REGULAR GENERALIZED $\gamma$–CLOSED SETS
IN TOPOLOGICAL SPACES

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Abstract: In this paper, we introduce and study the notion of regular generalized $\gamma$–closed sets. Also, some of their properties are investigated. Further, the notion of regular generalized $\gamma$–open sets and the notion of regular generalized $\gamma$–neighbourhood are discussed. Moreover, some of its basic properties are introduced. Finally, we give an applications of $rg\gamma$–closed sets, say $\gamma$–regular $T_1$ spaces.

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1. INTRODUCTION AND PRELIMINARIES

N. Levine [15] introduced generalized the notion of closed sets in general topology as a generalization of closed sets. This notion was found to be useful and many results in general topology were improved. Many researchers like Balachandran et.al [4], Arockiarani [2], Dunham [8], Gnanambal [12], Malghan [17], Palaniappan et.al [19], Arya et.al [3] and Keskin et.al [13] have worked on generalized closed sets, their generalizations and related concepts in general topology. The purpose of this paper is to define and study the properties of regular generalized $\gamma$–closed (briefly, $rg\gamma$–closed) sets. Also, we define the concept of regular generalized $\gamma$–open (briefly, $rg\gamma$–open) sets and obtained some of its properties and results. Further, we define $\gamma$–regular $T_1$–spaces as the spaces in which every regular generalized $\gamma$–closed set is $\gamma$–closed and study its properties.

Throughout the paper $X$ and $Y$ denote the topological spaces $(X; \tau)$ and $(Y; \sigma)$ respectively and on which no separation axioms are assumed unless otherwise explicitly stated. For any subset $A$ of a space $(X, \tau)$, the closure of $A$, the interior of $A$, the $gpr$–closure of $A$, the $gpr$–interior of $A$, the $w$–closure of $A$, the $w$–interior of $A$,

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the \( gpr \)-closure of \( A \), the \( gpr \)-interior of \( A \) and the complement of \( A \) are denoted by \( cl(A) \), \( int(A) \), \( \gamma cl(A) \), \( \gamma - int(A) \), \( \omega cl(A) \), \( \omega - int(A) \), \( gpr - cl(A) \), \( gpr - int(A) \) and \( A^c \) (equivalently, \( X - A \) or \( X \setminus A \)) respectively. \( (X, \tau) \) will be replaced by \( X \) if there is no chance of confusion.

Let us recall the following definitions as pre-requesties.

**Definition 1.1:** A subset \( A \) of a space \( X \) is called:

(i) a preopen set [18] if \( A \subseteq int(cl(A)) \) and a preclosed set if \( cl(int(A)) \subseteq A \),

(ii) a semi-open set [14] if \( A \subseteq cl(int(A)) \) and a semi-closed set if \( int(cl(A)) \subseteq A \),

(iii) a \( \gamma \)-open [9] (equivlently, \( b \)-open[1 ] or \( sp \)-open [7 ] set if \( A \subseteq int(cl(A)) \cup cl(int(A)) \) and a \( \gamma \)-closed ( equivlently, \( b \)-closed or \( sp \)-closed) set if \( int(cl(A)) \cap cl(int(A)) \subseteq A \),

(iv) a regular open set [22] if if \( A \neq int(cl(A)) \) and a regular closed set if \( cl(int(A)) = A \),

(v) \( \pi \)-open [23] if \( A \) is the finite union of regular open sets and the complement of \( \pi \)-open set is said to be \( \pi \)-closed.

The intersection of all \( \gamma \)-open subsets of \( X \) containing \( A \) is called the \( \gamma \)-kernel of \( A \) and is denoted by \( \gamma \ker(A) \). The intersection of all preclosed ( resp. \( \gamma \)-closed ) subsets of \( X \) containing \( A \) is called the pre-closure (resp. \( \gamma \)-closure ) of \( A \) and is denoted by \( pcl(A) \) ( resp. \( \gamma cl(A) \)).

**Definition 1.2:** A subset \( A \) of a space \( X \) is called:

1. generalized closed set (breifly, \( g \)-closed ) [15] if \( cl(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is open in \( X \),

2. \( \gamma \)-generalized closed set (breifly, \( \gamma g \)-closed ) [11] if \( \gamma cl(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is open in \( X \),

3. generalized \( \gamma \)- closed set (breifly, \( g \gamma \)-closed )[10 ] if \( \gamma cl(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is \( \gamma \)-open in \( X \),

4. generalized preclosed set (breifly, \( gp \)-closed )[16] if \( pcl(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is open in \( X \),

5. regular generalized closed set (breifly, \( rg \)-closed )[19] if \( cl(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is regular open in \( X \),

6. generalized preregular closed set (breifly, \( gpr \)-closed )[12] if \( pcl(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is regular open in \( X \),
(7) \( \pi \)-generalized closed set (briefly, \( \pi g \)-closed) \([6]\) if \( \text{cl}(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is \( \pi \)-open in \( X \),

(8) weakly closed set, (briefly, \( \omega \)-closed) \([20]\) if \( \text{cl}(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is semi-open in \( X \).

The complements of the above mentioned closed sets are their respectively open sets.

2. BASIC PROPERTIES OF REGULAR GENERALIZED \( \gamma \)-CLOSED SETS

We introduce the following definitions.

**Definition 2.1:** A subset \( A \) of a space \( X \) is called a regular-\( \gamma \)-open set (briefly, \( r\gamma \)-open) if there is a regular open set \( H \) such that \( H \subseteq A \subseteq \text{cl}(H) \). The family of all regular-\( \gamma \)-open sets of \( X \) is denoted by \( R\gamma O(X) \).

