PRE-DFT PROCESSING WITH SPACE-TIME-FREQUENCY CODING

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Subcarrier based space processing was conventionally employed in Orthogonal Frequency Division Multiplexing (OFDM) systems under Multiple-Input and Multiple-Output (MIMO) channels to achieve optimal performance. At the receiver of such systems, multiple Discrete Fourier Transform (DFT) blocks, each corresponding to one receive antenna, are required to be used. This induces considerable complexity. In this paper, we propose a pre-DFT processing scheme for the receiver of MIMO-OFDM systems with space-time-frequency coding. With the proposed scheme, the number of DFT blocks at the receiver can be any number from one to the number of receives antennas, thus enabling effective complexity and performance tradeoff. Using the pre-DFT processing scheme, the number of input signals to the space-time-frequency decoder can be reduced compared with the subcarrier based space processing. Therefore, a high dimensional MIMO system can be shrunk into an equivalently low dimension one. Due to the dimension reduction, both the complexity of the decoder and the complexity of channel estimation can be reduced. In general, the weighting coefficients calculation for the pre-DFT processing scheme should be relevant to the specific space-time-frequency code employed. In this paper, we propose a simple universal weighting coefficients calculation algorithm that can be used to achieve excellent performance for most practical space-time-frequency coding schemes. This makes the design of the pre-DFT processing scheme independent of the optimization of the space-time-frequency coding, which is desirable for multiplatform systems.

Keywords: Orthogonal frequency division multiplexing (OFDM), multiple-input and multiple-output (MIMO), Space-time-frequency code.

1. INTRODUCTION

For high data rate wideband wireless communications, Orthogonal Frequency Division Multiplexing (OFDM) can be used with Multiple-Input and Multiple-Output (MIMO) technology to achieve superior performance. In conventional MIMO-OFDM systems, subcarrier based space processing [1]-[9] was employed to achieve optimal performance. However, it requires multiple discrete Fourier transform/inverse DFT (DFT/IDFT) blocks, each corresponding to one receive/transmit antenna. Even though DFT/IDFT can be efficiently implemented using fast Fourier transform/inverse FFT (FFT/IFFT), its complexity is still a major concern for OFDM implementation [10]. In addition, the use of multiple antennas requires the baseband signal processing components to handle multiple input signals, thus inducing considerable complexity for the decoder and the channel estimator at the receiver. To reduce the complexity of such systems, several schemes [10]-[15] were proposed in the literature.

For an OFDM system with multiple transmit antennas, the schemes mentioned above [13]-[15], explicitly or implicitly, assume that the channel state information (CSI) is known at the transmitter. In mobile communications, where the channel can vary rapidly, it is difficult to maintain the CSI at the transmitter up-to-date without substantial system overhead [16]. Space-time-frequency codes [1]-[9] were proposed for OFDM systems to fully take advantage of the frequency diversity and spatial diversity presented in frequency selective fading channels without the requirement of the availability of CSI at the transmitter. For such a system, traditional subcarrier based space processing induces considerable complexity due to the reasons mentioned before.

In this paper, we propose to use pre-DFT processing to reduce the receiver complexity of MIMO-OFDM systems with space-time-frequency coding. In our proposed scheme, the received signals at the receiver are first weighted and then combined before the DFT processing. Owing to the pre-DFT processing, the number of DFT blocks required at the receiver can be reduced, and a high dimensional MIMO system can be shrunk into an equivalently low dimension one. Both enable effective complexity reduction.

One important issue in the proposed pre-DFT processing scheme for MIMO-OFDM systems with space-time-frequency coding is the calculation of the weighting coefficients before the DFT processing. In general, the weighting coefficients calculation are specific to the space-time-frequency coding scheme. In this paper, we propose a universal weighting coefficients calculation algorithm that can be applied in most practical space-time-frequency codes such as those proposed...
in [1]-[4], [6], [8] and [9]. This makes the design of the pre-DFT processing scheme independent of the optimization of the space-time-frequency coding, which is desirable for multiplatform systems.

