v}_j = \beta v_j + (1 - \beta) v_l \) with \( \beta \in [0, 1] \) and \( v_l \) is the weight (WA) \( v_i \) ordered according to \( b_j \) that is, according to the \( j^{th} \) largest of the \( a_i \).

Note that other approaches for unifying the OWA and the WA are possible as it
was suggested by Merigó (2008) such as a similar approach than the immediate
probability. Thus, in the WA we get the immediate weighted OWA (IWOWA)
operator that could be defined, for example, by using \( \hat{v}_j = (w_j v / \sum_{j=1}^{n} w_j v) \) or by using
\( \hat{v}_j = [w_j + v] / \sum_{j=1}^{n} (w_j + v) \).

Note that in the literature we find a lot of extensions of weighted aggregation
functions such as those that use uncertain information represented in the form of
interval numbers, FNs or linguistic variables (Merigó, 2008).

### 2.3 Probabilistic Aggregation Functions

Probabilistic aggregation functions (or operators) are those functions that use
probabilistic information in the aggregation process. Some examples are the
aggregation with simple probabilities, the aggregation with belief structures
(Casanovas and Merigó, 2008; Merigó, 2008; 2009a; 2009b), the concept of immediate
probabilities (Engemann et al., 1996; Yager et al., 1995) and the probabilistic OWA
operator (Merigó, 2009d). The immediate probability is an approach that uses OWAs
and probabilities in the same formulation. It can be defined as follows.

**Definition 3:** An IPOWA operator of dimension \( n \) is a mapping \( \text{IPOWA} : \mathbb{R}^n \rightarrow \mathbb{R} \)
that has an associated weighting vector \( W \) of dimension \( n \) such that \( w_j \in [0, 1] \) and ,
according to the following formula:

\[
\text{IPOWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j
\]

where \( b_j \) is the \( j^{th} \) largest of the \( a_i \), each argument \( a_i \) has a probability \( v_i \) with and
\( v_i \in [0, 1] \), and \( v_j \) is the probability \( v_i \) ordered according to \( b_j \) that is, according to the \( j^{th} \)
largest of the \( a_i \).
Note that the IPOWA operator is a good approach for unifying probabilities and OWAs in some particular situations. But it is not always useful, especially in situations where we want to give more importance to the probabilities or to the OWA operators. In order to see why this unification does not seem to be a final model is considering other ways of representing \( \hat{\omega} \). For example, we could also use \( \hat{\omega} = \left[ w_j + v_j / \sum_{j=1}^{n} (w_j + v_j) \right] \) or other similar approaches.

Another approach for unifying probabilities and OWAs in the same formulation is the probabilistic OWA (POWA) operator (Merigó, 2008; 2009d). Its main advantage is that it is able to include both concepts considering the degree of importance of each case in the problem. It is defined as follows.

**Definition 4:** A POWA operator of dimension \( n \) is a mapping \( \text{POWA} : \mathbb{R}^n \rightarrow \mathbb{R} \) that has an associated weighting vector \( W \) of dimension \( n \) such that \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \), according to the following formula:

\[
\text{POWA}(a_1, \ldots, a_n) = \frac{1}{\lambda} \sum_{j=1}^{n} \hat{p}_j b_j \tag{6}
\]

where \( b_j \) is the \( j^{th} \) largest of the \( a_j \) each argument \( a_j \) has an associated probability \( p_j \) with \( \sum_{j=1}^{n} p_j = 1 \), \( \hat{p}_j = \beta w_j + (1 - \beta) p_j \) with \( \beta \in [0, 1] \) and \( p_j \) is the probability \( p_j \) ordered according to the \( j^{th} \) largest of the \( a_j \).

2.4 Generalized Aggregation Operators

There are a lot of aggregation operators that use the generalized mean such as the weighted generalized mean, the generalized OWA operator, the Minkowski distance and a lot of other cases. In this section we will give an example of how to use the generalized mean based on the OWA operator.

The generalized OWA (GOWA) operator was introduced by Yager in (2004). It generalizes a wide range of aggregation operators that includes the OWA operator with its particular cases, the ordered weighted geometric (OWG) operator, the ordered weighted harmonic averaging (OWHA) operator and the ordered weighted quadratic averaging (OWQA) operator. It can be defined as follows.