**Definition 2.2:** A subset \( A \) of a space \( X \) is called a regular generalized-\( \gamma \)-closed set (briefly, \( rg\gamma \)-closed) if \( \text{cl}(A) \subseteq H \) whenever \( A \subseteq H \) and \( H \) is regular-\( \gamma \)-open in \( X \). The family of all regular-\( \gamma \)-closed sets of \( X \) is denoted by \( RG\gamma C(X) \).

In the following proposition, we prove that the class of \( rg\gamma \)-closed sets has properly lies between the class of \( g\gamma \)-closed sets and the class of \( rg \)-closed sets.

**Proposition 2.3:** For a space \( (X, \tau) \), we have

(i) Every \( g\gamma \)-closed (resp. \( \omega \)-closed, closed, regular closed, \( \pi \)-closed) set is \( rg\gamma \)-closed,

(ii) Every \( rg\gamma \)-closed set is \( gpr \)-closed (resp. \( rg \)-closed).

**Proof:** The proof follows from definitions and the fact that every regular open sets are \( r\gamma \)-open.

The converse of the above proposition need not be true, as seen from the following examples.

**Example 2.4:** Let \( X = \{a, b, c, d, e\} \) with topology \( \tau = \{X, \emptyset, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\} \). Then,

1. the set \( A = \{a, d, e\} \) is \( rg\gamma \)-closed set but not \( g\gamma \)-closed set in \( X \),
2. the set \( B = \{b\} \) is \( rg\gamma \)-closed set but not \( \omega \)-closed set in \( X \),
3. the set \( D = \{a, b\} \) is \( rg \)-closed and \( gpr \)-closed set but not \( rg\gamma \)-closed set in \( X \).

**Remark 2.5:** From the above discussions and known results we have the following implications, in the following diagram by \( A \longrightarrow B \), we mean \( A \) implies \( B \) but not conversely and \( A \dashv B \) means \( A \) and \( B \) are independent of each other.
Remark 2.6: The union of two \(rg\gamma\)–closed subsets of \(X\) need not be a \(rg\gamma\)–closed subset of \(X\). Let \(X = \{a, b, c, d, e\}\) with a topology \(\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}\). Then two subsets \(\{c\}\) and \(\{d\}\) of \(X\) are \(rg\gamma\)–closed subsets but their union \(\{c, d\}\) is not \(rg\gamma\)–closed subset of \(X\).

Theorem 2.7: The arbitrary intersection of any \(rg\gamma\)–closed subsets of \(X\) is also \(rg\gamma\)–closed subset of \(X\).

Proof: Let \(\{A_i : i \in I\}\) be any collection of \(rg\gamma\)–closed subsets of \(X\) such that \(\bigcap A_i \subseteq H\) and \(H\) be \(rg\gamma\)–open in \(X\). But \(A_i\) is \(rg\gamma\)–closed subset of \(X\), for each \(i \in I\), then \(\gamma cl(A_i) \subseteq H\), for each \(i \in I\). Hence \(\bigcap \gamma cl(A_i) \subseteq H\), for each \(i \in I\). Therefore \(\gamma cl(\bigcap A_i) \subseteq H\). So, \(\bigcap A_i\) is \(rg\gamma\)–closed subset of \(X\).

Theorem 2.8: A subset \(A\) of a space \((X, \tau)\) is \(rg\gamma\)–closed if and only if for each \(A \subseteq H\) and \(H\) is \(rg\gamma\)–open, there exists a \(\gamma\)–closed set \(F\) such that \(A \subseteq F \subseteq H\).

Proof: Suppose that \(A\) is a \(rg\gamma\)–closed set, \(A \subseteq H\) and \(H\) is a \(rg\gamma\)–open set. Then \(\gamma cl(A) \subseteq H\). If we put \(F = \gamma cl(A)\), hence \(A \subseteq F \subseteq H\).

Conversely, Assume that \(A \subseteq H\) and \(H\) is a \(rg\gamma\)–open set. Then by hypothesis there exists a \(\gamma\)–closed set \(F\) such that \(A \subseteq F \subseteq H\). So, \(A \subseteq \gamma cl(A) \subseteq F\) and hence \(\gamma cl(A) \subseteq H\). Therefore \(A\) is \(rg\gamma\)–closed.

Theorem 2.9: If \(A\) is closed and \(B\) is a \(rg\gamma\)–closed subset of a space \(X\), then \(A \cup B\) is also \(rg\gamma\)–closed.

Proof: Suppose that \(A \cup B \subseteq H\) and \(H\) is a \(rg\gamma\)–open set. Then \(A \subseteq H\) and \(B \subseteq H\). But \(B\) is \(rg\gamma\)–closed, then \(\gamma cl(B) \subseteq H\) and hence \(A \cup B \subseteq A \cup \gamma cl(B) \subseteq H\). But \(A \cup \gamma cl(B)\) a \(\gamma\)–closed set, hence there exists a \(\gamma\)–closed set \(A \cup \gamma cl(B)\) such that \(A \cup B \subseteq A \cup \gamma cl(B) \subseteq H\). Thus by Theorem 2.8, \(A \cup B\) is \(rg\gamma\)–closed.

Theorem 2.10: If a subset \(A\) of \(X\) is a \(rg\gamma\)–closed set in \(X\), then \(\gamma cl(A) \setminus A\) does not contain any non-empty \(rg\gamma\)–open set in \(X\).

Proof: Assume that \(A\) is a \(rg\gamma\)–closed set in \(X\). Let \(H\) be a \(rg\gamma\)–open set such that \(H \subseteq \gamma cl(A) \setminus A\) and \(H \neq \phi\). New \(H \subseteq \gamma cl(A) \setminus A\). Therefore \(H \subseteq XA\) which implies \(A \subseteq XH\). Since \(H\) is a \(rg\gamma\)–open set, then \(XH\) is also \(rg\gamma\)–open in \(X\). But \(A\) is a \(rg\gamma\)–closed set in \(X\), then \(\gamma cl(A) \subseteq XH\). So, \(H \subseteq X \gamma cl(A)\). Also, \(H \subseteq \gamma cl(A)\). Hence, \(H \subseteq (\gamma cl(A)\)
\( \cap (X \gamma cl(A))) = \emptyset \). This shows that, \( H = \emptyset \) which is a contradiction. Therefore, \( \gamma cl(A) \backslash A \) does not contain any non-empty \( r\gamma \)-open set in \( X \).