In general, the weighting coefficients before the DFT processing can be calculated assuming that the CSIs are explicitly available. In this paper, we will show that the weighting coefficients can also be obtained using the signal space method [10], [11] without the explicit knowledge of the CSIs. This helps to reduce the complexity of channel estimation required by the space-time-frequency decoding since the number of equivalent channel branches required to be estimated in the proposed scheme can be reduced from the number of receive antennas to the number of DFT blocks. This paper is organized as follows. In Section 2, the proposed scheme for MIMO-OFDM systems is described. The calculation of weighting coefficients with/without explicit CSIs is introduced in Section 3 and 4, respectively. Some discussions about the proposed pre-DFT processing scheme are given in Section V. Simulation results are then presented in Sections 6. Finally, Section 7 concludes our work. Throughout this paper, the following notations will be used. (·)\( ^T \), (·)\( ^H \), and (·)\( ^\dagger \) in the superscripts denote conjugate, transpose and Hermitian transpose, respectively. Likewise, diag (x) denotes a diagonal matrix \( x \) with on its diagonal; rank (·) denotes the rank of (·); \( E(·) \) denotes the expectation of (·); \( \text{trace}(·) \) denotes the trace of matrix (·); and \( \lambda_i(·) \) denotes the \( i \)th largest eigen value of matrix (·).

2. SYSTEM MODEL

We investigate a MIMO-OFDM system with \( N \) subcarriers as shown in Fig. 1. In the system, there are \( F \) transmit antennas and \( M \) receive antennas. At the \( t \)th OFDM symbol period, the output of the space-time-frequency encoder is assumed to be as follows:

\[
C^{(t)} = C^{(t)}_{0,1}, ..., C^{(t)}_{N-1}, C^{(t)}_{0,F}, ..., C^{(t)}_{N-1,F}, T = 0, 1, ..., T - 1
\]

![Fig. 1: Pre-DFT Processing for a MIMO-OFDM System with Space-time-frequency Coding: (a) Transmitter; (b) Receiver](image)

Where \( C^{(t)}_{n,f}(t) \) is the coded information symbol at the \( n \)th subcarrier of the \( f \)th OFDM symbol period transmitted from the \( f \)th transmit antenna, and \( T \) is the number of OFDM symbols in a space-time-frequency codeword. When \( T = 1 \), the space-time-frequency code reduces to a space-frequency code.

After the IDFT processing, at the \( f \)th OFDM symbol period, the \( l \)th sample at the \( f \)th transmit antenna is given by

\[
S_{f,l}(f) = 1/N \sum_{n=0}^{N-1} C_{n,f}^{(t)} e^{j2\pi fn/N} \\
N_g \leq 1 < N, f = 1, ..., F, t = 0, ..., T - 1
\]

Where \( N_g \) is the length of the cyclic prefix, and we assume that \( (N_g + 1) < N \) to keep high transmission efficiency. In the following, we assume that the channel does not vary over the period of one space-time-frequency codeword (i.e., the period of TOFDM symbols). Furthermore, we assume that the channel impulse responses (CIRs) decay to zero during the cyclic extension, or \( L \leq (N_g + 1) < N \) where \( L \) is the maximum length of the CIRs. At the \( m \)th receive antenna, \( l \)th sample at the \( f \)th OFDM symbol period is then given by

\[
Y_{m,f}^{(t)} = \sum_{j=1}^{N_g} h_{f,m}^{(t)} S_{f,l}(f) + Z_{m,f}^{(t)}
\]

Where \( * \) denotes the convolution product, \( h_{f,m}^{(t)} \) denotes the CIR between the \( f \)th transmit antenna and the \( m \)th receive antenna, and \( Z_{m,f}^{(t)} \) denotes the additive white Gaussian noise (AWGN) component at the \( m \)th receive antenna.