**Definition 5:** A GOWA operator of dimension \( n \) is a mapping \( \text{GOWA} : \mathbb{R}^n \rightarrow \mathbb{R} \) that has an associated weighting vector \( W \) of dimension \( n \) such that the sum of the weights is 1 and \( w_j \in [0, 1] \), then:

\[
\text{GOWA}(a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \tag{7}
\]

where \( b_j \) is the \( j^{th} \) largest of the \( a_j \) and \( \lambda \) is a parameter such that \( \lambda \in (-\infty, \infty) \).

As it is demonstrated in Yager (2004), the GOWA operator is a mean operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded.
and idempotent. It can also be demonstrated that the GOWA operator has as special cases the maximum, the minimum, the generalized mean and weighted generalized mean. Other families of GOWA operators can be found in Yager (2004).

2.5 The FPWA Operator

The fuzzy probabilistic weighted averaging (FPWA) operator (Merigó, 2009e) is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation and in an uncertain environment that can be assessed with FNs. It is defined as follows.

**Definition 6**: Let $\Psi$ be the set of FNs. A FPWA operator of dimension $n$ is a mapping $FPWA: \Psi^n \rightarrow \Psi$ such that:

$$FPWA(\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} \tilde{v}_j \tilde{a}_j$$

(8)

where the $\tilde{a}_i$ are the argument variables represented in the form of FNs, each argument $\tilde{a}_i$ has an associated weight (WA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, and a probabilistic weight $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\tilde{v}_i = \beta p_i + (1 - \beta) v_i$ with $\beta \in [0, 1]$ and $\tilde{v}_i$ is the weight that unifies probabilities and WAs in the same formulation.

Note that it is also possible to formulate the FPWA operator separating the part that strictly affects the probabilistic information and the part that affects the WAs. This representation is useful to see both models in the same formulation but it does not seem to be as a unique equation that unifies both models. Note that if the weighting vector of probabilities or WAs is not normalized, i.e., $P = \sum_{i=1}^{n} p_i \neq 1$, or $V = \sum_{i=1}^{n} v_i \neq 1$, then, the FPWA operator can be expressed as:

$$f(\tilde{a}_1, \ldots, \tilde{a}_n) = \frac{\beta}{P} \sum_{j=1}^{n} p_j \tilde{a}_j + \frac{(1 - \beta)}{V} \sum_{i=1}^{n} v_i \tilde{a}_i .$$

(9)

The FPWA is monotonic, commutative, bounded and idempotent. For further reading on the FPWA, see Merigó (2008).

3. **THE FUZZY GENERALIZED PROBABILISTIC WEIGHTED AVERAGING OPERATOR**

The fuzzy generalized probabilistic weighted averaging (FGPWA) operator is a new aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation. Moreover, it is able to deal with an uncertain environment that can not be assessed with the usual exact numbers but it is possible to use FNs. The use of FNs permits to consider the best and worst possible scenarios and the possibility that the intermediate values will occur. Furthermore, in this approach we use fuzzy
generalized means in order to include a wide range of aggregation operators such as the FPWA, the fuzzy probabilistic weighted geometric average (FPWGA), the fuzzy probabilistic weighted harmonic average (FPWHA), the fuzzy probabilistic weighted quadratic average (FPWQA), and a lot of other cases. In this case, we also unify the fuzzy weighted generalized average (FWGA) with the fuzzy probabilistic generalized mean (FPGM), and we are able to include other unifications, for example, the fuzzy weighted geometric mean (FWGM) with the fuzzy probabilistic geometric mean (FPGM), the fuzzy weighted quadratic mean (FWQM) with the fuzzy probabilistic quadratic mean (FPQM), and so on. It is defined as follows.

