Robust $H_\infty$ Control for Uncertain Switched Time-delay Systems with Sampled-data Input

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ABSTRACT

The robust $H_\infty$ control for uncertain switched system with time delay and sampled-data input is investigated in this paper. The design of sampled-data input technique is based on a computer digital control theory and used instead of analog elements. Hence designing a practical and useful feedback control strategy is an important work in implementation. Time-varying delay approach is used to deal with the sampled data input problem. The upper bound for the sampled time of state is estimated from the proposed LMI optimization approach. A numerical example is illustrated to show the main results.

NOMENCLATURE

$A^T$ := Transpose of the matrix $A$.
$\text{trace}(A)$ := Trace of the matrix $A$.
$L^2[0, \infty]$ := Space of square integrable functions on $[0, \infty)$.
$\|x\|$ := Euclidean norm of the vector $x$.
$\|f\|_2 := \sqrt{\int_0^\infty \|f(t)\|^2 dt}, f(t) \in L^2[0, \infty)$
$P > 0$ (resp. $P < 0$) := Positive-definite (resp., negative-definite) symmetric matrix.
$A \leq B$ := $B - A$ is a positive semidefinite symmetric matrix.
$I$ := Identity matrix.
$\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ := $*$ = $B^T$ yields the symmetric form of the matrix.
$N := \{1, 2, 3, ..., N\}$.

1. INTRODUCTION

Switched system is composed of a class of subsystems and the switching signal is used to specify which subsystem is activated in each instant of time. Switched systems are often encountered in automated highway systems, automotive engine control systems, chemical processes, constrained robotics, power systems and power electronics, robot manufacture, and stepper motors [1-7]. Hence many complicated phenomena about switched systems are studied and proposed in recent years [4].

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Time delay is often encountered in various practical systems; such as aircraft stabilization, manual control, models of lasers, neural networks, nuclear reactors, ship stabilization, and systems with lossless transmission lines. On the other hand, time delay often generates instability and oscillations in many practical systems.

Sampled-data input is a practical and useful tool to implement some complicated control schemes; such as parallel distributed control in T-S fuzzy system [8] and switching control in switched system in [9]. Parallel distributed control (PDC) and switching control are often used to improve the stability and performance of T-S fuzzy systems and switched systems, but PDC and switching control are difficult to implement by analog elements and devices. Hence digital control will be an available consideration for implementing PDC and switching control. Suppose these controls are calculated by computer, then the input value will be held until next sample instant to reflect the input. This important issue to guarantee the stabilization and performance is arisen from the sampled time \( T > 0 \). For retaining the achieved purpose of design, we will assume that the sampled time is sufficiently small. Suppose that the states of system are measured by some feasible sensors, then the state values will be held until next measured instant to renew the state [2]-[6]. There are many researchers to consider this important issue for estimating sampled time \( T > 0 \) to stabilize and guarantee the performance of the systems.

In [8]-[12], a time-varying delay technique is used to represent the sampled-data input. This technique provides a useful analytic tool to estimate the upper bound of sampled time \( T > 0 \). In recent years, \( H_\infty \) control is a stabilization scheme with a performance index [8-9]. \( H_\infty \) control concept was proposed to reduce the effect of the disturbance input on the regulated output within a prescribed level and guarantee that the closed-loop system is stable. In [13], Lyapunov-Krasovskii functional with Leibniz-Newton formula had been used to find guaranteed control for T-S fuzzy systems by using time-varying delay input approach. In [14], some linear fractional perturbations are considered for T-S fuzzy time-delay systems. In this paper, the \( H_\infty \) control for switched time-delay systems with linear fractional perturbations is considered. Some additional nonnegative inequalities are introduced to improve the conservativeness of the proposed results in this paper.

2. PROBLEM FORMULATION AND MAIN RESULTS

Consider a continuous-time switched time-delay system with sampled input and nonlinear perturbations:

\[
\begin{align*}
\dot{x}(t) &= A_{1\sigma}x(t) + A_{2\sigma}x(t-\tau) + \Delta f_\sigma(x(t)) + B_{1\sigma}u(t) + B_{2\sigma}w(t), t \geq 0, \\
y(t) &= C_{1\sigma}x(t) + C_{2\sigma}x(t-\tau) + D_{1\sigma}u(t) + D_{2\sigma}w(t), t \geq 0, \\
x(t) &= \varphi(t), t \in [-\tau, 0],
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the system state, \( u(t) \in \mathbb{R}^p \) is the input, \( w(t) \in \mathbb{R}^q \) is the disturbance input, \( y(t) \in \mathbb{R}^r \) is the regulated output, time delay \( \tau \) is a nonnegative constant, \( \sigma \) is a switching signal which is a piecewise constant function and depends on \( t \), \( \overline{i} \) is a switching signal which is a piecewise constant function and depends on \( t \), \( \sigma \) takes its values in the finite set \( \mathcal{N} \), and the initial vector \( \varphi \in C_0 \), where \( C_0 \) is the set of continuous functions from \([-\tau, 0]\) to \( \mathbb{R}^n \). Matrices \( A_{i\sigma}, A_{2\sigma}, B_{1\sigma}, B_{2\sigma}, C_{1\sigma}, C_{2\sigma}, D_{1\sigma}, D_{2\sigma} \) are given. \( \Delta f_\sigma(x(t)) \) is a perturbed nonlinear function satisfying