The converse of the above theorem need not be true as seen from the following example.

**Example 2.11:** In Example 2.4, if \( A = \{ a, b \} \), then \( \gamma cl(A) \backslash A = \{ a, b \} \backslash \{ a, b \} = \{ c \} \) does not contain any non-empty \( r\gamma \)-open set in \( X \). But \( A \) is not a \( r\gamma \)-closed set in \( X \).

**Proposition 2.12:** If a subset \( A \) of \( X \) is a \( r\gamma \)-closed set in \( X \), then \( \gamma cl(A) \backslash A \) does not contain any regular closed (resp. regular open) set in \( X \).

**Proof:** Follows directly from Theorem 2.10 and the fact that every regular open set is \( r\gamma \)-open.

**Theorem 2.13:** For an element \( p \in X \), the set \( X \backslash \{ p \} \) is \( r\gamma \)-closed or \( r\gamma \)-open.

**Proof:** Suppose that \( X \backslash \{ p \} \) is not a \( r\gamma \)-open set. Then \( X \) is the only \( r\gamma \)-open set containing \( X \backslash \{ p \} \). This implies that \( \gamma cl(X\backslash \{ p \}) \subseteq X \). Hence \( X \backslash \{ p \} \) is a \( r\gamma \)-closed set in \( X \).

**Theorem 2.14:** If \( A \) is regular open and \( r\gamma \)-closed, then \( A \) is regular closed and hence \( \gamma \)-clopen.

**Proof:** Assume that \( A \) is regular open and \( r\gamma \)-closed. Since every regular open set is \( r\gamma \)-open and \( A \subseteq A \), we have \( \gamma cl(A) \subseteq A \). But \( A \subseteq \gamma cl(A) \). Therefore \( A = \gamma cl(A) \), that is, \( A \) is \( \gamma \)-closed. Since \( A \) is regular open, \( A \) is \( \gamma \)-open. Therefore \( A \) is regular closed and \( \gamma \)-clopen.

**Theorem 2.15:** If \( A \) is a \( r\gamma \)-closed set of \( X \) such that \( A \subseteq B \subseteq \gamma cl(A) \), then \( B \) is a \( r\gamma \)-closed set in \( X \).

**Proof:** Let \( H \) be a \( r\gamma \)-open set of \( X \) such that \( B \subseteq H \). Then \( A \subseteq H \). But \( A \) is a \( r\gamma \)-closed set of \( X \), then \( \gamma cl(B) \subseteq \gamma cl(\gamma cl(A)) = \gamma cl(A) \subseteq H \). Therefore \( B \) is a \( r\gamma \)-closed set in \( X \).

**Remark 2.16:** The converse of the above theorem need not be true in general. In Example 2.4, if we take a subset \( A = \{ b \} \) and \( B = \{ b, c \} \), then \( A \) and \( B \) are \( r\gamma \)-closed set in \( (X, \tau) \), but \( A \subseteq B \) is not subset in \( \gamma cl(A) \).

**Theorem 2.17:** Let \( A \) be a \( r\gamma \)-closed set in \( X \). Then \( A \) is \( \gamma \)-closed if and only if \( \gamma cl(A) \backslash A \) is \( r\gamma \)-open.

**Proof:** Suppose that \( A \) is \( \gamma \)-closed in \( X \). Then \( \gamma cl(A) = A \) and so \( \gamma cl(A) \backslash A = A = \emptyset \), which is \( r\gamma \)-open in \( X \). Conversely, Suppose that \( \gamma cl(A) \backslash A \) is a \( r\gamma \)-open set in \( X \). Since \( A \) is \( r\gamma \)-closed, then by Theorem 2.10, \( \gamma cl(A) \backslash A \) does not contain any non-empty \( r\gamma \)-open set in \( X \). Then \( \gamma cl(A) \backslash A = \emptyset \), hence \( A \) is a \( \gamma \)-closed set in \( X \).
Theorem 2.18: If $A$ is regular open and $rg$–closed, then $A$ is a $rg\gamma$–closed set in $X$.

Proof: Let $H$ be any $rg$–open set in $X$ such that $A \subseteq H$. Since $A$ is regular open and $rg$–closed, we have $\gamma cl(A) \subseteq A$. Then $\gamma cl(A) \subseteq A \subseteq H$. Hence $A$ is a $rg\gamma$–closed set in $X$.

Theorem 2.19: If a subset $A$ is both $rg$–open and $rg\gamma$–closed in a topological space $(X, \tau)$, then $A$ is $\gamma$–closed.

Proof: Assume that $A$ is both $rg$–open and $rg\gamma$–closed in a topological space $(X, \tau)$. Then $\gamma cl(A) \subseteq A$. Hence $A$ is $\gamma$–closed.

Theorem 2.20: If $A$ is both $rg$–open and $rg\gamma$–closed subset in $X$ and $F$ is a closed set in $X$, then $A \cap F$ is a $rg\gamma$–closed set in $X$.

Proof: Let $A$ be $rg$–open and $rg\gamma$–closed subset in $X$ and $F$ be a closed set in $X$. Then by Theorem 2.19, $A$ is $\gamma$–closed. So, $A \cap F$ is $\gamma$–closed. Therefore $A \cap F$ is a $rg\gamma$–closed set in $X$.