**Definition 7**: Let $\Psi$ be the set of FNs. A FGPWA operator of dimension $n$ is a mapping $FGPWA: \Psi^n \to \Psi$ such that:

$$FGPWA(\tilde{a}_1, \ldots, \tilde{a}_n) = \left( \frac{1}{n} \sum_{i=1}^{n} \hat{\nu}_i \tilde{a}_i^\lambda \right)^{1/\lambda}$$

(10)

where the $\tilde{a}_i$ are the argument variables represented in the form of FNs, each argument $\tilde{a}_i$ has an associated weight (FPWA) $\nu_i$ with $\sum_{i=1}^{n} \nu_i = 1$ and $\nu_i \in [0, 1]$, and a probabilistic weight $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{\nu}_i = \beta p_i + (1 - \beta) \nu_i$ with $\beta \in [0, 1]$, $\hat{\nu}_i$ is the weight that unifies probabilities and WAs in the same formulation and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that it is also possible to formulate the FGPWA operator separating the part that strictly affects the probabilistic information and the part that affects the WA. This representation is useful to see both models in the same formulation but it does not seem to be as a unique equation that unifies both models.

**Definition 8**: Let $\Psi$ be the set of FNs. A FGPWA operator of dimension $n$ is a mapping $FGPWA: \Psi^n \to \Psi$ of dimension $n$, if it has an associated probabilistic vector $P$, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$ and a weighting vector $V$ that affects the WA, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$FGPWA(\tilde{a}_1, \ldots, \tilde{a}_n) = \beta \left( \frac{1}{n} \sum_{j=1}^{n} p_j \tilde{a}_j^\lambda \right)^{1/\lambda} + (1 - \beta) \left( \frac{1}{n} \sum_{i=1}^{n} v_i \tilde{a}_i^\lambda \right)^{1/\lambda}$$

(11)

where the $\tilde{a}_i$ are the argument variables represented in the form of FNs, $\beta \in [0, 1]$ and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that sometimes, it is not clear how to reorder the arguments. Then, it is necessary to establish a criterion for comparing FNs. For simplicity, we recommend the following method. Select the FN with the highest value in its highest membership level, usually, when $\alpha = 1$. Note that if the membership level $\alpha = 1$ is an interval, then, we will calculate the average of the interval. If there is still a tie, then, we recommend the use of an average or a weighted average of the FN according to the interests of the decision maker. Note that other methods could be used (Kaufmann and Gil-Aluja, 1987; Merigó, 2008).
Note that different types of FNs can be used in the aggregation process according to the interests or necessities of the decision maker. For example, we could mention the following ones (Merigó, 2008):

- Triangular FNs (Interval-valued, generalized, etc.).
- Trapezoidal FNs (Interval-valued, generalized, etc.).
- L-R FNs (Interval-valued, generalized, intuitionistic, Type 2 and \( n \), etc.).
- Interval-valued FNs (triplets, quadruplets, etc.).
- Generalized FNs (simple, interval-valued, intuitionistic, Type 2 and \( n \), etc.).
- Intuitionistic FNs (simple, interval-valued, generalized, etc.).
- Type 2 and \( n \) FNs (simple, generalized, etc.).
- Etc.

Note that if the weighting vector of probabilities or WAs is not normalized, i.e., 
\[ P = \sum_{i=1}^{n} p_i \neq 1, \text{ or } V = \sum_{i=1}^{n} v_i \neq 1, \] 
then, the FGPWA operator can be expressed as:

\[
\text{FGPWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = \frac{\beta}{P} \left( \sum_{j=1}^{n} p_j \tilde{a}_j^\lambda \right)^{1/\lambda} + \frac{(1-\beta)}{V} \left( \sum_{i=1}^{n} v_i \tilde{a}_i^\lambda \right)^{1/\lambda}. \tag{12}
\]

If \( B \) is a vector corresponding to the ordered arguments \( b_j \), we shall call this the ordered argument vector and \( W^T \) is the transpose of the weighting vector, then, the GPWA operator can be expressed as:

\[
\text{FGPWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = W^T B. \tag{13}
\]

The FGPWA is monotonic, bounded and idempotent. It is monotonic because if \( \tilde{a} \geq i_j \), for all \( \tilde{a} \), then, FGPWA \((\tilde{a}_1, \ldots, \tilde{a}_n) \geq \text{FGPWA}(i_1, \ldots, i_n) \). It is bounded because the FGPWA aggregation is delimited by the minimum and the maximum. That is, \( \min \{\tilde{a}\} \leq \text{FGPWA}(\tilde{a}_1, \ldots, \tilde{a}_n) \leq \max \{\tilde{a}\} \). It is idempotent because if \( \tilde{a}_i = \tilde{a} \), for all \( \tilde{a}_j \), then, FGPWA \((\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{a} \).