At the receiver, before the DFT processing, the \( M \) data streams from the output of the \( M \) receive antennas are weighted and then combined to form \( Q \) branches. After the guard interval removal, the weighted and combined signals are then applied to the DFT processors. Note that there are \( Q \) branches, and hence the number of DFT blocks required at the receiver is \( Q \). As a result, compared to the conventional receiver structure [1]-[9], where \( M \) DFT blocks are used, the number of DFT blocks employed at the receiver can be reduced when pre-DFT processing is used. For the \( q \)th branch, the output of the DFT processor at the \( f \)th OFDM symbol period is given by

\[
V_{m,q}^{(t)} = \sum_{f=1}^{F} \sum_{m=1}^{M} w_{m,q} H_{n,f}^{(m)} C_{n,f}^{(t)} + \sum_{m=1}^{M} w_{m,q} Z_{m,f}^{(t)}
\]

Where

\[
H_{n,f}^{(m)} = \sum_{l=0}^{L-1} h_{f,m}^{(t)} e^{-j2\pi ln/N},
\]

\[
Z_{m,f}^{(t)} = \sum_{l=0}^{N-1-N_g} z_{m,f}^{(t)} e^{-j2\pi ln/N},
\]

And \( w_{m,q} \) is the weighting coefficient at the \( m \)th receive antenna of the \( q \)th branch. In order to keep the noise white and its variance at different branch the same, we assume that the weighting coefficients are normalized (i.e., \( \Omega^T \Omega = I_Q \), where \( \Omega \) is an \( M \times Q \) matrix with the \( (m,q) \)th entry given by \( w_{m,q} \), and \( I_Q \) is a \( Q \times Q \) identify matrix).
3. WEIGHTING COEFFICIENTS CALCULATION WITH EXPPLICIT CSI

In this section, we will present a way to calculate the weighting coefficients for the proposed pre-DFT processing scheme. When the ML decoder is employed, the pair-wise error probability (PEP) can be used to denote system performance, which is further determined by the pair-wise codeword distance [11]. The pair-wise codeword distance $d^2(C, E|H)$ between a favored coded sequence,

$$ E = [E^{(0)}, E^{(1)}, \ldots, E^{(T-1)}]^T $$

$$ E^{(q)} = [e^{(q)}_{0,1}, \ldots, e^{(q)}_{N-1,1}, \ldots, e^{(q)}_{0,F}, \ldots, e^{(q)}_{N-1,F}] $$

$(\forall t \in [0, T-1])$ and the transmitted coded sequence

$$ C = [C^{(0)}, C^{(1)}, \ldots, C^{(T-1)}]^T $$

Where $C^{(t)} (\forall t \in [0, T-1])$ is defined in (1), is given by

$$ d^2(C, E|H) = \sum_{q=1}^{Q} \sum_{n=0}^{N-1} \sum_{m=1}^{M} \sum_{k=0}^{M} w_{n,k} W_{m/k}^T $$

$$ \sum_{f=1}^{F} H_n^{(m,f)} (H_n^{*} f) \begin{pmatrix} c_{n,f}^{(q)} - \epsilon_{n,f}^{(q)} \end{pmatrix} \begin{pmatrix} c_{n,f}^{(q)} - \epsilon_{n,f}^{(q)} \end{pmatrix}^T \tag{7} $$

According to [11], minimizing the pair-wise error probability is equivalent to maximizing the pair-wise codeword distance given by (7). A close observation of (7) indicates that the optimal weighting coefficients are related to the specific codeword pair. To make the weighting coefficients and the codeword pair independent, we average (7) over all codeword’s pair ensemble. As a result, we get

$$ d^2(C, E|H) = \sum_{q=1}^{Q} \sum_{n=0}^{N-1} \sum_{m=1}^{M} W_{n,m} W_{m,n}^T $$

$$ \sum_{n=0}^{N-1} \sum_{f=1}^{F} H_n^{(m,f)} H_n^{*} f \begin{pmatrix} c_{n,f}^{(q)} - \epsilon_{n,f}^{(q)} \end{pmatrix} \begin{pmatrix} c_{n,f}^{(q)} - \epsilon_{n,f}^{(q)} \end{pmatrix} \tag{8} $$

