INVERSE AND DIRECT SYSTEM IN CATEGORY OF FUZZY MODULES

Cigdem Gunduz (Aras)¹ and Sadi Bayramov²

Abstract: In this study, we first define the concept of inverse and direct system in category of fuzzy modules. We investigate whether or not limit of inverse (direct) system of exact sequences of fuzzy modules is exact. Later, we show that direct system limit of exact sequences of fuzzy modules is exact. Generally, limit of inverse system of exact sequences is not exact. Then we define the notion \( \lim (\) which is first derived functor of the inverse limit functor.

1. INTRODUCTION

Recently, methods of homology algebra had a widespread application in fuzzy topology. Lopez-Permouth and Malik [7] introduced category of fuzzy modules which is shown the notion \( R–fzmod \). \( R–fzmod \) has products, coproducts, kernel and cokernels but it is not an abelian category. Also, projective, injective and free fuzzy left \( R \)–modules are characterized. Zahedi and Ameri [12] defined fuzzy exact sequence in category of fuzzy modules and obtained some results about these notions. Same researchers [1] introduced the notions of fuzzy (co)homology, fuzzy exact sequence of fuzzy complexes and fuzzy homotopy. Up to the present, inverse and direct systems and their limits are defined in different categories. Furthermore, a series of their properties are investigated. Sheng-Gang Li [10, 11] defined inverse (direct) systems of topological spaces and their limits and obtained their properties for the case of category \( L–Top \). M. Ghadiri and B. Davvaz [4] introduced direct system and direct limit of \( H \)–modules. Violeta Leoreanu [6] proved that the direct limit (inverse limit) of an \( SHR \) direct (respectively, inverse) family of \( SHR \) semigroups is an \( SHR \) semigroup.

In this study, we firstly give the concept of inverse and direct systems in the category of fuzzy modules and prove that their limits always exist in this category. We investigate whether or not the limits of inverse and direct system of exact sequences of fuzzy modules are exact. Later, we show that direct system limit of exact sequences of fuzzy modules is exact. Generally, the inverse system limit of exact sequences is not exact. Then we define the notion \( \lim (\) which is first derived functor of the inverse limit functor. Finally by using this notion \( \lim (\) we prove that the inverse system limit of exact sequences of fuzzy modules is exact and we investigate

¹ Department of Mathematics, Kocaeli University, 41380, Kocaeli, Turkey. E-mail: carasgunduz@gmail.com
² Department of Mathematics, Kafkas University, 36100, Kars, Turkey. E-mail: baysadi@gmail.com
that \( \lim_{n \to \infty} (M_n, \mu_n) = 0 \).

2. PRELIMINARIES

Firstly, let us give some necessary definitions and use the same notations as in [7]. Let \( R \) be a ring (possibly without 1). A fuzzy left (right) \( R \)–module is defined to be a pair \((M, \mu)\) where \( M \) is a left (right) \( R \)–module and \( \mu : M \to [0, 1] \) is a function satisfying

(i) \( \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \),

(ii) \( \mu(-x) = \mu(x) \),

(iii) \( \mu(0) = 1 \), and

(iv) \( \mu(rx) \geq \mu(x) \) for all \( x, y \in M \) and \( r \in R \).

As is standard in the fuzzy set literature, we will refer to \( \mu \) as the grade function of \((M, \mu)\). We also say that \( \mu \) is a modular grade function for \( M \).

Given two fuzzy \( R \)–modules \((M, \mu)\) and \((N, h)\) and an \( R \)–homomorphism \( f : M \to N \), we say that \( f \) is a fuzzy \( R \)–homomorphism from \((M, \mu)\) into \((N, h)\) if

\[
\mu(x) \geq \mu(f(x)) \geq h(f(x)) \quad \text{for all} \quad x \in M.
\]

Lemma 2.1: [7] Let \( M \) and \( N \) be \( R \)–modules and \( f : M \to N \) be an \( R \)–homomorphism.

(i) If \((M, \mu)\) is a fuzzy \( R \)–module, there exists a modular grade function \( \mu' \) on \( N \) such that for any modular grade function \( \eta \) on \( N \), \( \mu'(x) = \sup\{\mu(x) : f(x) = y\}, \eta(y) = \eta(f(y)) \), for all \( x \in M \).

(ii) If \((N, \eta)\) is a fuzzy \( R \)–module, there exists a modular grade function \( \eta' \) on \( M \) such that for any fuzzy \( R \)–module \((M, \mu)\), \( \eta'(x) = \inf\{\mu(x) : f(x) = y\}, \eta(y) = \eta(f(y)) \), for all \( x \in M \).

Lemma 2.2: [7] (i) Given modules \( \{M_i | i \in I\} \) and \( N \) and a family of \( R \)–homomorphisms \( kA = \{f_i : M_i \to N | i \in I\} \), if \( \{(M_i, \mu_i) | i \in I\} \) are fuzzy modules, there exists a smallest grade function \( \eta = \mu^{\text{dim}} = \mu^{(i)} \) such that, for all \( i \in I \), \( f_i : (M_i, \mu_i) \to (N, \eta) \).

(ii) Given modules \( M \) and \( \{N_i | i \in I\} \) and \( R \)–homomorphisms \( B = \{g_i : M \to N_i | i \in I\} \), if \( (N_i, \eta_i) \) are fuzzy modules, then there exists a largest grade function \( \mu = \eta^{\text{dim}} = \eta^{(i)} \) such that, for all \( i \in I \), \( g_i : (M, \mu) \to (N_i, \eta_i) \).