\[
\|\Delta f_\sigma(x(t))\| \leq \|F_i x(t)\|, \quad i \in \{1, 2, \cdots, N\},
\]

where \( F_i \) is a given constant matrix. Define the following functions \( \lambda_i(\sigma), \forall \ i \in \mathcal{N} \), that will be used to represent our system:

\[
\lambda_i(\sigma) = \begin{cases} 1, & \sigma = i, \\ 0, & \text{otherwise}, \end{cases} \quad i \in \mathcal{N},
\]
where \( \sigma \) is defined in (1). The state equation of switched system is rewritten as

\[
x(t) = \sum_{i=1}^{m} \lambda_i(\sigma) \cdot \left[ A_{i1}x(t) + A_{i2}x(t - \tau) + A_{i3}x(t) + B_{i1}u(t) + B_{i2}w(t) \right], \quad t \geq 0,
\]

\[
y(t) = \sum_{i=1}^{m} \lambda_i(\sigma) \cdot \left[ C_{i1}x(t) + C_{i2}x(t - \tau) + D_{i1}u(t) + D_{i2}w(t) \right], \quad t \geq 0.
\]

The following state feedback control is used to stabilize the switched system:

\[
u(t) = -K_i x(kT), \quad \sigma = i, \quad kT \leq t < (k+1)T,
\]

where \( K_i \in \mathbb{R}^{p \times n} \) will be designed. The final feedback control is inferred as

\[
u(t) = -\sum_{i=1}^{m} \lambda_i(\sigma) \cdot K_i x(t-kT), \quad kT \leq t < (k+1)T. \tag{3a}
\]

With (3a), the sampled-data control input can be described as follows:

\[
u(t) = -\sum_{i=1}^{m} \lambda_i(\sigma) \cdot K_i x(t-h(t)), \quad kT \leq t < (k+1)T, \tag{3b}
\]

where \( h(t) \) is specified by

\[
h(t) = t - kT, \quad kT \leq t < (k+1)T,
\]

and \( 0 \leq h(t) < T, \quad t \geq 0 \). The original system can be rewritten as follows:

\[
x(t) = \sum_{i=1}^{N} \lambda_i(\sigma) \cdot \left[ A_{i1}x(t) + A_{i2}x(t - \tau) + A_{i3}x(t) - B_{i1}K_i x(t-h(t)) + B_{i2}w(t) \right], \quad t \geq 0, \tag{4a}
\]

\[
y(t) = \sum_{i=1}^{N} \lambda_i(\sigma) \cdot \left[ C_{i1}x(t) + C_{i2}x(t - \tau) - D_{i1}K_i x(t-h(t)) + D_{i2}w(t) \right], \quad t \geq 0. \tag{4b}
\]

**Definition 1.** For a prescribed level of disturbance attenuation \( \gamma > 0 \), find a sampled input (3) satisfying the following conditions:

(i) With \( w(t) = 0 \), the closed-loop system (4) with (2) is asymptotically stabilizable by sampled-data input in (3).

(ii) With zero initial condition (i.e. \( \phi = 0 \)), the signals \( w(t) \) and \( y(t) \) are bounded by

\[
\int_{0}^{T} \| y(t) \| dt \leq \gamma^2 \int_{0}^{T} \| w(t) \| dt, \quad \text{(i.e., } \| y \|_2 \leq \gamma \cdot \| w \|_2 \text{), } \forall \ w \in L_2[0, \infty), \ w \neq 0, \)
\]

For all \( \gamma > 0 \). In above conditions, system (4) with (2) is said to be stabilizable by sampled data input (3) and disturbance attenuation \( \gamma \), and the control law (3) is said to be a sampled-data \( H_\infty \) control for system (4) with (2).

**Lemma 1:** (Schur complement of [15]). For a given matrix \( S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} \) with \( S_{11} = S_{11}^T, \ S_{22} = S_{22}^T \), then the following conditions are equivalent:
(1) $S < 0,$

(2) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0.$

Now we present a result to design the sampled-data $H_\infty$ control (3) for system (4) with (2).

**Theorem 1.** Suppose for a given constant $\eta > 0$, the following optimization problem:

\[
\text{Minimize } \tilde{\gamma},
\]

subject to the following LMIs:

\[
\begin{bmatrix}
\dot{\hat{Q}}_{11} & \dot{\hat{Q}}_{12} \\
* & \dot{\hat{Q}}_{22}
\end{bmatrix} > 0, \quad \begin{bmatrix}
\hat{R}_{11} & \hat{R}_{12} \\
* & \hat{R}_{22}
\end{bmatrix} > 0, \quad \hat{R}_{11} > \hat{Q}_{11}, \quad \hat{R}_{12} > \hat{Q}_{12}, \quad \hat{P}_1 > \hat{Q}_{21}, \quad \hat{P}_2 > \hat{Q}_{22},
\]

\[
T \cdot \hat{P}_2 < \eta^{-1} \cdot \hat{P}_0,
\]

\[
\tilde{\gamma} = \frac{1}{2} \left[ \hat{P}_1^T \hat{P}_1 + \hat{P}_2^T \hat{P}_2 - \hat{Q}_{11} - \hat{Q}_{12} - \hat{Q}_{21} - \hat{Q}_{22} \right]
\]

