A Nonlinear Generalized Standard Solid Model for Viscoelastic Materials

Marc Delphin MONSIA

From: Département de Physique Université d'Abomey-Calavi, 09 B.P. 305
Cotonou, République du Bénin E-mail: monsiamarc@yahoo.fr

ABSTRACT: A single differential constitutive equation is derived from a standard nonlinear solid model consisting of a polynomial elastic spring in series with a classical Voigt model for the prediction of time-dependent nonlinear stress of some viscoelastic materials. Under a nonlinear exponential strain history the constitutive laws gives a description of the stress-time equation as a second-order polynomial in hyperbolic tangent function which is useful to reproduce the stiffening effect and softening response of some viscoelastic materials. Numerical examples are performed to show the sensitivity of the model to material parameters and the validity of the model.

Keywords: Constitutive equation, hyperbolic tangent function, standard linear solid

1. INTRODUCTION

Most materials show both viscous and elastic behavior, that is, are viscoelastic materials and exhibit time-dependent and history-dependent properties. Their mechanical response varies under different loading histories. The problem in mechanical modeling is that at finite deformation, viscoelastic materials show often a nonlinear mechanical behavior, which cannot be described by the well-known established linear viscoelastic theory. Thus, an extension of small-strain theory to finite deformation model in viscoelasticity has become necessary. Many models with different success have been proposed to describe the time-dependent nonlinear behavior of viscoelastic materials [1]. Theoretical viscoelastic models are very important for prediction, simulation, description and design in different disciplines of sciences such as engineering research, medical sciences and biology. Most successful predictive models are shown to be founded on extension of the classical linear rheological models to finite deformation [2 – 4]. In other words, to build nonlinear viscoelastic models, investigators often modify the simple classical Maxwell model, Voigt model or their different combinations, by introducing nonlinear elastic springs and/or nonlinear dashpots. Other way consists to consider the elasticity and/or viscous modules as variable coefficients. Corr et al [2] extending the Maxwell fluid model to finite deformations, constructed a new viscoelastic model which is useful to describe the time-dependent properties of some viscoelastic materials. Recently, Monsia [3] utilizing a power series expansions method which consists in an extended Voigt model to finite deformation, developed a hyperlogistic equation which represents successfully the time-dependent mechanical properties of a variety of viscoelastic materials, in other words the strain stiffening and softening response. More recently, Monsia [4] using a second-order polynomial elastic spring in series with a classical Voigt model, which is an extended form of the standard linear solid to finite strain, formulated a hyperlogistic-type equation to reproduce the nonlinear time-dependent stress response of some viscoelastic materials.

In this study, a nonlinear rheological model with constant material parameters for representing successfully the nonlinear time-dependent stress response of a class of viscoelastic materials subjected to a nonlinear exponential strain history is proposed. The model is made, contrary to our previous study [4], of a second-order polynomial elastic spring in series with a classical linear Voigt model as shown in Figure 1. The obtained differential constitutive law gives under a nonlinear exponential strain history, the nonlinear time-dependent stress induced in the material studied as a second-order polynomial in hyperbolic tangent function, which appears powerful to represent any S-shaped curve. The model exhibits then the strain stiffening and softening response noted in some viscoelastic materials at finite strains. Numerical examples are performed to show the sensitivity of the model to material parameters and the validity of the model. In particular, the model is shown to be sensitive to the maximum strain experienced and relaxation times.

Figure 1. Viscoelastic Model
2. MECHANICAL MODEL

A. Theoretical Formulation

In this part we describe the theoretical rheological model and derive the governing differential constitutive law including the polynomial restoring force effects. Most rheological material properties are nonlinear time-dependent. For this, a best description of these materials may proceed from the use of nonlinear theories. To build our proposed viscoelastic model, we start from the classical standard linear solid for including the nonlinear polynomial elastic term. Thus, the mechanical properties of the considered material is divided into two parts: a nonlinear polynomial spring (with stiffness $E_2$ and $E_3$) which captures the nonlinear pure elastic response, and a classical Voigt element made of a linear spring (with stiffness $E_1$) in parallel with a linear dashpot ($\eta$), capturing the time-dependent history response. From the mathematical point of view, the polynomial stiffness term is included in a model in order to loss the linearity in the differential constitutive equation that represents the dynamic properties of the mechanical system studied. Due to the fact that the elements are in series the total stress $\sigma$ and the total strain $\varepsilon$ can be written as

$$
\begin{align*}
\sigma &= E_1 \varepsilon_1 + \eta \varepsilon_2 \\
\sigma &= E_2 \varepsilon_2 + E_3 \varepsilon_2^2 \\
\varepsilon &= \varepsilon_1 + \varepsilon_2
\end{align*}
$$