Definition 2.3: [1] A fuzzy chain complex \( \theta_c = \{(\theta^a_c, \delta_a) \}_{a \in \Lambda} \) over \( \Lambda \) is an object in \( \text{fgm}^{\text{\Lambda}} \) together with a fuzzy endomorphism \( \delta : \theta_c \to \theta_c \) of degree \(-1\) with

Remark 2.4: [1] Let \( \theta_c = \{(\theta^a_c, \delta_a) \}_{a \in \Lambda} \) be a fuzzy chain complex. The condition \( \delta \delta = 0 \) implies that \( \text{Im} \, \delta_{\alpha+1} \subseteq \ker \delta_a \), \( \alpha \in \mathbb{Z} \). Hence, we can associate with \( \theta_c \) the fuzzy graded module
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\[ H(\theta_c) = \{ H_n(\theta_c) \}, \]

where \( H_n(\theta_c) = \bar{\theta}_n / \bar{\theta}_n \) is the fuzzy quotient of \( \ker \bar{\theta}_n \) by \( \text{im} \bar{\theta}_{n+1} \), \( n \in \mathbb{Z} \). Then \( H(\theta_c) (H_n(\theta_c)) \) is called the \((n)th\) fuzzy homology module of \( \theta_c \) (of course, if \( \Lambda = \mathbb{Z} \) we can speak of the \((n)th\) fuzzy homology group of \( \theta_c \)).

**Theorem 2.5:** [1] For each \( n, H_n(-) : \text{FComp} \to \Lambda - \text{fzmod} \) is an additive functor.

**Definition 2.6:** [1] A fuzzy homotopy \( \tilde{\Sigma} : \tilde{\phi} \to \tilde{\psi} \) between two fuzzy chain maps \( \tilde{\psi}, \tilde{\phi} : \theta_c \to \nu_D \) is a morphism of degree +1 of fuzzy graded modules \( \tilde{\Sigma} : \theta_c \to \nu_D \) such that \( \psi - \phi = \sum \hat{\Sigma}_n \hat{\phi}_n \), i.e., such that for \( n \in \mathbb{Z} \)

\[ \tilde{\psi}_n - \tilde{\phi}_n = \hat{\phi}_n + \Sigma \hat{\phi}_n. \]

**Proposition 2.7:** [1] If two fuzzy chain maps \( \tilde{\psi}, \tilde{\phi} : \theta_c \to \nu_D \) are fuzzy homotopic, then

\[ H(\tilde{\phi}) = H(\tilde{\psi}) : H(\theta_c) \to H(\nu_D). \]

\( R - \text{fzmod} \) denotes the category of all fuzzy \( R \) modules.

**Definition 2.8:** Any functor \( D : \Lambda^{\text{op}} \to R - \text{fzmod} \) (\( D : \Lambda \to R - \text{fzmod} \)), where \( \Lambda \) is a directed set (considered as a category), is called the inverse (direct) system of fuzzy modules; the limit of \( D \) is called limit of inverse (direct) system [2].

Let

\[ (M, \mu) = \left\{ \left\{(M_a, \mu_a) \right\}_{a \in \Lambda}, \left\{ F^a_{\mu} : (M_a, \mu_a) \to (M_{a'}, \mu_{a'}) \right\}_{a \to a'} \right\} \quad (2.1) \]

be inverse system of fuzzy modules, \( A = \left\{ p_{\beta} \prod_{a \in \Lambda} M_a \to M_{\beta} \right\}_{\beta \in \Lambda} \) be a family of projections, and \( \left( \prod_{a \in \Lambda} M_{a}, \mu_A \right) \) be the direct product of the fuzzy modules [7]. Then we get the fuzzy module

\[ \left( \text{lim}_{\alpha} M_{a}, \mu_{\alpha} \right) \left( \text{lim}_{\alpha} M_{a} \right). \]

**Theorem 2.9:** Every inverse system in representation (2.1) has limit in the category of \( R - \text{fzmod} \), and this limit is equal to the fuzzy module

\[ \left( \text{lim}_{\alpha} M_{a}, \mu_{\alpha} \right) \left( \text{lim}_{\alpha} M_{a} \right). \]
Proof: It suffices to show that, there exists a unique homomorphism of fuzzy modules \( \overline{\psi} : (N, \eta) \to \mathcal{F}_\mathcal{I} \circ \overline{\varphi}_\mathcal{I} \) which has the following commutative diagram:

Here for every fuzzy module \((N, \eta)\) and \(\alpha < \alpha'\), \(\{ \overline{\varphi}_\alpha : (N, \eta) \to (M_\alpha, \mu_\alpha) \}_{\alpha \in \Lambda}\) is the family of homomorphisms of fuzzy modules which makes up following commutative diagram:

\[
\begin{array}{ccc}
(N, \eta) & \xrightarrow{\overline{\varphi}_\alpha} & (M_\alpha, \mu_\alpha) \\
\downarrow{\overline{\varphi}_\alpha} & & \downarrow{\overline{\varphi}_\alpha} \\
(M_\alpha, \mu_\alpha) & \xrightarrow{\mu_\alpha} & (M_{\alpha'}, \mu_{\alpha'})
\end{array}
\]

and also \(\overline{\pi}_\alpha : \left( \lim_{\alpha} M_\alpha, \mu_\alpha \right) \to (M_\alpha, \mu_\alpha)\) is canonical projection. We define \(\psi : N \to \lim_{\alpha} M_\alpha\) as the homomorphism of modules such that for every \(x \in N\)

\[
\Psi(x) = \{ \varphi_\alpha(x) \}_{\alpha \in \Lambda}.
\]