for all $i \in \{1, \ldots, N\},$ (5d),

has a solution with positive constants $\tilde{\gamma}, \varepsilon_i$, positive definite symmetric matrices $\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{Q}_{22}$, $\hat{R}_{22} \in \mathcal{H}^{R_{22}}$, $\hat{Q}_{11} \in \mathcal{H}^{R_{11}}$, $\hat{R}_{11} \in \mathcal{H}^{R_{11}}$, matrices $\hat{Q}_{12} \in \mathcal{H}^{R_{12}}$, $\hat{R}_{12} \in \mathcal{H}^{R_{12}}$, $\hat{K}_i \in \mathcal{R}^{p \times n}$ where $*$ represents the symmetric form in the matrix and

\[
\begin{align*}
\dot{\hat{X}}_{11i} & = A_{1i} \hat{P}_1 + \hat{P}_0 A_{1i}^T + \hat{P}_1 + \hat{P}_3, \\
\dot{\hat{X}}_{12i} & = -B_{1i} \hat{K}_i, \\
\dot{\hat{X}}_{14i} & = A_{2i} \hat{P}_0, \\
\dot{\hat{X}}_{15i} & = \varepsilon_i \cdot I,
\end{align*}
\]

\[
\dot{\hat{X}}_{16i} = B_{2i} + \dot{\hat{X}}_{17i}^T = \hat{P}_0 C_{1i}^T, \\
\dot{\hat{X}}_{18i} = \hat{P}_0 A_{4i}^T, \\
\dot{\hat{X}}_{19i} = \hat{P}_0 F_i^T, \\
\dot{\hat{X}}_{27i} = -\hat{K}_i^T D_i^T,
\]

\[
\begin{align*}
\dot{\hat{X}}_{28i} & = -\hat{K}_i^T B_{1i}, \\
\dot{\hat{X}}_{33i} & = -\hat{P}_1, \\
\dot{\hat{X}}_{44i} & = \hat{P}_0 C_{2i}, \\
\dot{\hat{X}}_{48i} & = \hat{P}_0 A_{2i}^T, \\
\dot{\hat{X}}_{55i} & = -\varepsilon_i \cdot I,
\end{align*}
\]

\[
\dot{\hat{X}}_{58i} = \varepsilon_i \cdot I, \\
\dot{\hat{X}}_{66i} = -\hat{P}_1 \cdot \hat{X}_{66i} = -\hat{P}_3, \\
\dot{\hat{X}}_{67i} = \hat{P}_0 C_{2i}, \\
\dot{\hat{X}}_{68i} = \hat{P}_0 A_{2i}^T, \\
\dot{\hat{X}}_{77i} = -I, \\
\dot{\hat{X}}_{88i} = -\eta \cdot \hat{P}_0, \\
\dot{\hat{X}}_{99i} = -\varepsilon_i \cdot I,
\]

\[
\dot{\hat{\Gamma}} = T \cdot \hat{R}_{11} + \hat{Q}_{12} [I \ -I \ 0] + [I \ -I \ 0] \hat{Q}_{12} + \hat{R}_{12} [0 \ I \ -I] + [0 \ I \ -I] \hat{R}_{12}.
\]

Then the system (4) with (2) is stabilizable by sampled-data input (3) with $K_i = \hat{K}_i \hat{P}_0^{-1}$ and disturbance attenuation $\gamma = \sqrt{\tilde{\gamma}}$.

**Proof.** Define the Lyapunov functional
where \( P_0 = \hat{P}_0^{-1} \), \( P_i = \hat{P}_0^{-1} \hat{P}_i \hat{P}_0^{-1} \), \( i \in \{1, 2, 3\} \), are positive definite symmetric matrices. The time derivatives of \( V(x_i) \), along the trajectories of system (1) with (2) satisfy

\[
\dot{V}(x_i) = \sum_{j=1}^{m} \lambda_i(\sigma) \cdot \left\{ x^T(t)(P_0 A_{ii} + A_{ii}^T P_0) x(t) + 2 x^T(t) P_0 \left[ A_{ii} x(t - \tau) + \Delta f_i(x(t)) + B_{ii} w(t) \right] \right\} \\
- \sum_{j=1}^{m} \lambda_i(\sigma) \cdot \left\{ 2 x^T(t) P_0 B_{ii} K_i x(t - h(t)) \right\} + x^T(t) P_i x(t) - x^T(t - T) P_i x(t - T) \\
+ T \cdot \dot{x}^T(t) P_2 \dot{x}(t) - \left[ \int_{t-T}^{t-h(t)} \dot{x}^T(s)P_2 \dot{x}(s) ds + \int_{t-h(t)}^{t} \dot{x}^T(s)P_2 \dot{x}(s) ds \right] \\
+ x^T(t) P_i x(t) - x^T(t - \tau) P_i x(t - \tau).
\]

(6b)

Now we define a vector by

\[
X^T(t) = \begin{bmatrix} x^T(t) & x^T(t - h(t)) & x^T(t - T) \end{bmatrix}.
\]

By Leibniz-Newton formula and LMI (5b), we have

\[
\int_{t-h(t)}^{t-h(t)} \left[ X(t) \hat{X}(s) \right]^T \begin{bmatrix} Q_{11} & Q_{12} \\ \ast & Q_{22} \end{bmatrix} \left[ X(t) \hat{X}(s) \right] ds \\
= h(t) \cdot X^T(t) Q_{11} X(t) + 2 X^T(t) Q_{12} [x(t) - x(t - h(t))] + \int_{t-h(t)}^{t} \dot{x}^T(s) Q_{22} \dot{x}(s) ds \geq 0,
\]