(1)

where $\varepsilon_1$ and $\varepsilon_2$ are the strains of the Voigt element and the nonlinear spring, respectively, $\eta$ is the viscosity module. $E_1$ is the elasticity module of the linear spring and, $E_2$ and $E_3$ are respectively the elasticity modules of the polynomial spring, and the dot denotes the time derivative. By making substitutions, appropriate for each individual element, we may deduce from Eq. (1) the law

$$
\varepsilon_2 + \left( \frac{E_2 + E_3}{\eta} \right) \varepsilon_2 + \frac{E_3}{\eta} \varepsilon_2^3 = \frac{E_1}{\eta} \varepsilon + \varepsilon
$$

(2)

Eq. (2) represents mathematically in the single differential form the relation between the total strain $\varepsilon$ and the strain $\varepsilon_2$ of the nonlinear spring. This equation is a first-order nonlinear ordinary differential equation in $\varepsilon_2$ for a given strain history $\varepsilon$.

B. Dimensionalization

If $M$, $L$ and $T$ denote the mass, length and time dimension, respectively, the dimension of the stress varies as $ML^{-1}T^2$. The strain $\varepsilon$ is a dimensionless quantity. Consequently, in Eq. (2) the coefficients $E_1$, $E_2$ and $E_3$ possess the same dimension with the stress $\sigma$, that of $\eta$ varies as $ML^{-1}T$.

C. Derivation of Polynomial Time-Dependent Stress

We derive in this section the time-dependent stress as a second-order polynomial in hyperbolic tangent or logistic-type function for the variety of viscoelastic materials studied. For this, we consider the term of right hand side of Eq. (2) to be constant and equal to $k_m$, to say, that the material under consideration is subjected to the time-dependent strain history

$$
\varepsilon(t) = \varepsilon \left( 1 - \exp \left( -\frac{t}{\tau_1} \right) \right)
$$

... (3)

with

$$
\varepsilon_m = \frac{\eta k_m}{E_1}
$$

$$
\tau_1 = \frac{\eta}{E_1}
$$

Then Eq. (2) becomes

$$
\varepsilon_2 = -\frac{1}{\tau_3} \varepsilon_2^3 - \frac{1}{\tau_2} \varepsilon_2 + \frac{1}{\tau_1} \varepsilon_m
$$

(4)

where $\tau_1$, $\tau_2$, $\tau_3$ are respectively, a relaxation time and a nonlinear material coefficient. Eq. (4) is a first-order Riccati nonlinear ordinary differential equation which can be solved analytically with the suitable boundary conditions of the mechanical problem considered in hyperbolic tangent function or logistic-type function. Using following suitable boundary conditions that satisfy the dynamic of the viscoelastic material under study, that is to say

$$
t \rightarrow 0, \lim_{t \rightarrow 0} \varepsilon_2(t) = \varepsilon_0
$$

$$
t \rightarrow +\infty, \lim_{t \rightarrow +\infty} \varepsilon_2 = \varepsilon_{2\text{max}}
$$

we can obtain as solution

$$
\varepsilon_2(t) = \frac{\tau_3}{2\tau_2} \left( 1 + \sqrt{1 + 4 \frac{\varepsilon_m}{\tau_1 \tau_3} \frac{\varepsilon_2^2}{\tau_2}} \right)
$$

$$
\frac{\tau_1}{\sqrt{1 + 4 \frac{\varepsilon_m}{\tau_1 \tau_3} \frac{\varepsilon_2^2}{\tau_2}}} + \frac{2\tau_2}{\tau_2 + \tau_3} \left( 1 - \frac{1}{\tau_2} \varepsilon_2 \right) \exp \left( -\left( \frac{1}{\tau_2} \varepsilon_2 \right) \right)
$$