Then, is the \(\overline{\psi} : (N, \eta) \to \left( \lim_{\alpha} M_\alpha, \mu_\alpha \right)\) an homomorphism of fuzzy modules? Since \(\overline{\varphi}_\alpha : (N, \eta) \to (M_\alpha, \mu_\alpha)\) is the homomorphism of fuzzy modules for every \(\alpha \in \Lambda\), the condition

\[
\mu_\alpha(\varphi_\alpha(x)) \geq \eta(x)
\]

is satisfied, for \(\forall x \in N\). Therefore, we obtain the condition

\[
\mu_\alpha(\varphi_\alpha(x)) = \bigwedge_{\alpha \in \Lambda} \mu_\alpha(\varphi_\alpha(x)) \geq \eta(x).
\]

Since \(\psi\) is unique homomorphism, \(\overline{\psi}\) is also unique homomorphism.

It’s clear that \(\lim_{\alpha}\) is a functor from the category of inverse system of fuzzy modules to the category of \(R - \text{fzmod}\).

We investigate whether or not limit of inverse system of exact sequences of fuzzy modules is exact.

Example 2.10: Let \(M_n = \mathbb{Z}, M'_n = \mathbb{Z}, M''_n = \mathbb{Z}_2\) be modules an \(\mathbb{Z}\) ring, \(\forall R \in \mathbb{N}\). Then,

\[
\begin{align*}
M &= \{ (M_n)_{n \in \mathbb{N}} : \{ p^{n+1}(m) = 3m \} \} \\
M' &= \{ (M'_n)_{n \in \mathbb{N}} : \{ q^{n+1}(m) = 3m \} \}
\end{align*}
\]
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\( M'' = \left\{ \left( M'_a \right)_{a \in \Lambda}, \left\{ f_{a+1}^a(m) = [m] \right\} \right\} \)

are inverse systems of modules and

\[
\begin{align*}
  f &= \{ f_a : M'_a \to M'_a | f_a(m) = 2m \} \\
  g &= \{ g_a : M'_a \to M'_a | g_a(m) = [m] \}
\end{align*}
\]

are morphisms of inverse systems. The following sequence

\[
0 \to M' \xrightarrow[f]{} M \xrightarrow[g]{} M'' \to 0
\]

is short exact sequence of inverse system of \( \mathbb{Z} - \) modules. Then the following sequence

\[
0 \to \left( M'_a, \mu'_a \right) \xrightarrow{f} \left( M'_a, \mu_a \right) \xrightarrow{g} \left( M''_a, \mu''_a \right) \to 0
\]

is also short exact sequence of fuzzy modules, where \( \mu_a = (\chi_{(0)})_{M'_a}, \mu'_a = (\chi_{(0)})_{M'_a}, \mu''_a = (\chi_{(0)})_{M''_a} \) [12].

Hence, the sequence

\[
0 \to \left( M'_a, \mu'_a \right) \xrightarrow{f} \left( M'_a, \mu_a \right) \xrightarrow{g} \left( M''_a, \mu''_a \right) \to 0
\]

is short exact sequence of inverse system of fuzzy modules. Taking the limits of this sequence, the sequence

\[
0 \to 0 \to 0 \to \left( \mathbb{Z}_2, \mu'' \right) \to 0
\]

is not exact.

We get inverse system in (2.1). We define the following homomorphism of modules

\[
d : \prod M_a \to \prod M_a
\]

by the formula

\[
d \left( \left\{ x_a \right\} \right) = \left\{ x_a - p_a^x (x_a) \right\}_{a \in \Lambda}.
\]

Here, is the \( \overrightarrow{d} : \left( \prod_{a \in \Lambda} M_a, \mu_A \right) \to \left( \prod_{a \in \Lambda} M_a, \mu_A \right) \) an homomorphism of fuzzy modules? Indeed,

\[
\begin{align*}
  \mu_a \left( d \left( \left\{ x_a \right\} \right) \right) &= \mu_a \left( \left\{ x_a - p_a^x (x_a) \right\} \right) = \alpha \in \Lambda \left( x_a - P^x_a (x_a) \right) \\
  &\geq \alpha \in \Lambda \min \left\{ \mu_a(x_a), \mu_a \left( p_a^x (x_a) \right) \right\}.
\end{align*}
\]

Since \( \mu_a (p_a^x (x_a)) \geq \mu_a (x_a) \),

\[
\begin{align*}
  \mu_a \left( d \left( \left\{ x_a \right\} \right) \right) &\geq \alpha \in \Lambda \min \left\{ \mu_a (x_a), \mu_a (x_a) \right\} = \alpha \in \Lambda (\mu_a (x_a) \land \mu_a (x_a)) \\
  &= \alpha \in \Lambda \mu_a (x_a) = \mu_a \left( \left\{ x_a \right\} \right).
\end{align*}
\]

Since \( \overrightarrow{d} \) is the homomorphism of fuzzy modules, we can de\^-ne fuzzy modules \( \text{ker} \overrightarrow{d} \) and
For the inverse system of modules
\[ \left\{ M_a \right\}_{a \in \Lambda}, \ \{ \beta \rightarrow M_{a'} \}_{a' < a} \right\} \]
\[ \lim_{\alpha} \left( \prod_{a \in \Lambda} M_a \mid Imd \right) [9]. \]

If \( \pi: \prod_{a \in \Lambda} M_a \rightarrow \lim_{\alpha} M_a \) is the canonical epimorphism, it can be defined fuzzy module
\[ \left( \lim_{\alpha} M_a, (\mu_A)^{x} \right). \]

**Definition 2.11:** \( \left( \lim_{\alpha} M_a, (\mu_A)^{x} \right) \) is called “first derived functor” of the inverse limit functor of inverse system of fuzzy modules given in (2.1). This functor is denoted by \( \lim_{\alpha} \left( \cdot \right) \).