(6c)

\[
\int_{t-T}^{t-h(t)} \left[ X(t) \hat{X}(s) \right]^T \begin{bmatrix} R_{11} & R_{12} \\ \ast & R_{22} \end{bmatrix} \left[ X(t) \hat{X}(s) \right] ds \\
= [T - h(t)] \cdot X^T(t) R_{11} X(t) + 2 X^T(t) R_{12} [x(t - h)] - x(t - T)] + \int_{t-T}^{t-h(t)} \dot{x}^T(s) R_{22} \dot{x}(s) ds \geq 0,
\]

(6d)

where

\[
\begin{bmatrix} Q_{11} & Q_{12} \\ \ast & Q_{22} \end{bmatrix} = \begin{bmatrix} \hat{P}_0^{-1} & 0 & 0 & 0 \\ 0 & \hat{P}_0^{-1} & 0 & 0 \\ 0 & 0 & \hat{P}_0^{-1} & 0 \\ 0 & 0 & 0 & \hat{P}_0^{-1} \end{bmatrix}, \quad \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \ast & \hat{Q}_{22} \end{bmatrix} = \begin{bmatrix} \hat{P}_0^{-1} & 0 & 0 & 0 \\ 0 & \hat{P}_0^{-1} & 0 & 0 \\ 0 & 0 & \hat{P}_0^{-1} & 0 \\ 0 & 0 & 0 & \hat{P}_0^{-1} \end{bmatrix} > 0,
\]

\[
\begin{bmatrix} R_{11} & R_{12} \\ \ast & R_{22} \end{bmatrix} = \begin{bmatrix} \hat{P}_0^{-1} & 0 & 0 & 0 \\ 0 & \hat{P}_0^{-1} & 0 & 0 \\ 0 & 0 & \hat{P}_0^{-1} & 0 \\ 0 & 0 & 0 & \hat{P}_0^{-1} \end{bmatrix}, \quad \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \ast & \hat{R}_{22} \end{bmatrix} = \begin{bmatrix} \hat{P}_0^{-1} & 0 & 0 & 0 \\ 0 & \hat{P}_0^{-1} & 0 & 0 \\ 0 & 0 & \hat{P}_0^{-1} & 0 \\ 0 & 0 & 0 & \hat{P}_0^{-1} \end{bmatrix} > 0.
\]

From the condition (2), we have
\[ x^T(t)F_i^TF_i x(t) - \Delta f_i^T(x(t))\Delta f_i(x(t)) \geq 0, \quad i \in \{1, 2, \ldots, N\}. \]  

From the output in (4b), we have

\[ y^T(t)y(t) = \sum_{i=1}^N \sum_{j=1}^N \lambda_i(\sigma) \cdot \lambda_j(\sigma) [C_{i,j}x(t) + C_{j,i}x(t - \tau) - D_{i,j}K_jx(t - h(t)) + D_{j,i}w(t)]^T \cdot [C_{i,j}x(t) + C_{j,i}x(t - \tau) - D_{i,j}K_jx(t - h(t)) + D_{j,i}w(t)] \]

\[ = \sum_{i=1}^N \lambda_i(\sigma) [C_{i,j}x(t) + C_{j,i}x(t - \tau) - D_{i,j}K_jx(t - h(t)) + D_{j,i}w(t)]^T \cdot [C_{i,j}x(t) + C_{j,i}x(t - \tau) - D_{i,j}K_jx(t - h(t)) + D_{j,i}w(t)] \]  

(8a)

By the similar derivation of (8a) and condition (5c) with \( P_2 = \hat{P}_0^{-1} \hat{P}_2 \hat{P}_0^{-1} \), we have

\[ T \cdot \dot{x}^T(t)P_2 \dot{x}(t) \leq \sum_{i=1}^N \lambda_i(\sigma) \cdot [A_{i,j}x(t) + A_{j,i}x(t - \tau) + \Delta f_i(x(t)) - B_{i,j}K_jx(t - h(t)) + B_{j,i}w(t)]^T \cdot \hat{P}_0^{-1} \cdot \hat{P}_0^{-1} \cdot [A_{i,j}x(t) + A_{j,i}x(t - \tau) + \Delta f_i(x(t)) - B_{i,j}K_jx(t - h(t)) + B_{j,i}w(t)] \]  

(8b)

From (5b) and (6)-(8), we have

\[ \dot{V}(x_i) + y^T(t)y(t) - y^2 \cdot w^T(t)w(t) + \sum_{i=1}^N \lambda_i(\sigma) \cdot \epsilon_i^{-1} \cdot \left[ x^T(t)F_i^TF_i x(t) - \Delta f_i^T(x(t))\Delta f_i(x(t)) \right] \]

\[ \leq \sum_{i=1}^N \lambda_i(\sigma) \cdot \left[ \Gamma_i^T(t) \cdot \Sigma_i \cdot \Gamma_i(t) - h(t) \cdot X^T(t)(R_{11} - Q_{11})X(t) \right] \]

\[ - \sum_{i=1}^N \lambda_i(\sigma) \cdot \left[ \int_{t-h(t)}^{t-h(t)} \dot{x}^T(s)(P_2 - R_{22})\dot{x}(s)ds + \int_{t-h(t)}^{t} \dot{x}^T(s)(P_2 - Q_{22})\dot{x}(s)ds \right] \]

\[ \leq \sum_{i=1}^N \lambda_i(\sigma) \cdot \left[ \Gamma_i^T(t) \cdot \Sigma_i \cdot \Gamma_i(t) \right] \]  