(5)
or
\[ \varepsilon_2(t) = \varepsilon_{2\min} + \frac{\varepsilon_{2\max} - \varepsilon_{2\min}}{1 + M \exp(-\mu t)} \] (6)

where
\[ \varepsilon_{2\min} = \frac{\tau_1}{2\tau_2} \left( 1 + \sqrt{1 + 4 \frac{\varepsilon_m \tau_2^4}{\tau_1 \tau_3}} \right) \]
\[ \varepsilon_{2\max} = \frac{\tau_1}{2\tau_2} \left( 1 - \sqrt{1 + 4 \frac{\varepsilon_m \tau_2^4}{\tau_1 \tau_3}} \right) \]

\[ M = \frac{2\tau_2 \varepsilon_{\max} + \tau_3 \left( 1 - \sqrt{1 + 4 \frac{\varepsilon_m \tau_2^4}{\tau_1 \tau_3}} \right) \right)}{2\tau_2 \varepsilon_{\max} + \tau_3 \left( 1 + \sqrt{1 + 4 \frac{\varepsilon_m \tau_2^4}{\tau_1 \tau_3}} \right) \right)} \]
and
\[ \mu = \left( \frac{1 + 4 \varepsilon_m \tau_2^4}{\tau_1 \tau_3} \right) \frac{1}{\tau_2} \]

Thus, from Eq. (1) the time-dependent nonlinear stress can be written as
\[ \sigma(t) = \eta \left( \frac{1}{\tau_2} - \frac{1}{\tau_1} \right) \varepsilon_2 + \eta \frac{\varepsilon_2^2}{\tau_3} \] (7)

Eq. (7) gives the time variation of the stress in the viscoelastic material studied. It models the time nonlinear stress as a second-order polynomial in hyperbolic tangent function or logistic-type function, which is powerful to reproduce any S-shaped curve.

3. NUMERICAL RESULTS AND DISCUSSION

In this part some numerical examples concerning the time-dependent stress are presented to illustrate the ability of the model to reproduce the mechanical response of the viscoelastic material studied. The dependence of time-stress function on the material parameters is also discussed. In the following of this work the numerical examples are considered with the coefficient \( \eta = 1 \). Figure 2 exhibits the typical time-dependent stress curve with an increasing until a peak asymptotical value, obtained from the Eq. (7) with the values of coefficients
\[ \tau_1 = 2, \tau_2 = 0.5, \tau_3 = 1, \varepsilon_m = 1, \varepsilon_{\max} = 0.05 \]

It can be observed from Figure 2 that the model is capable to represent accurately the typical time-dependent stress curve, as shown in [2 - 4]. The stress-time curve is nonlinear and illustrates the sigmoid mechanical behavior of the viscoelastic material considered. The plotting showing a nonlinear behavior indicates then the material stiffening followed by softening. The model predicts a mechanical response in which the slope, after reaching its maximum value at the inflexion point, declines gradually with increase time.

**Figure 2: Typical Stress-Time Plotting Exhibiting a Maximum Asymptotical value**

Figure 3 (a, b, c, d, e) shows the effects of material parameters on the time-stress response. The effects of these parameters are studied by varying one coefficient while keeping the other four constants. Figure 3 (a) illustrates how the relaxation time \( \tau_1 \) of the strain affects maximum value of the stress. The graph shows that an increasing \( \tau_1 \) decreases the maximum stress, the slope also decreases. But, an increasing \( \tau_1 \) has no significant effect on the time needed to reach the maximum stress. The red color corresponds to \( \tau_1 = 1 \), the blue to \( \tau_1 = 1.5 \), and the green to \( \tau_1 = 2 \). The other parameters are \( \tau_2 = 0.5, \tau_3 = 1, \varepsilon_m = 1, \varepsilon_{\max} = 0.05 \).