**Proposition 2.12:** \( \lim_{\alpha} \left( \cdot \right) \) is a functor from the category of inverse systems of fuzzy modules to the category of \( R - fzmod \).

**Proof:** For this reason, it suffices to show that for each the morphism
\[ (\lim_{\alpha} \lim_{\alpha} M_a, (\mu_A)^{x}) \rightarrow \left( \lim_{\alpha} N_{\beta}, (\eta_B)^{x} \right) \]
is the homomorphism of fuzzy modules. Since
\[ (\mu_A)^{x}(x + Imd) = \sup_{z \in Imd} \mu_A(x + z) \leq \sup_{z \in Imd} \eta_A(f(x + z)) \]
\[ = \sup_{z \in Imd} \eta_A(f(x) + f(z)) = \sup_{y \in Imd} \eta_A(f(x) + y) \]
\[ \leq \lim_{\alpha} f(x + y) = (\eta_A)^{x} \lim_{\alpha} f(x + Imd) \],
\( \lim_{\alpha} \left( \cdot \right) \) is a functor.

Let us take the following cochain complex
\[ \mathcal{O} \rightarrow (\prod M_a, \mu_A) \rightarrow (\prod M_a, \mu_A) \rightarrow \mathcal{O}. \]

Fuzzy cohomolgy modules which are not trivial of this cochain complex are ker \( d \) and co ker \( d \).

**Lemma 2.13:** \( \lim_{\alpha} (M_a, \mu_A) = \ker d \) and \( \lim_{\alpha} (M_a, \mu_A) = \ker d \).
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Proof: The proof of lemma is trivial.

We accept natural numbers set which is index set of inverse system.

Theorem 2.14: Let the sequence

\[
(M_1, \mu_1) \xleftarrow{P_1^2} (M_2, \mu_2) \xrightarrow{P_2^3} \ldots
\]

be inverse sequence of fuzzy modules. For each infinite subsequence of this sequence, \( \lim_{(1)} \) doesn’t change.

Proof: Let \( S = \{i, j, k, \ldots\} \) be infinite subsequence of natural numbers \( N \). From Lemma 2.13., \( \lim_{(1)} \) is defined by the following homomorphism of fuzzy modules as appropriate subsequence \( S \):

\[
\overline{T} : \left( \prod_{s \in S} M, \prod_{s \in S} \mu_s \right) \rightarrow \left( \prod_{s \in S} M, \prod_{s \in S} \mu_s \right).
\]

We may define

\[
f_0, f_1 : \prod_{n \in N} M \rightarrow \prod_{n \in N} M
\]

homomorphisms of modules with this formula:

\[
f_0(x_i, x_j, x_k, \ldots) = (P^i_1(x_i), P^i_2(x_j), P^i_{i+1}(x_k), \ldots, x_i, x_j, x_k, \ldots)
\]

\[
f_1(x_i, x_j, x_k, \ldots) = (0, 0, \ldots, x_i, x_j, 0, \ldots, x_k, 0, \ldots)
\]

Since

\[
\left( \prod_{n \in N} \mu_n \right)(P^i_1(x_i), P^i_2(x_j), P^i_{i+1}(x_k), \ldots, x_i, x_j, x_k, \ldots)
\]

\[
\mu_i(P^i_1(x_i)) \land \ldots \land \mu_{i+1}(P^i_{i+1}(x_k)) \land \mu_i(x_i) \land \mu_{i+1}(P^i_{i+1}(x_k)) \land \ldots \land \mu_j(x_j) \land \ldots
\]

\[
\geq \left( \mu_i(x_i) \land \ldots \land \mu_i(x_i) \land \mu_{i+1}(x_k) \land \ldots \land \mu_j(x_j) \land \ldots \right) = \bigwedge_{s \in S} \mu_s(x_s)
\]

and

\[
\left( \prod_{n \in N} \mu_n \right)(0, 0, \ldots, x_i, 0, \ldots, x_j, 0, \ldots) = \mu_i(0) \land \ldots \land \mu_i(x_i) \land \mu_{i+1}(0) \land \ldots \land \mu_j(x_j) \land \ldots
\]

\[
= \bigwedge_{s \in S} \mu_s(x_s),
\]

\[
\overline{T}_0, \overline{T}_1 : \left( \prod_{s \in S} M, \prod_{s \in S} \mu_s \right) \rightarrow \left( \prod_{n \in N} M, \prod_{n \in N} \mu_n \right)
\]

are homomorphisms of fuzzy modules. It is clear that following diagram is commutative:
\( (\prod_{n \in N} M_n, \Lambda_{\mu_n}) \xrightarrow{\bar{\varphi}} (\prod_{n \in S} M_n, \Lambda_{\mu_n}) \xrightarrow{\bar{\varphi}} (\prod_{n \in N} M_n, \Lambda_{\mu_n}) \)

i.e., \( \{\bar{\varphi}_0, \bar{\varphi}_1\} \) are morphisms fuzzy cochain complexes. Now, let us define

\[
g_0, g_1 : \prod_{n \in N} M_n \rightarrow \prod_{n \in S} M_n
\]

homomorphisms with this formula:

\[
g_0(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots)
\]

\[
g_1(x_1, x_2, x_3, \ldots) = (x_1 + P_i^1(x_{i+1}) + \ldots + P_i^{j-1}(x_{j-1}), x_j + P_j^1(x_{j+1}) + \ldots + P_j^{k-1}(x_{k-1}), \ldots)
\]