(9a)

where

\[ \Gamma_i^T(t) = \begin{bmatrix} X^T(t) & x^T(t - \tau) & \Delta f_i^T(x(t)) & w^T(t) \end{bmatrix}. \]

\[
\Sigma_i = \begin{bmatrix}
\Sigma_{11i} & \Sigma_{12i} & 0 & \Sigma_{14i} & \Sigma_{15i} & \Sigma_{16i} \\
* & 0 & 0 & 0 & 0 & 0 \\
* & * & \Sigma_{33i} & 0 & 0 & 0 \\
* & * & * & \Sigma_{44i} & 0 & 0 \\
* & * & * & * & \Sigma_{55i} & 0 \\
* & * & * & * & * & \Sigma_{66i}
\end{bmatrix} + \begin{bmatrix}
\Sigma_{17i} & \Sigma_{18i} \\
\Sigma_{27i} & \Sigma_{28i} \\
\Sigma_{37i} & \Sigma_{38i} \\
\Sigma_{47i} & \Sigma_{48i} \\
\Sigma_{57i} & \Sigma_{58i} \\
\Sigma_{67i} & \Sigma_{68i}
\end{bmatrix}^T + \begin{bmatrix}
\Sigma_{17i} & \Sigma_{18i} \\
\Sigma_{27i} & \Sigma_{28i} \\
\Sigma_{37i} & \Sigma_{38i} \\
\Sigma_{47i} & \Sigma_{48i} \\
\Sigma_{57i} & \Sigma_{58i} \\
\Sigma_{67i} & \Sigma_{68i}
\end{bmatrix} \cdot \left( \eta \cdot \hat{P}_0^{-1} \right) \cdot \begin{bmatrix}
\Pi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(9b)
\[ \Sigma_{11} = P_0 A_i + A_i T P_0 + P_1 + P_3 + \varepsilon_i^{-1} \cdot F_i T F_i, \quad \Sigma_{12} = -P_0 B_i K_i, \quad \Sigma_{14} = P_0 A_2, \quad \Sigma_{15} = P_0 , \]

\[ \Sigma_{16} = P_0 B_2, \quad \Sigma_{17} = C_i T , \quad \Sigma_{28} = -K_i D_i T, \quad \Sigma_{33} = -P_i , \]

\[ \Sigma_{44} = -P_3, \quad \Sigma_{47} = C_i T, \quad \Sigma_{55} = -\varepsilon_i^{-1} \cdot I, \quad \Sigma_{58} = I, \quad \Sigma_{66} = -\vec{\nu} \cdot I, \]

\[ \Pi = T \cdot R_{12} + Q_{12} [I - I \ 0] + [I - I \ 0] T R_{12}^T + R_{12} [0 \ I - I] + [0 \ I - I] T R_{12}^T. \]

Pre- and post-multiplying the matrix \( \Sigma \) in (9b) by

\[
\text{diag} \begin{bmatrix} P_0^{-1} & P_0^{-1} & P_0^{-1} & \varepsilon_i \cdot I & I \end{bmatrix} = \text{diag} \begin{bmatrix} \hat{P}_0 & \hat{P}_0 & \hat{P}_0 & \hat{P}_0 & \varepsilon_i \cdot I & I \end{bmatrix} > 0
\]

with

\[
\vec{\nu} = \gamma^2, \quad \hat{P}_0 = P_0^{-1}, \quad \hat{P}_i = P_0^{-1} P_i P_0^{-1}, \quad i \in \{1, 2, 3\}, \quad \hat{K}_i = K_i P_0^{-1},
\]

we have

\[
\begin{bmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} & 0 & \hat{\Sigma}_{14} & \hat{\Sigma}_{15} & \hat{\Sigma}_{16} \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & \hat{\Sigma}_{33} & 0 & 0 & 0 \\ * & * & * & \hat{\Sigma}_{44} & 0 & 0 \\ * & * & * & * & \hat{\Sigma}_{55} & 0 \\ * & * & * & * & * & \hat{\Sigma}_{66} \end{bmatrix} + \begin{bmatrix} \hat{\Sigma}_{17} & \hat{\Sigma}_{18} \end{bmatrix} T + \begin{bmatrix} \hat{\Sigma}_{18} \end{bmatrix} + \begin{bmatrix} \hat{\Sigma}_{27} & \hat{\Sigma}_{28} \end{bmatrix} + \begin{bmatrix} \hat{\Sigma}_{28} \end{bmatrix} = \begin{bmatrix} \hat{\Sigma}_{33} & \hat{\Sigma}_{44} & \hat{\Sigma}_{55} & \hat{\Sigma}_{66} \end{bmatrix}, \quad (10)
\]

where \( \hat{\Sigma}_{40}, \ k, l \in \{1, 2, \cdots, 8\} \), \( \hat{\Pi} \) are defined in (5e). By Lemma 1, LMI \( \hat{\Sigma}_i < 0 \) in (5d) is equivalent to \( \hat{\Sigma}_i < 0 \) in (10). The condition \( \hat{\Sigma}_i < 0 \) in (10) is also equivalent to \( \Sigma_i < 0 \) in (9b) for all \( i \in \{1, \cdots, m\} \). From (7) and (9a) with the condition \( \Sigma_i < 0 \) and \( w(t) = 0 \), there exists \( \rho > 0 \) such that

\[
\dot{V}(x_i)\big|_{w(t)=0} \leq -\rho \cdot \|x(t)\|^2.
\]