Theorem 2.21: If $A$ is an open set and $H$ is a $\gamma$–open set in a topological space $(X, \tau)$, then $A \cap H$ is a $rg\gamma$–closed set in $X$.

Proof: Let $A$ be an open and $g$–closed subset in $X$ and $A \subseteq H$, where $H$ is a $rg$–open set in $X$. Then by hypothesis $\gamma cl(A) \subseteq A$, that is, $\gamma cl(A) \subseteq H$. Thus $A$ is a $rg\gamma$–closed set in $X$.

Remark 2.23: If $A$ is both open and $rg\gamma$–closed subset in $X$, then $A$ need not be $g$–closed set in $X$. In Example 2.4, the subset $\{a, d, e\}$ is an open and $rg\gamma$–closed but not $g$–closed.

Theorem 2.24: For a topological space $(X, \tau)$, if $R\gamma O(X, \phi) = \{X, \phi\}$, then every subset of $X$ is a $rg\gamma$–closed subset in $X$.

Proof: Let $(X, \tau)$ be a space and $R\gamma O(X, \tau) = \{X, \phi\}$. Let $A$ be a subset of $X$. (i) if $A = \phi$, then $A$ is a $rg\gamma$–closed subset in $X$. (ii) if $A \neq \phi$, then $X$ is the only a $rg$–open set in $X$ containing $A$ and so $\gamma cl(A) \subseteq X$. Hence $A$ is a $rg\gamma$–closed subset in $X$.

The converse of Theorem 2.24 need not be true in general as seen from the following example.

Example 2.25: Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \phi, \{a, b\}, \{c, d\}\}$. Then every subset of $(X, \tau)$ is a $rg\gamma$–closed subset in $X$. But $R\gamma O(X, \tau) = \tau$.

Theorem 2.26: For a topological space $(X, \tau)$, then $R\gamma O(X, \tau) \subseteq \{F \subseteq X : F$ is closed$\}$ if and only if every subset of $X$ is a $rg\gamma$–closed subset in $X$. 

Proof: Suppose that $R\gamma O(X, \tau) \subseteq \{F \subseteq X : F \text{ is closed}\}$. Let $A$ be any subset of $X$ such that $A \subseteq H$, where $H$ is a $r\gamma$-open set in $X$. Then $H \in R\gamma O(X, \tau) \subseteq \{F \subseteq X : F \text{ is closed}\}$. That is $H \in \{F \subseteq X : F \text{ is closed}\}$. Thus $H$ is a $\gamma$-closed set. Then $\gamma cl(H) = H$. Also, $\gamma cl(A) \gamma cl(H) \subseteq H$. Hence $A$ is a $rg\gamma$-closed subset in $X$.

Conversely, Suppose that every subset of $(X, \tau)$ is a $rg\gamma$-closed subset in $X$. Let $H \in R\gamma O(X, \tau)$.

**Definition 2.27**: The intersection of all $r\gamma$-open subsets of $(X, \tau)$ containing $A$ is called the regular $\gamma$-kernel of $A$ and is denoted by $r\gamma \ker(A)$.

**Lemma 2.28**: For any subset $A$ of a topological space $(X, \tau)$, then $A \subseteq r\gamma \ker(A)$.

**Proof**: Follows directly from Definition 2.27.

**Lemma 2.29**: Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. If $A$ is a $r\gamma$-open in $X$, then $r\gamma \ker(A) = A$.

**Lemma 2.30**: Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. Then $\gamma \ker(A) \subseteq r\gamma \ker(A)$.

**Proof**: Follows from the implication $R\gamma O(X, \tau) \subseteq \gamma O(X, \tau)$.

3. REGULAR GENERALIZED $\gamma$-OPEN SETS

In this section, we introduce and study the concept of regular generalized $\gamma$-open (briefly, $rg\gamma$-open) sets in topological spaces and obtain some of their properties.

**Definition 3.1**: A subset $A$ of a topological space $(X, \tau)$ is called a regular generalized $\gamma$-open (briefly, $rg\gamma$-open) set in $X$ if $A^c$ is a $rg\gamma$-closed subset in $X$. We denote the family of all $rg\gamma$-open sets in $X$ by $RG\gamma O(X)$.

The following theorem gives equivalent definitions of a $rg\gamma$-open set.

**Theorem 3.2**: Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then the following statements are equivalent:

(i) $A$ is a $rg\gamma$-open set,

(ii) for each $r\gamma$-closed set $F$ contained in $A$, $F \subseteq \gamma \int(A)$,

(iii) for each $r\gamma$-closed set $F$ contained in $A$, there exists a $\gamma$-open set $H$ such that $F \subseteq H \subseteq A$.

**Proof**: (i) $\Rightarrow$ (ii). Let $F \subseteq A$ and $F$ be a $r\gamma$-closed set. Then $X \setminus A \subseteq X \setminus F$ which is $r\gamma$-open of $X$, hence $\gamma cl(X \setminus A) \subseteq X \setminus F$. So, $F \subseteq \gamma \int(A)$.

(ii) $\Rightarrow$ (iii). Suppose that $F \subseteq A$ and $F$ be a $r\gamma$-closed set. Then by hypothesis, $F \subseteq \gamma \int(A)$. Set $H = \gamma \int(A)$, hence there exists a $\gamma$-open set $H$ such that $F \subseteq H \subseteq A$. 

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(iii) $\Rightarrow$ (i). Assume that $X \setminus A \subseteq V$ and $V$ is $r\gamma$–open of $X$. Then by hypothesis, there exists a $\gamma$–open set $H$ such that $X \setminus V \subseteq H \subseteq A$, that is, $X \setminus A \subseteq X \setminus H \subseteq V$. Therefore, by Theorem 2.8, $X \setminus A$ is $rg\gamma$–closed in $X$ and hence $A$ is $rg\gamma$–open in $X$.