Where the over bar stands for the average over all the code words pair ensemble. In order to rewrite (8) into a matrix form, let $C_n$ be an $F \times T$ matrix with the $(f, t)$th entry given by $c_{n,f}^{(q)}$, $E_n$ be an matrix with the $(f, t)$th entry given by $e_{n,f}^{(q)}$ and $H_n$ $(n = 0, \ldots, N-1)$ be an $M \times F$ matrix with the $(m,f)$th entry given by $H_n^{(m,f)}$. With these definitions, (8) can be written into

$$ d^2(C, E|H) = \text{trace} \left( \Omega^T \Phi \Omega' \right) \tag{9} $$

Where

$$ \Phi = \sum_{n=0}^{N-1} H_n K_n H_n^H \tag{10} $$

With

$$ K_n = (C_n - E_n)(C_n - E_n)^H \tag{11} $$

Let the eigenvalues of $\Phi$ be $\lambda_i (q = 1, \ldots, M)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_M$ and $\omega_q (q = 1, \ldots, Q)$ be the $i$th column of $\Omega$. It is well known that when $\omega_q (q = 1, \ldots, Q)$ are the conjugate of the eigenvectors of $\Phi$ corresponding to the eigenvalues $\lambda_q (q = 1, \ldots, M)$, the maximum of $d^2(C, E|H)$ is achieved and is given by [22].

$$ (d^2(C, E|H))_{\max} = \sum_{q=1}^{Q} \lambda_q (\Phi) \tag{12} $$

In general, to obtain $\Phi$ in (14), we need both knowledge of the CSIs and the space-time-frequency code since $k_n$ is dependent on the specific space-time-frequency code. However, since the channel information is not available at the transmitter, the space-time-frequency coding scheme should not favor or bias a particular sub-carrier or a particular transmit antenna. As a result, in the following, it will be shown that for most practical space-time-frequency codes, it is reasonable to assume that $\Phi$ is in the following form:

$$ \Phi = k \sum_{n=0}^{N-1} H_n H_n^H \tag{13} $$

Where $k$ is a constant that is independent of $n$. As a result, the weighting coefficients (i.e., $\omega_q (q = 1, \ldots, Q)$), which are the conjugate of the eigenvectors of $\Phi$, are independent of the specific space-time-frequency coding scheme. For the space-time-frequency codes proposed in [8] and [9], for example, as shown in Appendix A, $k$ can be expressed as follows:

$$ K_n = K_{1, \text{diag} (\beta_{2,1})} \tag{14} $$

Where $k_i$ is a constant number independent of $n$, $\beta_{2,1} = [0, \ldots, 1, 0, \ldots, 0]$ is an $F$ – dimensional standard basis standard with 1 in its $\tau$ $(n)$th component and 0 elsewhere, and $\tau(n)$ is determined by the space-frequency coding scheme. Using (14), as shown in Appendix A, $\Phi$ can be proved to be in the form of (13) with $k = k_{i}/F$. For space-time-frequency codes where the orthogonal space time block code (STBC) [17], [18] is employed (e.g., the codes proposed in [1]-[4]) as an inner code. Using the orthogonal property of STBC, we can easily prove that

$$ K_n = \text{diag} \left( 2 \left| c_{n,1}^{(q)} - e_{n,1}^{(q)} \right|^2, \ldots, 2 \left| c_{n,F}^{(q)} - e_{n,F}^{(q)} \right|^2 \right) \tag{15} $$

It is reasonable to assume that the signals at the input of the inner encoder have the same distribution for different subcarriers and different transmit antennas, especially when an interleaver is employed between the outer encoder and the inner encoder. As a result, $k$ can be written as

$$ K_n = KL_i \tag{16} $$

Therefore, (10) can also be simplified into (13) for these codes. For a general space-time-frequency code such as that proposed in [6], simulation results in Section VI will also show that excellent performance can be achieved by using the weighting coefficients calculated based on $\Phi$ given by (13).