We conclude that the switched system (4) with (2), and \( w(t) = 0 \) is asymptotically stabilizable by sampled-data input in (3) with (5). Integrating the equation in (9a) from 0 to \( \Xi > 0 \) with \( \Sigma_i < 0 \), we have

\[
V(x_2) - V(\phi) + \int_0^3 \|y(t)\|^2 dt - \gamma^2 \cdot \int_0^3 \|w(t)\|^2 dt \leq 0.
\]

With zero initial condition (i.e. \( \phi = 0 \) in (1c)), we have

\[
V(\phi) = 0, \quad V(x_2) \geq 0, \quad \int_0^3 \|y(t)\|^2 dt \leq \gamma^2 \cdot \int_0^3 \|w(t)\|^2 dt, \quad w(t) \in L_2[0, \infty).
\]

By Definition 1, the system (4) with (2) is stabilizable by sampled-data input (3) with \( K_i = \hat{K}_i \hat{P}_0^{-1} \), and disturbance attenuation \( \gamma = \sqrt{\vec{\nu}} \).
Consider the system (1) with nonlinear and linear fractional perturbations in the following form:

\[
\dot{x}(t) = \tilde{A}_i(t)x(t) + \tilde{A}_2(t)x(t - \tau) + \Delta f_\sigma(x(t)) + \tilde{B}_i(t)y(t) + \tilde{B}_2(t)y(t), \quad t \geq 0,
\]

(11a)

\[
y(t) = \tilde{C}_i(t)x(t) + \tilde{C}_2(t)x(t - \tau) + \tilde{D}_i(t)y(t) + \tilde{D}_2(t)y(t), \quad t \geq 0,
\]

(11b)

\[
x(t) = \varphi(t), \quad t \in [-\tau, 0],
\]

(11c)

where \(\tilde{A}_i(t), \tilde{A}_2(t), \tilde{B}_i(t), \tilde{B}_2(t), \tilde{C}_i(t), \tilde{C}_2(t), \tilde{D}_i(t), \) and \(\tilde{D}_2(t)\) are defined by

\[
\begin{bmatrix}
\tilde{A}_i(t) & \tilde{A}_2(t) \\
\tilde{B}_i(t) & \tilde{B}_2(t) \\
\tilde{C}_i(t) & \tilde{C}_2(t) \\
\tilde{D}_i(t) & \tilde{D}_2(t)
\end{bmatrix} = \begin{bmatrix}
A_i(t) + \Delta A_i(t) & A_2(t) + \Delta A_2(t) \\
B_i(t) + \Delta B_i(t) & B_2(t) + \Delta B_2(t) \\
C_i(t) + \Delta C_i(t) & C_2(t) + \Delta C_2(t) \\
D_i(t) + \Delta D_i(t) & D_2(t) + \Delta D_2(t)
\end{bmatrix},
\]

(12a)

where constant matrices \(A_i, A_2, B_i, B_2, C_i, C_2, D_i,\) and \(D_2\) are given, \(\Delta A_i(t), \Delta A_2(t), \Delta B_1(t), \Delta B_2(t), \Delta C_i(t), \Delta C_2(t), \Delta D_i(t),\) and \(\Delta D_2(t)\) are some perturbed matrices and satisfying

\[
\begin{bmatrix}
\Delta A_i(t) & \Delta A_2(t) & \Delta B_1(t) & \Delta B_2(t) \\
\Delta C_1(t) & \Delta C_2(t) & \Delta D_i(t) & \Delta D_2(t)
\end{bmatrix} = \begin{bmatrix}
M_{1,1} & 0 & 0 & 0 \\
0 & M_{2,1} & 0 & 0 \\
0 & 0 & M_{3,1} & 0 \\
0 & 0 & 0 & M_{4,1}
\end{bmatrix} \begin{bmatrix}
N_{1,1} & N_{1,2} & N_{1,3} & N_{1,4} \\
N_{2,1} & N_{2,2} & N_{2,3} & N_{2,4} \\
N_{3,1} & N_{3,2} & N_{3,3} & N_{3,4} \\
N_{4,1} & N_{4,2} & N_{4,3} & N_{4,4}
\end{bmatrix}, \quad i \in \{1, 2, \ldots, N\},
\]

(12b)

\[
\begin{bmatrix}
Q_{1,1} & 0 & 0 & 0 \\
0 & Q_{2,1} & 0 & 0 \\
0 & 0 & Q_{3,1} & 0 \\
0 & 0 & 0 & Q_{4,1}
\end{bmatrix} < I,
\]

(12c)

where \(M_{1,1}, M_{2,1}, M_{3,1}, M_{4,1}, N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, N_{2,1}, N_{2,2}, N_{2,3}, N_{2,4}, N_{3,1}, N_{3,2}, N_{3,3}, N_{3,4}, N_{4,1}, N_{4,2}, N_{4,3}, N_{4,4}\) are some given constant matrices with appropriate dimensions. \(\Delta A_i(t)\) and \(\Delta A_2(t)\) are some given constant matrices representing the parameter perturbations which satisfies

\[
F_{x_i}(t) = \Theta_{x_i} < 0, \quad h \in \{x, y\},
\]

(12d)

\[
\Delta f_\sigma(x(t)) \leq I, \quad h \in \{x, y\}.
\]

(12e)

**Lemma 2.** [14] Suppose that \(\Delta(t)\) is defined in (12c) and satisfying (12d)-(12e), then for real matrices \(U_i, W_i\) and \(X_i\) with \(X_i = X_i^T\), the following statements are equivalent:

(I) The inequality is satisfied

\[
X_i + U_i\Delta_i(t)W_i + W_i^T\Delta_i(t)U_i^T < 0,
\]

(II) There exists a scalar \(\mu_i > 0\), such that

\[
\begin{bmatrix}
X_i & \mu_i \cdot U_i^T & W_i^T \\
* & -\mu_i \cdot I & \mu_i \cdot \Theta_i \\
* & * & -\mu_i \cdot I
\end{bmatrix} < 0,
\]

(12f)

where the matrix \(\Theta_i\) is defined in (12d).