**Theorem 3.3:** For a topological space $(X, \tau)$, then every $\omega$– open set is $rg\gamma$–open.

**Proof:** Let $A$ be a $\omega$– open set in $X$. Then $A^c$ is $\omega$–closed in $X$. Hence by Proposition 2.3, $A^c$ is $rg\gamma$–closed in $X$. Therefore, $A$ is $rg\gamma$–open.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.4:** In Example 2.4, the subset $A = \{b\}$ is $rg\gamma$–open but not $\omega$– open.

**Theorem 3.5:** For a topological space $(X, \tau)$, then every open ( resp. regular-open) set is $rg\gamma$–open but not conversely.

**Proof:** Follows from S. John [21] ( resp. Stone [22] ) and Theorem 3.3.

**Theorem 3.6:** Let $(X, \tau)$ be a topological space. Then every $rg\gamma$–open set of $X$ is $rg$–open (resp. $gpr$–open).

**Proof:** (i) Let $A$ be a $rg\gamma$–open set of $X$. Then $A^c$ is $rg\gamma$–closed set of $X$. Then by Proposition 2.3, $A^c$ is a $rg$–closed set of $X$. Therfore $A$ is $rg$–open in $X$.

(ii) For the case of a $gpr$–open set is similar to (i).

The converse of the above theorem need not be true as seen from the following example.

**Example 3.7:** In Example 2.4, the subset $\{a, b\}$ is $rg$–open (resp. $gpr$–open) but not a $rg\gamma$–open set in $X$.

**Theorem 3.8:** The arbitrary union of $rg\gamma$–open sets of $X$ is $rg\gamma$–open.

**Proof:** Obvious from Theorem 2.7.

**Remark 3.9:** The intersection of two $rg\gamma$–open subsets of $X$ need not be a $rg\gamma$–open subset of $X$. In Remark 2.6, two subsets $\{a, b, d, e\}$ and $\{a, b, c, e\}$ of $X$ are $rg\gamma$–open subsets but their intersection $\{a, b, e\}$ is not $rg\gamma$–open subset of $X$.

**Theorem 3.10:** If $A$ is an open and $B$ is a $rg\gamma$–open subsets of a space $X$, then $A \cap B$ is also $rg\gamma$–open.

**Proof:** Follows from Theorem 2.9.

**Proposition 3.11:** If $\gamma – int(A) \subseteq B \subseteq A$ and $A$ is a $rg\gamma$–open set of $X$, then $B$ is $rg\gamma$–open.

**Proposition 3.12:** Let $A$ be a $r\gamma$–closed and a $rg\gamma$–open set of $X$. Then $A$ is $\gamma$–open.
Proof: Assume that \( A \) is a \( r\gamma \)-closed and a \( r\gamma \)-open set of \( X \). Then \( A \subseteq \gamma - \text{int}(A) \) and hence \( A \) is \( \gamma \)-open.

**Theorem 3.13:** For a space \((X, \tau)\), if \( A \) is a \( r\gamma \)-closed set, then \( \gamma \text{cl}(A) \backslash A \) is \( r\gamma \)-open.

**Proof:** Suppose that \( A \) is a \( r\gamma \)-closed set and \( F \) is a \( r\gamma \)-closed set contained in \( \gamma \text{cl}(A) \backslash A \). Then by Theorem 2.8, \( F = \emptyset \) and hence \( F \subseteq \gamma - \text{int}(\gamma \text{cl}(A) \backslash A) \). Therefore, \( \gamma \text{cl}(A) \backslash A \) is \( r\gamma \)-open.

**Theorem 3.14:** If a subset \( A \) of a space \((X, \tau)\) is \( r\gamma \)-open, then \( H = X \), whenever \( H \) is \( r\gamma \)-open and \( \text{int}(A) \cup A^c \subseteq H \).

**Proof:** Assume that \( A \) is \( r\gamma \)-open in \( X \). Let \( H \) be \( r\gamma \)-open and \( \text{int}(A) \cup A^c \subseteq H \). This implies that \( H^c \subseteq (\text{int}(A) \cup A^c)^c = (\text{int}(A))^c \cap A \), that is, \( H^c \subseteq (\text{int}(A))^c \backslash A^c \). Thus \( H^c \subseteq \text{cl}(A)^c \backslash A^c \), since \( (\text{int}(A))^c = \text{cl}(A)^c \). Now \( H^c \) is \( r\gamma \)-open and \( A^c \) is \( r\gamma \)-closed, hence by Theorem 2.10, it follows that \( H^c = \emptyset \). Then \( H = X \).

The converse of the above theorem is not true in general as seen from the following example.

**Example 3.15:** Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{X, \emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\} \). Then \( \text{RG}O(X) = \{X, \emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{b, d\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\} \) and \( \text{rg}O(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, d\}, \{b, c\}, \{b, c, d\}\} \). If we take \( A = \{b, c, d\} \), then \( A \) is not \( \text{rg}\gamma \)-open. However \( \text{int}(A) \cup A^c = \{b, c\} \cup \{a\} = \{a, b, c\} \). So, for some \( r\gamma \)-open \( H \), we have \( \text{int}(A) \cup A^c = \{a, b, c\} \subseteq H \), gives \( H = X \), but \( A \) is not \( \text{rg}\gamma \)-open.

**Theorem 3.16:** For a topological space \((X, \tau)\), then every singleton of \( X \) is either \( \text{rg}\gamma \)-open or \( r\gamma \)-open.

**Proof:** Let \((X, \tau)\) be a topological space. Let \( p \in X \). To prove that \( \{p\} \) is either \( \text{rg}\gamma \)-open or \( r\gamma \)-open. That is to prove \( X \setminus \{p\} \) is either \( \text{rg}\gamma \)-closed or \( r\gamma \)-open, which follows directly from Theorem 2.13.