4. WEIGHTING COEFFICIENTS CALCULATION WITHOUT EXPPLICIT CSI

In the following, we propose a way to obtain the weighting coefficients (i.e., the eigenvectors of $\Phi$) without explicit CSI.
This is especially important for differential modulation, where the CSI is not supposed to be explicitly known at the receiver. For coherent modulation, when CSI is not explicitly required for the weighting coefficients calculation, the complexity of channel estimation can be reduced since the number of equivalent channel branches required to be estimated is now reduced from the number of receive antennas to the number of DFT branches. Note that the covariance matrix of the received signal vector 

$$\gamma_{l,i} = \begin{bmatrix} \gamma_{l,i}^{(1)} & \gamma_{l,i}^{(2)} & \cdots & \gamma_{l,i}^{(M)} \end{bmatrix}$$

Can be given by

$$R = E[\gamma_{l,i} \gamma_{l,i}^H]$$

$$\rho_{m,m'} = E\left[ q_{m}^{(m)} (q_{m'}^{(m)})^* \right]$$

$$= \frac{1}{N} \sum_{l=1}^{L} \sum_{n=1}^{N_0} \left( z_{l,m} h_{n,m} f_{n,m} f_{l,m} \right) \left( h_{n,m'} f_{n,m'} f_{l,m'} \right)^* + N_0 \delta(m-m')$$

(18)

Similar to the SIMO case proposed in [10], when a large number of subcarriers are used, it is reasonable to assume that the transmitted signals are white, that is

$$E(S_{m'}^{(m),f} S_{m'}^{(m'),f}) = E(\delta(f-f') \delta(v-v'))$$

(19)

Where $E$ is the average energy of the coded symbol. Hence, by substituting (19) into (18) and after some manipulations, $\rho_{m,m'}$ can be proven to be given by

$$\rho_{m,m'} = E_s / N \left( \sum_{l=1}^{L} \sum_{n=1}^{N_0} \left( h_{n,m} f_{l,m} \right) \left( h_{n,m'} f_{l,m'} \right)^* + N_o \delta(m-m') \right)$$

(20)

Where $N_o$ is the variance of the noise. Using (13), we then have

$$\rho_{m,m'} = E_s / N_o \left( (\Phi_{m,m}) + N_o \delta(m-m') \right)$$

(21)

Where $\Phi_{m,m'}$ is the $(m, m')$ th entry of $\Phi$. From (21), it can be easily seen that the eigenvectors of $\Phi$ are the same as those of $R$. As a result, we can obtain the weighting coefficients directly from $R$ without explicit knowledge of CSI.

5. COMPLEXITY CONSIDERATION

The proposed MIMO-OFDM system consists of pre-DFT weighting and combining, weighting coefficients calculation, DFT-processing, channel estimation, and ML decoding. By weighting and combining before the DFT processing, the number of branches to be handled by the ML decoder is reduced from $M$ to $Q$. As a result, compared with the subcarrier based processing [1]-[9], the complexity of ML decoding can be reduced. As for the complexity coming from the DFT processing, the pre-DFT weighting and combining, the ratio of the number of multiplications needed between the proposed scheme and the subcarrier based scheme is as follows:

$$\eta = \frac{QN \log_2 N + QMN}{MN \log_2 N}$$

(22)