For,

\[
\left(\Lambda_{\mu_n}\right)(x_1, x_2, x_3, \ldots) = \mu_n(x_1) \wedge \mu_n(x_2) \wedge \ldots \geq \Lambda_{\mu_n}(x_n)
\]

and

\[
\left(\Lambda_{\mu_s}\right)(x_1, P_i^1(x_{i+1}) + \ldots + P_i^{j-1}(x_{j-1}), x_j + \ldots + P_j^{k-1}(x_{k-1}), \ldots)
\]

\[
= \mu_s(x_1) + P_i^1(x_{i+1}) + \ldots + P_i^{j-1}(x_{j-1}) \wedge \mu_s(x_j + \ldots + P_j^{k-1}(x_{k-1})) \wedge \ldots
\]

\[
\geq \min \left\{\mu_s(x_1), \mu_i\left(P_i^1(x_{i+1})\right), \ldots, \mu_{i-1}\left(P_{i-1}^1(x_{i-1})\right)\right\} \wedge \ldots
\]

\[
\geq \min \left\{\mu_s(x_1), \mu_{i+1}(x_{i+1}), \ldots, \mu_{j-1}(x_{j-1})\right\} \wedge \ldots
\]

\[
= \Lambda_{\mu_n} \mu_m(x_n) \geq \Lambda_{\mu_n}(x_n).
\]

Thus \( \bar{g}_0, \bar{g}_1 : \left(\prod_{n \in N} M_n, \Lambda_{\mu_n}\right) \rightarrow \left(\prod_{n \in S} M_n, \Lambda_{\mu_s}\right) \) are homomorphisms of fuzzy modules and

\[ D' \circ \bar{g}_0 = \bar{g}_1 \circ \bar{\varphi} \]

are satisfied, i.e., \( \{\bar{g}_0, \bar{g}_1\} \) are homomorphisms of cochain complexes. It is clear that

\[ \bar{g}_0 \circ \bar{f}_0 = \bar{g}_1 \circ \bar{f} = \bar{f} \left(\prod_{n \in S} M_n, \Lambda_{\mu_s}\right) \]

Hence we give

\[ D : \prod_{n \in N} M_n \rightarrow \prod_{n \in N} M_n \]

homomorphisms of modules with this formula:

\[
D(x_1, x_2, \ldots) = (x_1 + P_1^2(x_2) + \ldots + P_1^{i-1}(x_{i+1}), x_2 + P_2^3(x_3) + \ldots + P_2^{i-1}(x_{i+1}), \ldots, x_{i-1}, 0,
\]
For

\[(x_1 + P^2(x_2) + ... + P^{i-1}(x_i), x_2 + P^2(x_3) + ... + P^{i-1}(x_i), ..., x_i, 0, ...)
\]

By using simplicity of calculation, it is shown that \(D\) is a fuzzy chain homotopy between \(f \circ g\) and \(g \circ f\) homomorphisms. Then the following cohomology modules of fuzzy cochain complexes

\[0 \rightarrow \left( \prod_{n \in \mathbb{N}} M_n, \bigwedge_{n \in \mathbb{N}} \mu_n \right) \xrightarrow{\partial} \left( \prod_{n \in \mathbb{N}} M_n, \bigwedge_{n \in \mathbb{N}} \mu_n \right) \rightarrow 0\]

\[0 \rightarrow \left( \prod_{s \in S} M_s, \bigwedge_{s \in S} \mu_s \right) \xrightarrow{\partial'} \left( \prod_{s \in S} M_s, \bigwedge_{s \in S} \mu_s \right) \rightarrow 0\]

are quasi isomorphic [1]. Since \(\lim\) is first cohomology module, the theorem is proved.

Since \(\lim (M_n, \mu_n) = \ker \partial\) and \(P^{n+1}(x_{n+1}) = x_n\) is satisfied for each \(x_n \in \lim M_n\),

\[\mu_n(x_n) = \mu_n(P^{n+1}(x_{n+1})) \geq \mu_{n+1}(x_{n+1})\]

i.e. for each \(x_n \in \ker \partial\), the sequence \(\{\mu_n(x_n)\}\) is decreasing sequence.

**Theorem 2.15:** For \(\forall \{x_n\} \in \ker d''\), if \(\lim_{n \to \infty} u_n(x_n) = 0\) and the following diagram is short exact sequence of inverse system of fuzzy modules

\[\begin{array}{ccccccccc}
0 & \longrightarrow & \phi_{\infty, \infty} & \longrightarrow & \phi_{\infty, \infty} & \longrightarrow & \phi_{\infty, \infty} & \longrightarrow & 0 \\
0 & \longrightarrow & \phi_{\infty, \infty} & \longrightarrow & \phi_{\infty, \infty} & \longrightarrow & \phi_{\infty, \infty} & \longrightarrow & 0 \\
\end{array}\]
then the sequence

\[ 0 \rightarrow \lim_{\leftarrow} (M'_n, \mu'_n) \rightarrow \lim_{\leftarrow} (M_n, \mu_n) \rightarrow \lim_{\leftarrow} (M'_n, \mu'_n) \rightarrow \lim_{\leftarrow} (M_n, \mu_n) \rightarrow 0 \]

is exact.