4. FAMILIES OF FGPWA OPERATORS

First of all, we are going to consider the two main cases of the FGPWA operator that are found by analyzing the coefficient \( \beta \). Basically:

- If \( \beta = 0 \), then, we get the fuzzy weighted generalized mean (FWGM).
- If \( \beta = 1 \), the fuzzy generalized probabilistic aggregation (FGPA) operator.
- Note that if \( v_i = 1/n \), for all \( i \), then, we get the unification between the fuzzy generalized mean and the FGPA operator (fuzzy arithmetic generalized probabilistic aggregation (FAGPA) operator).
• And if \( p_i = 1/n \), for all \( j \), then, we get the unification between the fuzzy generalized mean and the FGWA operator (fuzzy arithmetic generalized weighted average (FAGWA) operator).

If we analyze different values of the parameter \( \lambda \), we obtain another group of particular cases such as the usual FPWA operator, the FPWGA operator, the FPWQA operator and the FPWHA operator.

**Remark 1:** When \( \lambda = 1 \), the FGPWA operator becomes the FPWA operator.

\[
\text{FGPWA}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \frac{1}{n} \sum_{j=1}^{n} \hat{v}_j \hat{a}_i .
\] (14)

Note that if \( p_j = 1/n \), for all \( \bar{a}_i \), we get the fuzzy arithmetic weighted average (FAWA) and if \( v_i = 1/n \), for all \( \bar{a}_r \), we get the fuzzy arithmetic probabilistic aggregation (FAPA) operator.

**Remark 2:** When \( \lambda = 0 \), the FGPWA operator becomes the fuzzy probabilistic weighted geometric averaging (FPWGA) operator.

\[
\text{FGPWA}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \prod_{j=1}^{n} \bar{a}_i^{\hat{v}_j} .
\] (15)

Note that if \( p_j = 1/n \), for all \( \bar{a}_r \), we get the fuzzy probabilistic geometric arithmetic weighted geometric average (FPGAWGA) and if \( v_i = 1/n \), for all \( \bar{a}_i \), we get the fuzzy probabilistic geometric arithmetic mean (FPGAM) operator. Note that if \( \beta = 1 \), we get the fuzzy probabilistic geometric aggregation (FPGA).

**Remark 3:** When \( \lambda = -1 \), we get the fuzzy probabilistic weighted harmonic averaging (FPWHA) operator.

\[
\text{FGPWA}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \frac{1}{\sum_{j=1}^{n} \hat{v}_j \bar{a}_j} .
\] (16)

Note that if \( p_j = 1/n \), for all \( \bar{a}_r \), we get the fuzzy probabilistic harmonic arithmetic weighted harmonic average (FPHAWHA) and if \( v_i = 1/n \), for all \( \bar{a}_i \), we get the fuzzy probabilistic harmonic arithmetic mean (FPHAM) operator. Note that if \( \beta = 1 \), we get the fuzzy probabilistic harmonic aggregation (FPHA).

**Remark 4:** When \( \lambda = 2 \), we get the fuzzy probabilistic weighted quadratic averaging (FPWQA) operator.

\[
\text{FGPWA}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \left( \sum_{j=1}^{n} \hat{v}_j \bar{a}_j^2 \right)^{1/2} .
\] (17)

Note that if \( p_j = 1/n \), for all \( \bar{a}_r \), we get the fuzzy probabilistic quadratic arithmetic weighted quadratic average (FPQAWQA) and if \( v_i = 1/n \), for all \( \bar{a}_i \), we get the fuzzy
probabilistic quadratic arithmetic mean (FPQAM) operator. Note that if $\beta = 1$, we get the fuzzy probabilistic quadratic aggregation (FPQA).