From Theorem 1 with the switched system in (11) with (2), (12) and Lemma 2, we can obtain the following results.

**Theorem 2.** Suppose for a given constant \(\eta > 0\), the following optimization problem:

\[
\text{Minimize } \overline{\gamma},
\]

(13a)

Subject to

(i) (5b), (5c),
Consider the switched system (11) with (2) and (12), the stability LMI condition in (5d) should have a solution with positive constants $\bar{\gamma}$, $\varepsilon$, positive definite symmetric matrices $\hat{P}_0$, $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$, $\hat{Q}_{12}$, $\hat{R}_{12} \in \mathbb{R}^{n \times n}$, $\hat{Q}_{1i} \in \mathbb{R}^{3n \times 3n}$, $\hat{R}_{1i} \in \mathbb{R}^{3n \times 3n}$, matrices $\hat{Q}_{12} \in \mathbb{R}^{3n \times 3n}$, $\hat{R}_{12} \in \mathbb{R}^{3n \times 3n}$, $\hat{K}_i \in \mathbb{R}^{p \times n}$, where $\hat{\Sigma}_{kl}^j$, $k, l \in \{1, 2, \cdots, 9\}$, and $\hat{\Pi}$ are defined in (5e). Then the system (11) with (2) and (12) is stabilizable by sampled-data input (3) with $K_i = \hat{K}_i \hat{P}_0^{-1}$ and disturbance attenuation $\gamma = \sqrt{\bar{\gamma}}$.

**Proof.** Consider the switched system (11) with (2) and (12), the stability LMI condition in (5d) should be rewritten as follows:

$$
\begin{bmatrix}
\dot{\hat{\Sigma}}_{11} & \dot{\hat{\Sigma}}_{12} & 0 & \dot{\hat{\Sigma}}_{14i} & \dot{\hat{\Sigma}}_{15i} & \dot{\hat{\Sigma}}_{16i} & \dot{\hat{\Sigma}}_{17i} & \dot{\hat{\Sigma}}_{18i} & \dot{\hat{\Sigma}}_{19i} & e_i \cdot M_{xi} & 0 & \dot{\hat{P}}_0 N_{11}^T & \dot{\hat{P}}_0 N_{15}^T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

has a solution with positive constants $\bar{\gamma}$, $\varepsilon$, positive definite symmetric matrices $\hat{P}_0$, $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$, $\hat{Q}_{12}$, $\hat{R}_{12} \in \mathbb{R}^{n \times n}$, $\hat{Q}_{1i} \in \mathbb{R}^{3n \times 3n}$, $\hat{R}_{1i} \in \mathbb{R}^{3n \times 3n}$, matrices $\hat{Q}_{12} \in \mathbb{R}^{3n \times 3n}$, $\hat{R}_{12} \in \mathbb{R}^{3n \times 3n}$, $\hat{K}_i \in \mathbb{R}^{p \times n}$, where $\hat{\Sigma}_{kl}^j$, $k, l \in \{1, 2, \cdots, 9\}$, and $\hat{\Pi}$ are defined in (5e). Then the system (11) with (2) and (12) is stabilizable by sampled-data input (3) with $K_i = \hat{K}_i \hat{P}_0^{-1}$ and disturbance attenuation $\gamma = \sqrt{\bar{\gamma}}$.

Proof. Consider the switched system (11) with (2) and (12), the stability LMI condition in (5d) should be rewritten as follows:

$$
\begin{bmatrix}
M_{xi} \\
0 \\
\vdots \\
0
\end{bmatrix}
+ \\
\begin{bmatrix}
\dot{\hat{\Sigma}}_{11} & \dot{\hat{\Sigma}}_{12} & 0 & \dot{\hat{\Sigma}}_{14i} & \dot{\hat{\Sigma}}_{15i} & \dot{\hat{\Sigma}}_{16i} & \dot{\hat{\Sigma}}_{17i} & \dot{\hat{\Sigma}}_{18i} & \dot{\hat{\Sigma}}_{19i} & e_i \cdot M_{xi} & 0 & \dot{\hat{P}}_0 N_{11}^T & \dot{\hat{P}}_0 N_{15}^T
\end{bmatrix}
\begin{bmatrix}
\Delta_{xi}(t) \\
0 \\
\vdots \\
0
\end{bmatrix}
=$$