**4. SOME APPLICATIONS ON \( \text{rg}\gamma \)-CLOSED AND \( \text{rg}\gamma \)-OPEN SETS**

In this section, we introduce and study the concept of regular generalized \( \gamma \)-neighbourhood (briefly, \( \text{rg}\gamma \)-nbd) in topological spaces by using the concept of \( \text{rg}\gamma \)-open sets. Further, we prove that every neighbourhood of a point \( p \) in \( X \) is \( \text{rg}\gamma \)-nbd of \( p \) but conversely not true.

**Definition 4.1:** Let \((X, \tau)\) be a topological space and let \( p \in X \). A subset \( N \) of \( X \) is said to be a regular generalized \( \gamma \)-neighbourhood (briefly, \( \text{rg}\gamma \)-nbd) of \( p \) if and only if there exists a \( r\gamma \)-open set \( H \) such that \( p \in H \subseteq N \).
**Definition 4.2:** A subset $N$ of a topological space $(X, \tau)$ is called a regular generalized $\gamma$-neighbourhood (briefly, $rg\gamma$–nbd) of a subset $A$ of a space $(X, \tau)$ if and only if there exists a $r\gamma$–open set $H$ such that $A \subseteq H \subseteq N$.

**Remark 4.3:** The $rg\gamma$–nbd of $N$ of a point $p$ of a space $X$ need not be a $rg\gamma$–open in $X$.

**Example 4.4:** Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $RG\gamma O(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}$. Note that $\{a, c\}$ is not a $rg\gamma$–open set, but it is a $rg\gamma$–nbd of $\{a\}$. Since $\{a\}$ is a $rg\gamma$–open set such that $a \in \{a\} \subseteq \{a, c\}$.

**Theorem 4.5:** Every nbd $N$ of $p \in X$ is a $rg\gamma$–nbd of $X$.

**Proof:** Let $N$ be a nbd of point $p$. To prove that $N$ is a $rg\gamma$–nbd of $p$. Then by definition of nbd, there exists an open set $H$ such that $p \in H \subseteq N$. Since every open set is $r\gamma$–open, hence there exists a $r\gamma$–open set $H$ such that $p \in H \subseteq N$. Therefore $N$ is a $rg\gamma$–nbd of $p$.

The converse of the above theorem is not true as seen from the following example.

**Example 4.6:** In Example 4.4, the set $\{a, c\}$ is $rg\gamma$–nbd of the point $c$, since $\{c\}$ is a $r\gamma$–open set such that $c \in \{c\} \subseteq \{a, c\}$. However, the set $\{a, c\}$ is not a nbd of the point $c$, since there is no exists an open set $H$ such that $c \in H \subseteq \{a, c\}$.

**Theorem 4.7:** If $N$ is a $rg\gamma$–open set of a space $(X, \tau)$, then $N$ is a $rg\gamma$–nbd of each of its points.

**Proof:** Assume that $N$ is a $rg\gamma$–open set. Let $p \in X$. We need to prove that $N$ is a $rg\gamma$–nbd of $p$. For $N$ is a $r\gamma$–open set such that $p \in N \subseteq N$. Since $p$ is an arbitrary point of $N$, it follows that $N$ is a $rg\gamma$–nbd of each of its points.

The converse of the above theorem is not true in general as seen from the following example.

**Example 4.8:** In Example 4.4, the set $\{a, d\}$ is a $rg\gamma$–nbd of the point $a$, since $\{a\}$ is the $r\gamma$–open set such that $a \in \{a\} \subseteq \{a, d\}$. Also, the set $\{a, d\}$ is a $rg\gamma$–nbd of the point $d$, since $\{d\}$ is the $r\gamma$–open set such that $d \in \{d\} \subseteq \{a, d\}$, that is $\{a, d\}$ is a $rg\gamma$–nbd of each of its points. However the set $\{a, d\}$ is not a $r\gamma$–open set in $X$.

**Theorem 4.9:** If $F$ is a $rg\gamma$–closed set of a space $(X, \tau)$ and $p \in F^c$, then there exists a $rg\gamma$–nbd $N$ of $p$ such that $N \cap F = \emptyset$.

**Proof:** Let $F$ be a $rg\gamma$–closed subset of a space $(X, \tau)$ and $p \in F^c$. Then $F^c$ is a $r\gamma$–open set of $X$. So, by Theorem 4.7, $F^c$ contains a $rg\gamma$–nbd of each of its points. Hence, there exists a $rg\gamma$–nbd $N$ of $p$ such that $N \subseteq F^c$, that is, $N \cap F = \emptyset$.

**Definition 4.10:** Let $p$ be a point in a space $X$. The set of all a $rg\gamma$–nbd of $p$ is called the $rg\gamma$–nbd system at $p$ and is denoted by $rg\gamma – N(p)$. 
Theorem 4.11: Let \((X, \tau)\) be a topological space and for each \(p \in X\). Let \(rg_\gamma - N(p)\) be the collection of all \(rg_\gamma - nbds\) of \(p\). Then the following statements are hold.

1. \(rg_\gamma - N(p) \neq \emptyset\), for every \(p \in X\),
2. If \(N \in rg_\gamma - N(p)\), then \(p \in N\),
3. If \(N \in rg_\gamma - N(p)\) and \(N \subseteq M\), then \(M \in rg_\gamma - N(p)\),
4. If \(N \in rg_\gamma - N(p)\) and \(M \in rg_\gamma - N(p)\), then \(N \cap M \in rg_\gamma - N(p)\),
5. If \(N \in rg_\gamma - N(p)\), then there exists \(M \in rg_\gamma - N(p)\) such that \(M \subseteq N\) and \(M \in rg_\gamma - N(q)\), for every \(q \in M\).