**Remark 5.** When $\lambda = 3$, we get the fuzzy probabilistic weighted cubic averaging (FPWCA) operator.

$$FGPWA (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \frac{1}{n} \sum_{i=1}^{n} \hat{v}_i \hat{a}_i^3 \right)^{1/3}.$$  \hspace{1cm} (18)

Note that if $p_j = 1/n$, for all $a_j$ we get the fuzzy cubic arithmetic weighted cubic average (FCAWCA) and if $v_i = 1/n$, for all $a_i$ we get the fuzzy cubic probabilistic cubic average (FCPCA) operator. Note that if $\beta = 1$, we get the fuzzy probabilistic cubic aggregation (FPCA).

**Remark 6:** Note that other families could be obtained by using different values in the parameter $\lambda$. And mixing different classes for each part of the aggregation, for example, we could obtain $\lambda = 1$ for the probabilities and $\lambda = 2$ for the WA. Thus, we would get the fuzzy probabilistic weighted quadratic average (FPWQA). And in a similar way we could form a lot of other cases such as the fuzzy probabilistic quadratic weighted geometric average (FPQWGA), the fuzzy probabilistic geometric weighted average (FPGWA), the fuzzy probabilistic geometric weighted quadratic average (FPGQWA), and so on.

5. **QUASI-ARITHMETIC MEANS IN THE FPWA OPERATOR**

Following Merigó (2008) and others (Beliakov, 2005; Fodor et al., 1995) it is possible to further generalize the generalized means by using quasi-arithmetic means. For the case of the FPWA operator, we obtain the quasi-arithmetic FPWA (Quasi-FPWA) operator. It can be defined as follows.

**Definition 9:** Let $\Psi$ be the set of FNs. A Quasi-FPWA operator of dimension $n$ is a mapping $QFPWA: \Psi^n \rightarrow \Psi$ that has an associated weighting vector $W$ of dimension $n$ such that the sum of the weights is 1 and $w_j \in [0, 1]$, then:

$$Quasi-FPWA (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = g^{-1} \left( \sum_{i=1}^{n} \hat{v}_i g(\tilde{a}_i) \right).$$ \hspace{1cm} (19)

where the $\tilde{a}_i$ are the argument variables represented in the form of FNs, each argument $\tilde{a}_i$ has an associated weight (PWA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, and a probabilistic weight $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{v}_i = \beta p_i + (1 - \beta) v_i$ with $b \in [0, 1]$, $\hat{v}_i$ is the weight that unifies probabilities and WAs in the same formulation and $g(b)$ is a strictly continuous monotone function.

Note that if the weighting vector of probabilities or WAs is not normalized, i.e., $P = \sum_{i=1}^{n} p_i \neq 1$, or $V = \sum_{i=1}^{n} v_i \neq 1$, then, the Quasi-FPWA operator can be expressed as:
Quasi-FPWA $(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) = \frac{\beta}{P} \left(g^{-1} \left( \sum_{i=1}^{n} p_i g(\bar{a}_i) \right) \right) + \frac{(1-\beta)}{V} \left(g^{-1} \left( \sum_{i=1}^{n} v_i g(\bar{a}_i) \right) \right). \quad (20)

Note that all the properties and particular cases commented in the FGPWA operator, are also included in this generalization. Thus, we could study all the particular cases of Section 4 and a lot of other cases.

6. ILLUSTRATIVE EXAMPLE

In the following, we are going to develop a brief illustrative example of the new approach in a financial decision making problem regarding investment selection.

Assume a company wants to invest some money the next period and they are looking for its optimal investment. The key question to answer is where to locate the money and they consider that it should be interesting to enter in a new market. By now, they are operating in Europe and North America. Therefore, their main choices are the following.

• Invest in the Asian market: $A_1$.
• Invest in the South American market: $A_2$.
• Invest in the African market: $A_3$.
• Invest in all the three regions: $A_4$.
• Do not invest in any region: $A_5$.

After careful review of the information, the group of experts of the enterprise establishes the following general information about the investments. They consider that the key factor for the selection process is the economic situation for the next period. Thus, depending on the economic situation, the results obtained by investing in each region will be different.