**Proof:** For an inverse system of fuzzy modules \( \{(M_n, \mu_n)\}_{n \in \mathbb{N}} \)

\[ C = 0 \rightarrow \left( \prod_{n \in \mathbb{N}} M_n, \mu_A \right) \rightarrow \left( \prod_{n \in \mathbb{N}} M'_n, \mu_A \right) \rightarrow 0 \rightarrow \cdots \]

is a cochain complexes of fuzzy modules.

\[ H^0(C) = \lim_{\leftarrow} (M_n, \mu_n), \ H^1(C) = \lim_{\leftarrow} (M'_n, \mu'_n), \ H^k(C) = 0, \ k \geq 2 \quad (2.2) \]

are fuzzy cohomology modules of this complexes. Similarly, for the inverse systems of fuzzy modules \( \{(M'_n, \mu'_n)\}, \{(M''_n, \mu''_n)\} \), we can constitute the following cochain complexes

\[ C' = 0 \rightarrow \left( \prod_{n \in \mathbb{N}} M'_n, \mu'_A \right) \rightarrow \left( \prod_{n \in \mathbb{N}} M''_n, \mu'_A \right) \rightarrow 0 \rightarrow \cdots \]

\[ C'' = 0 \rightarrow \left( \prod_{n \in \mathbb{N}} M''_n, \mu''_A \right) \rightarrow \left( \prod_{n \in \mathbb{N}} M'''_n, \mu''_A \right) \rightarrow 0 \rightarrow \cdots \]

It is clear that fuzzy cohomology modules of this complexes is the form in (2.2).

From the condition of this theorem, the following sequence

\[ 0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0 \]

is short exact sequence of cochain complexes of fuzzy modules. But generally the following sequence of fuzzy cohomology modules of this sequence

\[ 0 \rightarrow H^0(C') \rightarrow H^0(C) \rightarrow H^0(C'') \rightarrow H^1(C') \rightarrow H^1(C) \rightarrow H^1(C'') \rightarrow H^2(C') \rightarrow \cdots \]

is not exact, because \( \delta \) is usually not homomorphism of fuzzy modules [1]. Since \( H^0(C'') = \ker d'' \) and \( \lim_{n \to \infty} \mu''_n(\alpha'_n) = 0 \), grade function \( \mu'' \) of fuzzy module \( (H^0(C''), \mu'') \) is iqual to grade function \( \mu''_n \) [7]. Thus \( \delta \) is homomorphism of fuzzy modules. Therefore the sequence

\[ 0 \rightarrow H^0(C') \rightarrow H^0(C) \rightarrow H^0(C'') \rightarrow H^1(C') \rightarrow H^1(C) \rightarrow H^1(C'') \rightarrow H^2(C') \rightarrow \cdots \]

is exact. By using the (2.2), we obtain the following exact sequence of fuzzy modules

\[ 0 \rightarrow \lim_{\leftarrow} (M'_n, \mu'_n) \rightarrow \lim_{\leftarrow} (M_n, \mu_n) \rightarrow \lim_{\leftarrow} (M'_n, \mu'_n) \rightarrow \lim_{\leftarrow} (M_n, \mu_n) \rightarrow \cdots \]

\[ \rightarrow \lim_{\leftarrow} (M'_n, \mu'_n) \rightarrow \lim_{\leftarrow} (M_n, \mu_n) \rightarrow 0 \]
**Lemma 2.16:** Given the following inverse system of fuzzy modules

\[( M_1, \mu_1 ) \leftarrow \not\Delta \leftarrow ( M_2, \mu_2 ) \leftarrow \not\Delta \ldots \]

If each homomorphisms \( \overline{\Phi}_n \) are epimorphisms, then \( \lim_{\leftarrow} ( M_n, \mu_n ) = 0 \).

**Proof:** The proof is obvious, since

\[\overline{d} : \prod_{n=1}^{\infty} ( M_n, \mu_n ) \rightarrow \prod_{n=1}^{\infty} ( M_n, \mu_n )\]

is an epimorphism.

**Definition 2.17:** Given inverse system of \( R - fzmod \) in (2.3), for every integer \( n \), if there exists \( m \geq n \) such that

\[\text{Im} [( M_n, \mu_n ) \rightarrow ( M_m, \mu_m )] = \text{Im} [( M_m, \mu_m ) \rightarrow ( M_n, \mu_n )] \quad (\forall i \geq m),\]

then it is said that the inverse system in (2.3) satisfies the condition of Mittag-Leffler.

**Theorem 2.18:** If the inverse system in (2.3) satisfies the condition of Mittag-Leffler, then

\[\lim_{\leftarrow} ( M_n, \mu_n ) = 0.\]

**Proof:** Let us denote by \( M'_n = \text{Im} \Phi'_i \) for large \( i \). Then from condition of the theorem, the homomorphism \( \Phi_{n[M_{i+1}]} \) carries the module \( M_{i+1} \) on the module \( M'_n \). Then \( \Phi_{n[M_{i+1}]} \) is an epimorphism.

Thus for large \( i \), the homomorphisms

\[\overline{\Phi}_n : ( M_{n+1}, \mu_{n[M_{i+1}]} ) \rightarrow ( M'_n, \mu_{n[M_{i+1}]} )\]

are epimorphisms. Then by using Lemma 2.16, we have \( \lim_{\leftarrow} ( M'_n, \mu'_n ) = 0 \). Here \( \mu'_n = \mu_{n[M'_n]} \).