From (22), it can be seen that, when $\log_2 N >> M$, $\eta$ is close to $O(Q/M)$. From (12), it is easy to see that the number of DFT blocks at the receiver, $Q$, is determined by the rank of $\Phi$. After some manipulations, we have

$$\text{rank}(\Phi) \leq \min(\text{rank} (\hat{R}^{1/2}), M, FL)$$

Where

$$\hat{R}^{1/2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \cdots \frac{1}{2} \\
\ & \ & \ \\
\ & \ & \ \\
\ & \ & \ \\
\ & \ & \ \\
\ & \ & \ \\
\ & \ & \ \\
\ & \ & \ \\
\end{bmatrix}$$

(23)

While $R_l = R_l^{1/2}$ are the receive correlation matrix as defined in [19]. From (23), we can see that $\Phi$ is singular when $\hat{R}^{1/2}$ is not of full row rank or FL is smaller than $M$. In this case, the number of DFT blocks required can be smaller than the number of receive antennas to achieve optimal performance. On the other hand, when $\Phi$ is nonsingular, it is still possible to achieve good performance with a limit number of DFT blocks due to the small contribution of the small eigenvalue to the average pair-wise codeword distance $d^2(C, E \mid H)$.

6. SIMULATION RESULTS

In the considered MIMO-OFDM system, the number of subcarriers in an OFDM symbol is 64 ($N = 64$) and the length of the guard interval is 12 ($N = 12$). In the simulations, we assume that there are four receive antennas at the receiver and two or four transmit antennas at the transmitter. Further, we assume that the channel is quasi-static and perfect channel information is available at the receiver. Without special notation, the optimal lines in the figures are obtained using ML decoders based on subcarrier space processing as the corresponding references. Further, $E/N_0$ in all figures is a shorthand for $E/N_0$ per receive antenna.


In this part, we consider the code proposed in [1], where full diversity order provided by the fading channel can be achieved with low trellis complexity. As in [1], we use the optimal rate 2/3 TCM codes [21] designed for flat fading channels. For simplicity, only the 4-state 8PSK TCM code is used and the parity check matrix is $(6 4 7)$ in octal form.

When the two-ray equal gain Rayleigh fading channel model is employed, the bit error rate (BER) performance of the proposed scheme is shown in Fig. 2. It can be observed that, with the increase of the number of DFT blocks at the receiver, better performance can be achieved. When the number of DFT blocks is three or four, the performance is close to optimal.
When the weighting coefficients are obtained based on the signal space method as discussed in Section 4, the performance of the proposed scheme over two-ray equal gain Rayleigh fading channel is also shown in Fig. 2. In the simulations, $P$ is set to 64. From Fig. 2, we can see that the performance of the proposed scheme using the signal space method is almost the same as that with complete CSI.


The space-time-frequency code proposed in [8] can achieve full diversity without any rate reduction. In our simulations, the codeword of the space-frequency code $C$ is given by Eqn. (3.1) in [8], and QRD-M algorithm is employed as the space-time-frequency decoder [23]. The performance of the proposed scheme over a six-ray exponential decay quasi static Rayleigh fading channel is shown in Fig. 3. In Fig. 4, the general space-time-frequency code proposed in [6] is employed with 16-state trellis and QPSK modulation [21]. It can be seen from Fig. 3 and Fig. 4 that similar results can be obtained as those in Part A irrespective for channel type and system configuration. As a result, the weighting coefficients obtained in Section 3 can also be applied here.

7. CONCLUSION

In this paper, a pre-DFT processing scheme was proposed for a MIMO-OFDM system with space-time-frequency coding. With the proposed scheme, system complexity and performance can be effectively traded off. A simple weighting coefficients calculation algorithm was also derived. Theoretical analysis and simulation results have shown that the algorithm can be applied for most existing practical space-time-frequency codes. Using the proposed scheme, we have also shown that it is possible to use a very limited number of DFT blocks to achieve near optimal system performance.