• $S_1$: Very bad economic situation.
• $S_2$: Bad economic situation.
• $S_3$: Regular economic situation.
• $S_4$: Good economic situation.
• $S_5$: Very good economic situation.

The results are shown in Table 1. Note that the results represent the benefits obtained if the state of nature $S_i$ occurs and we select the investment $A_j$. 
### Table 1

<table>
<thead>
<tr>
<th>Characteristics of the Investments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
</tr>
<tr>
<td>$A_1$ (60, 70, 80)</td>
</tr>
<tr>
<td>$A_2$ (20, 30, 40)</td>
</tr>
<tr>
<td>$A_3$ (40, 50, 60)</td>
</tr>
<tr>
<td>$A_4$ (80, 90, 100)</td>
</tr>
</tbody>
</table>

With this information, it is possible to develop different methods based on the FGPWA operator for selecting an investment. In this example, we will consider the fuzzy probabilistic aggregation, the fuzzy weighted average (FWA), the fuzzy arithmetic mean (FAM), the fuzzy arithmetic probabilistic aggregation (FAPA), the fuzzy arithmetic weighted average (FAWA) and the fuzzy PWA (FPWA) operator.

We assume that $\beta = 0.4$ and the following weights: $P = (0.3, 0.2, 0.2, 0.1)$ and $V = (0.3, 0.3, 0.2, 0.1, 0.1)$. The results are shown in Table 2.

### Table 2

<table>
<thead>
<tr>
<th>Aggregated Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>FProb. FWA FAM FAPA FAWA FPWA</td>
</tr>
<tr>
<td>$A_1$ (57, 67, 77)</td>
</tr>
<tr>
<td>$A_2$ (52, 62, 72)</td>
</tr>
<tr>
<td>$A_3$ (56, 66, 76)</td>
</tr>
<tr>
<td>$A_4$ (50, 60, 70)</td>
</tr>
<tr>
<td>$A_5$ (59, 69, 79)</td>
</tr>
</tbody>
</table>

As we can see, depending on the particular type of FGPWA used, the results and the decisions may be different.

### Table 3

<table>
<thead>
<tr>
<th>Ordering of the Investments</th>
</tr>
</thead>
<tbody>
<tr>
<td>FProb. Ordering</td>
</tr>
<tr>
<td>$A_1 \sim A_5 \sim A_4 \sim A_3 \sim A_2$</td>
</tr>
<tr>
<td>FWA Ordering</td>
</tr>
<tr>
<td>$A_1 \sim A_3 \sim A_4 \sim A_5 \sim A_2$</td>
</tr>
<tr>
<td>FAM Ordering</td>
</tr>
<tr>
<td>$A_2 \sim A_1 \sim A_3 \sim A_4 \sim A_5$</td>
</tr>
</tbody>
</table>

As we can see, depending on the particular type of FGPWA used, the results and the decisions may be different.

## 7. CONCLUSIONS

We have introduced a new model that unifies the probability and the weighted average in the same formulation considering the degree of importance that each
concept has in the analysis. We have called it the FGPWA operator. We have seen that it is able to deal with uncertain environments that can be assessed with FNs providing a more complete representation of the decision problem. Furthermore, we have seen that this model uses generalized means providing a more robust formulation of the aggregation operator that includes a wide range of aggregation operators such as the GPWA, the FPWA, the FPWGA, the FPWQA, the FPWHA, the FAWA, the FAPA, and a lot of other cases. We have further generalized the FGPWA by using quasi-arithmetic means, obtaining the Quasi-FPWA operator.

We have also developed an application of the new approach in a financial decision making problem. We have studied an investment selection problem where a company is looking for its optimal investment. We have seen that depending on the particular type of FGPWA operator used, the results may be different. However, we should note that the manipulation of the results is not so flexible here as it was in the OWA operator (Merigó, 2008a, Yager, 1988).

In future research, we expect to develop further extensions to this approach by using more general formulations and considering other characteristics in the problem such as the use of uncertain information. We will also consider the use of distance measures in the analysis and a lot of other applications in business decision making and other fields such as statistics and engineering.

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