Let us take the following sequence of the inverse system of quotient modules

\[\left( M_{[M_{i+1}]} , \tilde{\mu}_i \right) \leftarrow \left( M_{[M_{i+1}]} , \tilde{\mu}_j \right) \leftarrow \ldots \quad (2.4)\]

For every \( n \), there exists \( m \geq n \) such that the homomorphism \( M_{[M_{i+1}]} \rightarrow M_{[M_{i+1}]} \) is zero homomorphism. Then \( \lim_{\leftarrow} ( M_{[M_{i+1}]} , \tilde{\mu}_n ) = 0 [2] \). Consequently the limit of inverse system in (2.4) is equal to 0. Therefore, \( \lim_{\leftarrow} ( M_{[M_{i+1}]} , \tilde{\mu}_n ) = 0 \) as well. Then let us consider the following short exact sequence of inverse system in category \( R - fzmod \)

\[0 \rightarrow \{( M'_n, \mu'_n )\} \rightarrow \{( M_n, \mu_n )\} \rightarrow \{( M_{[M_{i+1}]} , \tilde{\mu}_n )\} \rightarrow 0 \quad (2.5)\]

Granting that \( \lim_{\leftarrow} ( M_{[M_{i+1}]} ) = 0 \), we can apply to theorem 2.15 for the sequence (2.5), then we have the exact sequence
\[ 0 \to \lim_{\leftarrow} (M'_a, \mu'_a) \to \lim_{\leftarrow} (M_a, \mu_a) \to \lim_{\leftarrow} (M^\#, \mu^\#_a) \to \lim_{\leftarrow}^{(1)} (M'_a, \mu'_a) \to \lim_{\leftarrow}^{(1)} (M^\#, \mu^\#_a) \to 0 \]

Since \( \lim_{\leftarrow}^{(1)} (M'_a, \mu'_a) = 0 \), \( \lim_{\leftarrow} (M^\#, \mu^\#_a) = 0 \) and \( \lim_{\leftarrow}^{(1)} (M^\#, \mu^\#_a) = 0 \), then the sequence (2.6) would look like

\[ 0 \to \lim_{\leftarrow} (M'_a, \mu'_a) \to \lim_{\leftarrow} (M_a, \mu_a) \to 0 \to \lim_{\leftarrow}^{(1)} (M'_a, \mu'_a) \to 0 \to 0. \]

This proves that \( \lim_{\leftarrow}^{(1)} (M_a, \mu_a) = 0 \).

Now we will investigate the direct system of fuzzy modules. Let

\[
\left( \bar{M}, \bar{\mu} \right) = \left( \{M_a, \mu_a\}_{a \in \Lambda}, \left\{ \bar{P}_a : (M_a, \mu_a) \to (M'_a, \mu'_a) \right\}_{a \in \Lambda} \right)
\]

be direct system of fuzzy modules, \( \bigoplus_{\alpha} M_a, (\mu^\#) \) be fuzzy module and \( \pi : \bigoplus_{\alpha} M_a \to \lim_{\alpha} M_a \) be canonical epimorphism. Then we get the fuzzy module \( \lim_{\alpha} (\mu^\#)^\pi \) [7].

**Theorem 2.19:** Every direct system in representation (2.7) has limit on the category of \( R-\text{fzmod} \) and this limit is equal to the fuzzy module \( \lim_{\alpha} (\mu^\#)^\pi \).

**Proof:** It suffices to show that, there exists a unique homomorphism of fuzzy modules

\( \Psi : \left( \lim_{\alpha} M_a, (\mu^\#)^\pi \right) \to (N, \eta) \) which makes commutative following diagram:

\[
\begin{array}{ccc}
\Psi & \to & (N, \eta) \\
\downarrow & \downarrow & \downarrow \\
(\mu_a, \eta_a) & \overset{\Psi_a}{\to} & \left( \lim_{\alpha} M_a, (\mu^\#)^\pi \right) \\
\end{array}
\]

where \( \Psi = \{ \bar{\Psi}_a : (M_a, \mu_a) \to (N, \eta) \}_{a \in \Lambda} \) is the family of homomorphisms of fuzzy modules which makes commutative following diagram:

\[
\begin{array}{ccc}
\Psi & \to & (N, \eta) \\
\downarrow & \downarrow & \downarrow \\
(\mu_a, \eta_a) & \overset{\Psi_a}{\to} & (M_a, \mu_a) \\
\end{array}
\]
Inverse and Direct System in Category of Fuzzy Modules

and also \( \overline{i}_a : (M_a,\mu_a) \to \left( \bigoplus_{\alpha} M_{a_{\alpha}},\mu^a \right) \) are usual injections and \( \pi_a = \pi \circ i_a \). For every \( x \in \lim_{\alpha} M_a \), there exists \( x_a \in M_a \) such that \( \pi_a(x_a) = x \). If \( \pi_{a_{\alpha}}(x_{a_{\alpha}}) = x \) for each \( \pi_{a_{\alpha}} \in M_{a_{\alpha}} \), then \( \varphi_{a_{\alpha}}(x_{a_{\alpha}}) \) is equal to \( \varphi_a(x_a) \). We define the homomorphism \( \Psi : \lim_{\alpha} M_a \to N \) is defined by \( \Psi(x) = \varphi_{a_{\alpha}}(x_{a_{\alpha}}) \). Now, we can check that \( \overline{\Psi} \) is the homomorphism of fuzzy modules. For each \( x \in \lim_{\alpha} M_a \), let \( \pi \circ i_a(x_a) = x \). Here

\[
\left( \mu^a \right)^\pi(x) = \sup \left\{ \varphi_a(x_a) : \pi_a = x \right\}.
\]

Therefore,

\[
\eta(\Psi(x)) = \eta(\varphi_a(x_a)) \geq \mu_a(x_a).
\]

Since this inequality is satisfied for each \( x_a \) which satisfies \( \pi_a(x_a) = x \) we write the inequality

\[
\eta(\Psi(x)) \geq (\mu^a)^\pi(x).
\]

From the definition, it is obvious that above diagram is commutative.