REFERENCES


**APPENDIX A**

**PROOF OF (13) FOR THE SPACE-FREQUENCY CODES PROPOSED IN [8] AND [9]**

The codeword of the space-time-frequency codes proposed in [8] and [9] includes only one OFDM symbol. Therefore, they are space-frequency codes with \( T = 1 \). Based upon [8] and [9], when \( f \neq f' \), we have

\[
C_{n,f} - \epsilon_{n,f}^{(0)} = 0
\]  
(A.1)

As a result, \( k \) in Eqn. (11) is a diagonal matrix for any \( n \).

When \( f = f' \), we have

\[
C_{n,f} - \epsilon_{n,f}^{(0)} = \epsilon_{n,f}^{(0)} - \epsilon_{n,f}^{(0)}
\]  
(A.2)

For any \( n \), from [8] and [9], there exists only one \( f \) that makes \( C_{n,f} \) nonzero. As a result, this \( f \) is a function of \( n \), i.e., \( f = \tau(n) \). From [8] and [9], when \( f = \tau(n) \), \( \epsilon_{n,f}^{(0)} \) is also nonzero.

It is then reasonable to assume that \( C_{n,f} - \epsilon_{n,f}^{(0)} \) have the same distribution for any \( n \). As a result, for the space frequency codes proposed in [8] and [9], \( k_{n} \) is in the form of (14).

Specifically, for the space-frequency code proposed in [8], \( \tau(n) \) is in the following form

\[
\tau(n) = \left[ \frac{n}{\Gamma} \right] X_{F} + 1
\]  
(A.3)

Where \( \Gamma \) can be any fixed integer between 1 and \( L \), and \( \left[ \alpha \right] \) is the nearest integer less than or equal to \( \alpha \). For simplicity, here \( N \) is assumed to be an integer multiple of \( FT \). As a result, the \( (m,m') \) entry of \( \Phi \) can be written as (A.4) as shown in the next page. Note that

\[
\sum_{\mu=0}^{N/FT-1} e^{-j2\pi(l'-l)/Ft} e^{(0)}/N
\]

\[
= \left\{ \begin{array}{ll}
\frac{N}{FT} \left( l-l' \right) \mod \frac{N}{FT} = 0 \\
0, \left( l-l' \right) \mod \frac{N}{FT} \neq 0
\end{array} \right.
\]  
(A.5)

Where \( m' \) is the modulus function. In practical applications, \( N/FT > L \). As a result, \( \Phi_{m,m'} \) can be given by

\[
\Phi_{m,m'} = \frac{NK_{l}}{FT} \sum_{l=0}^{N-1} \sum_{l'=0}^{L-1} h_{l}^{(m,f)} h_{l}^{(m',f) *} 
\]  
(A.6)

On the other hand, it can be easily proved that

\[
N \sum_{l=0}^{L-1} \sum_{l'=0}^{L-1} h_{l}^{(m,f)} h_{l}^{(m',f) *}
= \sum_{n=0}^{N-1} \sum_{f=0}^{F} H_{n}^{(m,f)} H_{n}^{(m',f) *}
\]  
(A.7)

As a result, we have

\[
\Phi_{m,m'} = \frac{K_{l}}{F} \sum_{n=0}^{N-1} \sum_{f=0}^{F} H_{n}^{(m,f)} H_{n}^{(m',f) *}
\]  
(A.8)

And Eqn. (13) is satisfied. Similarly, we can prove that Eqn. (13) is also satisfied for the space-frequency code proposed in [9].

\[
\Phi_{m,m'} = \frac{K_{l}}{F} \sum_{n=0}^{N-1} \sum_{f=0}^{F} \sum_{l=0}^{L-1} \sum_{n=0}^{N-1} \sum_{f=0}^{F} H_{l}^{(m,f)} H_{l}^{(m',f) *}
\]  
(A.9)

\[
\sum_{n=0}^{N-1} \sum_{f=0}^{F} \sum_{l=0}^{L-1} \sum_{n=0}^{N-1} \sum_{f=0}^{F} H_{l}^{(m,f)} H_{l}^{(m',f) *}
\]  
(A.10)