Proof:

1. Since \(X\) is a \(rg_\gamma - open\) set, then \(X\) is a \(rg_\gamma - nbd\) for every \(p \in X\). Hence there exists at least one \(rg_\gamma - nbd\) (namely, \(X\)), for each \(p \in X\). Therefore \(rg_\gamma - N(p) \neq \emptyset\), for every \(p \in X\).
2. If \(N \in \), then \(N\) is a \(rg_\gamma - nbd\) of \(p\). Hence by definition of \(rg_\gamma - nbd\), \(p \in N\).
3. Let \(N \in \) and \(N \subseteq M\). Then there is a \(rg_\gamma - open\) set \(H\) such that \(p \in H \subseteq N\). Since \(N \subseteq M\), then \(p \in H \subseteq M\) and so \(M\) is a \(rg_\gamma - nbd\) of \(p\). Hence \(M \in \).
4. Let \(N \in \) and \(M \in \). Then by definition of \(rg_\gamma - nbd\), there exist \(rg_\gamma - open\) sets \(H_1\) and \(H_2\) such that \(p \in H_1 \subseteq N\) and \(p \in H_2 \subseteq M\). Hence \(p \in H_1 \cap H_2 \subseteq N \cap M\) ...
5. If \(N \in \), then there exists a \(rg_\gamma - open\) set \(M\) such that \(p \in M \subseteq N\). Since \(M\) is a \(rg_\gamma - open\) set, then \(M\) is a \(rg_\gamma - nbd\) of each of its points. Therefore \(M \in \), for every \(q \in M\).

Theorem 4.12: For each \(p \in X\), let \(rg_\gamma - N(p)\) be a non empty collection of subsets of \(X\) satisfying the following conditions.

1. If \(N \in \) implies that \(p \in N\),
2. If \(N \in \) and \(M \in \) implies that \(N \cap M \in \).

Let \(\tau\) consists of the empty set \(\emptyset\) and all those non-empty subsets \(H\) of \(X\) having the property that \(p \in H\) implies that there exists an \(N \in \) such that \(p \in H \subseteq N\). Then \(\tau\) is a topology on \(X\).

Proof:

1. Since \(\emptyset \in \) (given), now we show that \(X \in \tau\). Let \(p\) be an arbitrary element of \(X\). Since \(rg_\gamma - N(p)\) is non empty, then there is an \(N \in \) and so by (1), \(p \in N\). Since \(N\) is a subset of \(X\), then we have \(p \in N \subseteq X\). Hence \(X \in \tau\).
2. Let \(H_1 \in \tau\) and \(H_2 \in \tau\). If \(p \in H_1 \cap H_2\), then \(p \in H_1\) and \(p \in H_2\). Since \(H_1\) is
\( \tau \) and \( H_2 \in \tau \), hence there exists \( N \in \text{rg}_\gamma - N(p) \) and \( M \in \text{rg}_\gamma - N(p) \) such that \( p \in N \subseteq H_1 \) and \( p \in M \subseteq H_2 \). Then \( p \in N \cap M \subseteq H_1 \cap H_2 \). But by (2), \( N \cap M \in \text{rg}_\gamma - N(p) \). Hence \( H_1 \cap H_2 \in \tau \).

(3) Let \( H_i \in \tau \) for every \( i \in I \). If \( p \in \cup \{H_i : i \in I\} \), then \( p \in H_i \), for some \( i \in I \). Since \( H_i \in \tau \), then there exists an \( N \in \text{rg}_\gamma - N(p) \) such that \( p \in N \subseteq H_i \) and consequently, \( p \in N \subseteq \cup \{H_i : i \in I\} \). Hence \( \cup \{H_i : i \in I\} \in \tau \). Therefore \( \tau \) is a topology on \( X \).

In the following, we introduce the notion of \( \gamma \)-regular \( T_{1\frac{1}{2}} \)-spaces.

**Definition 4.13:** A space \((X, \tau)\) is called a \( \gamma \)-regular \( T_{1\frac{1}{2}} \)-space if every \( \text{rg}_\gamma \)-closed set is \( \gamma \)-closed.

**Example 4.14:** Any set with an indiscrete topology is an example for a \( \gamma \)-regular \( T_{1\frac{1}{2}} \)-space.

**Remark 4.15:** The notions of \( \gamma \)-regular \( T_{1\frac{1}{2}} \) and \( T_{1\frac{1}{2}} \)-spaces are independent of each other.

**Example 4.16:** Let \( X = \{a, b, c\} \) with topologies \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) and \( \sigma = \{X, \phi, \{c\}, \{a, b\}\} \). Then \((X, \tau)\) is \( T_{1\frac{1}{2}} \) but not \( \gamma \)-regular \( T_{1\frac{1}{2}} \) while as \((X, \sigma)\) is \( \gamma \)-regular \( T_{1\frac{1}{2}} \) but not \( T_{1\frac{1}{2}} \).

**Theorem 4.17:** For a topological space \((X, \tau)\), the following conditions are equivalent:

1. \( X \) is \( \gamma \)-regular \( T_{1\frac{1}{2}} \),
2. Every singleton of \( X \) is either regular closed or \( \gamma \)-open.

**Proof:** (1) \( \Rightarrow \) (2). Let \( p \in X \) and assume that \( \{p\} \) is not regular closed.

Then \( X \setminus \{p\} \) is not regular open and hence \( X \setminus \{p\} \) is \( \text{rg}_\gamma \)-closed. Hence, by hypothesis, \( X \setminus \{p\} \) is \( \gamma \)-closed and thus \( \{p\} \) is \( \gamma \)-open.