has a solution with positive constants $\bar{\gamma}$, $\varepsilon$, positive definite symmetric matrices $\hat{P}_0$, $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$, $\hat{Q}_{12}$, $\hat{R}_{12} \in \mathbb{R}^{n \times n}$, $\hat{Q}_{1i} \in \mathbb{R}^{3n \times 3n}$, $\hat{R}_{1i} \in \mathbb{R}^{3n \times 3n}$, matrices $\hat{Q}_{12} \in \mathbb{R}^{3n \times 3n}$, $\hat{R}_{12} \in \mathbb{R}^{3n \times 3n}$, $\hat{K}_i \in \mathbb{R}^{p \times n}$, where $\hat{\Sigma}_{kl}^j$, $k, l \in \{1, 2, \cdots, 9\}$, and $\hat{\Pi}$ are defined in (5e). Then the system (11) with (2) and (12) is stabilizable by sampled-data input (3) with $K_i = \hat{K}_i \hat{P}_0^{-1}$ and disturbance attenuation $\gamma = \sqrt{\bar{\gamma}}$.
where $\hat{\Sigma}_i$ is defined in (5d). By Lemma 2, condition (13b) will imply $\hat{\Sigma}_i < 0$. This proof is completed.

**Remark 1.** For a given constant $\eta > 0$, the optimization problem in Theorems 1 and 2 can be solved by the LMI Toolbox of Matlab. Simple “for loop” can be used to find the minimization of $\bar{\eta}$.

**Remark 2.** If $\gamma > 0$ is a known parameter, we can use the LMIs (5b), (5c), and (13b) for a given value $\bar{\eta} = \gamma^2$ to find the feasible solution.
3. NUMERICAL EXAMPLE

Consider the switched system (11) with the following parameters:

\[ N = 2, \quad A_{11} = \begin{bmatrix} -1.8 & 0.1 \\ 0.3 & -1.6 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -1.9 & 0 \\ -0.1 & -1.5 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} -0.6 & 0.05 \\ -0.2 & -0.5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.4 & 0.1 \\ -0.25 & -0.6 \end{bmatrix}, \]

\[ B_{11} = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, \]

\[ C_{11} = [1 \ 0], \quad C_{12} = [0.8 \ -0.1], \quad C_{21} = [-0.8 \ 0.6], \]

\[ C_{22} = [-0.2 \ 1], \quad D_{1i} = 0.3, \quad D_{2i} = -0.5, \quad F_i = 0.01, \quad \tau = 5, \quad M_{x_i} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}, \quad M_{y_i} = 0.01, \]

\[ \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \end{bmatrix}, \quad \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad N_{5i} = N_{6i} = 0, \quad N_{7i} = N_{8i} = 0, \]

\[ \Theta_{x_i} = 0.1 \cdot I, \quad \Theta_{y_i} = 0.1, \quad i = 1, 2. \quad (14) \]

(A) If the upper bound of sampled time is given by \( T = 0.2 \), the optimization problem in Theorem 2 with \( \eta = 29 \) has a feasible solution

\[ \bar{\gamma} = 0.2967, \quad \hat{K}_1 = \begin{bmatrix} 0.2286 & 0.094 \end{bmatrix}, \quad \hat{K}_2 = \begin{bmatrix} -0.0222 & -0.1943 \end{bmatrix}, \quad \hat{P}_0 = \begin{bmatrix} 0.4855 & 0.1913 \\ 0.1913 & 0.1868 \end{bmatrix}. \]

The system (11) with (2), (12), and (14) is stabilizable by sampled-data input in (3) with

\[ K_1 = \hat{K}_1 \hat{P}_0^{-1} = \begin{bmatrix} 0.4569 & 0.0356 \end{bmatrix}, \quad K_2 = \hat{K}_2 \hat{P}_0^{-1} = \begin{bmatrix} 0.6097 & -1.6642 \end{bmatrix}. \]

The disturbance attenuation is given by \( \gamma = \sqrt{\bar{\gamma}} = 0.5447 \). In this case, the \( H_\infty \) performance can be guaranteed via the stabilizing \( H_\infty \) control in (3) with the sampled time less than 0.2 second.

(B) If the upper bound of sampled time is given by \( T = 0.3 \), the optimization problem in Theorem 2 with \( \eta = 33 \) has a feasible solution

\[ \bar{\gamma} = 0.3002, \quad \hat{K}_1 = \begin{bmatrix} 0.1247 & 0.0736 \end{bmatrix}, \quad \hat{K}_2 = \begin{bmatrix} -0.0336 & -0.0986 \end{bmatrix}, \quad \hat{P}_0 = \begin{bmatrix} 0.4507 & 0.1891 \\ 0.1891 & 0.1813 \end{bmatrix}. \]

The system (11) with (2), (12), and (14) is stabilizable by sampled-data input in (3) with

\[ K_1 = \hat{K}_1 \hat{P}_0^{-1} = \begin{bmatrix} 0.1893 & 0.2084 \end{bmatrix}, \quad K_2 = \hat{K}_2 \hat{P}_0^{-1} = \begin{bmatrix} 0.2733 & -0.829 \end{bmatrix}. \]

The disturbance attenuation is given by \( \gamma = \sqrt{\bar{\gamma}} = 0.5479 \). In this case, the \( H_\infty \) performance can be guaranteed via the stabilizing \( H_\infty \) control in (3) with the sampled time less than 0.3 second. For larger upper bound of sampling time, the worse disturbance attenuation effect will be obtained.

4. CONCLUSIONS

The robust \( H_\infty \) control for uncertain switched time-delay systems with linear fractional perturbations has been investigated in this paper. Time-varying delay approach has been used instead of sampled-data input.
Some additional nonnegative inequalities have been applied to reduce the conservativeness of the obtained results.

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