We can easily show that \( \lim_{\alpha} \) is a functor from the category of direct system of fuzzy modules to the category of fuzzy modules.

Let

\[
\overline{M} = \left\{ \left( M_a,\mu_a \right)_{a \in \Lambda}, \left\{ f_a^a \right\}_{a \subseteq a'_{\alpha}} \right\}
\]

\[
\overline{M}' = \left\{ \left( M'_a,\mu'_a \right)_{a \in \Lambda}, \left\{ g_a^a \right\}_{a \subseteq a'_{\alpha}} \right\}
\]

\[
\overline{M}'' = \left\{ \left( M''_a,\mu''_a \right)_{a \in \Lambda}, \left\{ f''_a^a \right\}_{a \subseteq a'_{\alpha}} \right\}
\]

be direct systems of fuzzy modules, and the sequence

\[
\overline{M}' \xrightarrow{\tau} \overline{M} \xrightarrow{\pi} \overline{M}''
\]

be exact sequence of this system.

**Theorem 2.20:** If the direct limit functor is applied the sequence in (2.8), then the sequence

\[
\lim_{\alpha} (M'_a,\mu'_a) \to \lim_{\alpha} (M_a,\mu_a) \to \lim_{\alpha} (M''_a,\mu''_a)
\]

is exact as well.

**Proof:** Let the sequence (2.8) be exact. Then the ordinary sequence of \( R \) - module homomorphisms

\[
M'_a \to M_a \to M''_a
\]

is exact for \( \forall \alpha \in \Lambda \) [12]. Hence, the sequence

\[
\left\{ M'_a \right\}_a \xrightarrow{\left\{ f_a \right\}} \left\{ M_a \right\} \xrightarrow{\left\{ g_a \right\}} \left\{ M''_a \right\}
\]
is exact sequence of the direct system of ordinary modules. Therefore the limit of this exact sequence
\[
\lim_{\alpha} M'_\alpha \xrightarrow{f_\alpha} \lim_{\alpha} M_\alpha \xrightarrow{g_\alpha} \lim_{\alpha} M''_\alpha
\] (2.9)
is also exact. For the following sequence of fuzzy modules
\[
\left( \lim_{\alpha} M'_\alpha, (\mu^\alpha)^z \right) \xrightarrow{f_\alpha} \left( \lim_{\alpha} M_\alpha, (\mu^\alpha)^z \right) \xrightarrow{g_\alpha} \left( \lim_{\alpha} M''_\alpha, (\mu^{\alpha'})^z \right).
\]
\[
(\mu^\alpha)^z \xrightarrow{\text{Im } f_\alpha} (\mu^\alpha)^z \xrightarrow{\text{Ker } g_\alpha} (\mu^{\alpha'})^z
\]
is true, because sequence (2.9) is exact.

From theorem 2.20., we obtain following conclusion.

**Conclusion 2.21:** The functor of direct limit defenses monomorphism and epimorphism in the category of fuzzy modules.

Now, we want to constitute direct system of chain complexes.

Let \( \Lambda \) be directed set, for \( \Lambda \lambda \in \Lambda \)
\[
C(\lambda) = \left\{ \left( M_n^{(\alpha)}, (\mu^{(\alpha)}) \right), \bar{\delta}^{(\alpha)} : (M_n(\lambda), (\mu^\lambda)) \to (M_{n-1}(\lambda), (\mu^{\lambda-1}(\lambda))) \right\}
\]
be chain complexes of fuzzy modules and for \( \Lambda \lambda \times \mu, \bar{f}_{\lambda \mu} : C(\lambda) \to C(\mu) \) be morphism of chain complexes and \( \{ C(\lambda), \bar{f}_{\lambda \mu} \} \) be direct system of chain complexes.

**Theorem 2.22:** Homology module of limit of direct system of chain complexes of fuzzy modules is quasi isomorphic to limit of direct system of homology modules of chain complexes, i.e.
\[
H_\lambda \left( \lim_{\lambda} C(\lambda) \right) = \lim_{\lambda} H_\lambda (C(\lambda)).
\]

**Proof:** The proof of this theorem is done by using Conclusion 2.21. Hence,
\[
\lim_{\lambda} H_\lambda (C(\lambda)) = \lim_{\lambda} \left( \text{Ker } \bar{\delta}_{n+1}(\lambda), \bar{\mu}_n(\lambda) \right) = \left( \text{Ker } \bar{\delta}_n(\lambda), \mu_n \mid_{\text{Ker } \bar{\mu}_n(\lambda)} \right) \bigg|_{(\text{Im } \bar{\delta}_{n+1}(\lambda), \mu_{n+1}(\lambda))}
\]
\[
= \text{Ker } \lim_{\lambda} \bar{\delta}(\lambda) \bigg|_{\text{Im } \bar{\delta}_{n+1}(\lambda)} = H_\lambda \left( \lim_{\lambda} C(\lambda) \right).
\]
The proof is completed.
REFERENCES