(2) \( \Rightarrow \) (1). Let \( A \subseteq X \) be \( \text{rg}_\gamma \)-closed and \( p \in \gamma cl(A) \). We will show that \( p \in A \). For consider the following two cases:

Case (1). The set \( \{p\} \) is regular closed. Then, if \( p \notin A \), then there exists a regular closed set in \( \gamma cl(A) \setminus A \). Hence by Proposition 2.12, \( p \in A \).

Case (2). The set \( \{p\} \) is \( \gamma \)-open. Since \( p \in \gamma cl(A) \), then \( \{p\} \cap \gamma cl(A) \neq \phi \).

Thus \( p \in A \). So, in both cases, \( p \in A \). This shows that \( \gamma cl(A) \subseteq A \) or equivalently \( A \) is \( \gamma \)-closed.

**Definition 4.18:** A space \((X, \tau)\) is called:
(1) locally indiscrete (equivalently, partition space)[5] if every closed set is regular closed,
(2) $\gamma T_1$ space [11] if every $\gamma g$–closed set is $\gamma$–closed.

**Theorem 4.19:** For a topological space $(X, \tau)$, the following conditions are equivalent:

1. $X$ is $\gamma$–regular $T_1$,
2. $X$ is locally indiscrete and $\gamma T_1$.

**Proof:** (1) $\Rightarrow$ (2). Let $(X, \tau)$ be a $\gamma$–regular $T_1$ space. Then by Theorem 4.17, every singleton of $(X, \tau)$ is closed or $\gamma$–open. This implies that $(X, \tau)$ is a $\gamma T_1$ space.

(2) $\Rightarrow$ (1). Since the space $X$ is $\gamma T_1$, then every singleton of $(X, \tau)$ is closed or $\gamma$–open. But $(X, \tau)$ is locally indiscrete, hence all closed singletons are regular closed. So, by theorem 4.17, the space $X$ is $\gamma$–regular $T_1$.

**Remark 4.20:** Every $\gamma$–regular $T_1$ space is $\gamma T_1$. But the converse is not true as seen by the following example.

**Example 4.21:** Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, X, \{p\}, \{q\}, \{p, q\}\}$. Then every singleton of $(X, \tau)$ is closed or $\gamma$–open. Therefore $(X, \tau)$ is $\gamma T_1$, but not $\gamma$–regular $T_1$, since the $rg\gamma$–closed set $\{p, q\}$ is not $\gamma$–closed.

**Proposition 4.22:** For a space $(X, \tau)$, we have $\gamma O(X, \tau) \subseteq RG\gamma O(X, \tau)$.

**Proof:** Let $A$ be a $\gamma$–open set. Then $A^c$ is $\gamma$–closed and so $rg\gamma$–closed. This implies that $A$ is $rg\gamma$–open. Hence $\gamma O(X, \tau) \subseteq RG\gamma O(X, \tau)$.

**Theorem 4.23:** For a topological space $(X, \tau)$, the following conditions are equivalent:

1. $X$ is $\gamma$–regular $T_1$,
2. $\gamma O(X, \tau) = RG\gamma O(X, \tau)$.

**Proof:** (1) $\Rightarrow$ (2). Let $(X, \tau)$ be a $\gamma$–regular $T_1$ space and let $A \in RG\gamma O(X, \tau)$. Then $A^c$ is $rg\gamma$–closed. By hypothesis, $A^c$ is $\gamma$–closed and thus $A$ is $\gamma$–open this implies that $A \in \gamma O(X, \tau)$. Hence, $\gamma O(X, \tau) = RG\gamma O(X, \tau)$.

(2) $\Rightarrow$ (1). Let $\gamma O(X, \tau) = RG\gamma O(X, \tau)$ and let $A$ be a $rg\gamma$–closed set. Then $A^c$ is $rg\gamma$–open. Hence $A^c \in \gamma O(X, \tau)$. Thus $A$ is $\gamma$–closed thereby implying that $(X, \tau)$ is $\gamma$–regular $T_1$. 
**Theorem 4.24:** For a topological space \((X, \tau)\), the following conditions are equivalent:

1. \(X\) is \(\gamma\)-regular \(T_{\frac{1}{2}}\),
2. Every nowhere dense singleton of \(X\) is regular closed,
3. The only nowhere dense subset of \(X\) is the empty set,
4. Every subset of \(X\) is \(\gamma\)-open.

**Proof:**

(1) \(\Rightarrow\) (2). By Theorem 4.17, every singleton is \(\gamma\)-open or regular closed. Since non-empty nowhere dense set can not be \(\gamma\)-open at the same time, then every nowhere dense singleton of \(X\) is regular closed.

(2) \(\Rightarrow\) (1). Since every singleton is either \(\gamma\)-open or nowhere dense [10], then by hypothesis, every singleton of \(X\) is either \(\gamma\)-open or regular closed. Hence, by Theorem 4.17, \(X\) is \(\gamma\)-regular \(T_{\frac{1}{2}}\).

(2) \(\Rightarrow\) (3). Let \(A \subseteq X\) be a non-empty nowhere dense set. Then, there exists a nowhere dense singleton in \(X\) and hence by hypothesis, this singleton is regular closed. If the nowhere dense singleton \(\{p\}\) is regular closed, then \(\{p\} = cl(int\{p\}) = cl(\phi) = \phi\) which is the contradiction with this assumption.

(3) \(\Rightarrow\) (4). By hypothesis and the fact that singletons are either \(\gamma\)-open or nowhere dense, it follows that every singleton of \(X\) is \(\gamma\)-open. Since the union of \(\gamma\)-open sets is \(\gamma\)-open. Hence, every subset of \(X\) is \(\gamma\)-open.

(4) \(\Rightarrow\) (1). From the definition of a \(\gamma\)-regular \(T_{\frac{1}{2}}\) space and by hypothesis, every subset of \(X\) is \(\gamma\)-open and hence \(\gamma\)-closed.

**REFERENCES**

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