**Figure 3 (a): Stress Versus Time Curves at Various Values of Relaxation Time \( \tau_1 \)**

In Figure 3 (b) is shown the dependence of the stress on the relaxation time \( \tau_2 \). An increase \( \tau_2 \) decreases the peak stress, but has no significant effect on the time required to attain the maximum stress. An increasing \( \tau_2 \), decreases also the initial value of the stress, but the nonlinearity of the initial portion becomes more important. The red color corresponds to \( \tau_2 = 0.5 \), the blue to \( \tau_2 = 1 \), and the green to \( \tau_2 = 1.5 \). The other parameters are \( \tau_1 = 2, \tau_3 = 1, \varepsilon_m = 1, \varepsilon_{\max} = 0.05 \).

**Figure 3 (b): Stress-Time Curves with Different Values of the Relaxation Time Parameter \( \tau_2 \)**
The stress curves at various nonlinearity parameter $\tau_3$ for the material under study are shown in Figure 3 (c). An increasing $\tau_3$ decreases the maximum stress as well as the slope, but has no important effect on the time required to reach the maximum stress value. The red color corresponds to $\tau_3 = 1$, the blue to $\tau_3 = 1.5$, and the green to $\tau_3 = 2$. The other parameters are $\tau_1 = 2$, $\tau_2 = 0.5$, $\varepsilon_m = 1$, $\varepsilon_{\theta 2} = 0.05$.

Figure 3 (c): Stress-Time Curves with Different Values of the Nonlinearity Parameter $\tau_3$

Figure 3 (d) shows the sensitivity of the stress-time curve to the maximum strain $\varepsilon_m$. An increasing maximum strain has a high effect on the peak stress value. The maximum stress increases with increasing $\varepsilon_m$. The slope also increases with increasing maximum strain. An increasing maximum strain increases the time necessary to reach the maximum stress. But, an increase $\varepsilon_m$ has no important effect on the initial value of the stress. The red color corresponds to $\varepsilon_m = 0.5$, the blue to $\varepsilon_m = 1.5$, and the green to $\varepsilon_m = 3$. The other parameters are $\tau_1 = 2$, $\tau_2 = 0.5$, $\tau_3 = 1$, $\varepsilon_{\theta 2} = 0.05$.

Figure 3(d): Stress-Time Curves Showing the Effect of the Maximum Strain $\varepsilon_m$

We observe from Figure 3 (e) that an increasing initial value $\varepsilon_{\theta 2}$ increases the initial value of the stress but has no significant effect on the maximum stress as well as the time required to attain it. The red color corresponds to $\varepsilon_{\theta 2} = 0.0005$, the blue to $\varepsilon_{\theta 2} = 0.005$, and the green to $\varepsilon_{\theta 2} = 0.05$. The other parameters are $\tau_1 = 2$, $\tau_2 = 0.5$, $\tau_3 = 1$, $\varepsilon_m = 1$.

Figure 3 (e): Stress-Time Curves for Three Different Values $\varepsilon_{\theta 2}$

Predictive viscoelastic models are important tools in many different disciplines of sciences. In this study a simple model of nonlinear viscoelasticity taking into consideration the sensitivity of the mechanical response of the material to the maximum strain experienced and relaxation times of the strain is presented. Two working hypothesis governed the construction of this model. The first is that the material properties can be decomposed into nonlinear polynomial pure elastic component acting in series with a damping element capturing the time-dependent deviation from the equilibrium state. The second restriction is that the material deforms following a nonlinear exponential strain history. This hypothesis allows defining the time-dependent stress induced in the material as a second-order polynomial in hyperbolic tangent function, which can successfully reproduce any sigmoid curve as shown by the numerical examples. These predicted results by the proposed model are in very agreement with those published in the literature. It is worth mentioning that the present rheological model incorporates the model proposed by Monsia [4] as a special case.

4. CONCLUSION

The time and history dependence of stress and strain, the nonlinearity of the time-stress relationship, make difficult a complete characterization of viscoelastic materials. Then, investigators in their effort, attempt to develop viscoelastic models which at best approach the real mechanical response of these materials. To better understand and evaluate viscoelastic materials under large strain, a nonlinear generalized standard solid model has been proposed. According to the obtained results, this model is applicable to predict the nonlinear time-stress curve of a class of viscoelastic materials as a second-order polynomial in logistic-type or hyperbolic tangent function.
REFERENCES


