AN INVENTORY MODEL FOR DETERIORATING ITEMS WITH MULTI-VARIATE DEMAND AND PARTIAL BACKLOGGING UNDER INFLATION

A. K. Malik & Ashu Sharma

Abstract: This paper deals with an optimal replenishment policy for deteriorating items with multi-variate demand and partial backlogging. The proposed model allows for (1) multivariate demand replenishment cycles, (2) partial backlogging, and (3) time-varying shortage intervals. Consequently, the proposed model is in a general framework for that inventory models includes these above assumptions. We shown that the conditions the optimal replenishment schedule exists uniquely, and provide a good estimate for finding the optimal replenishment number.

1. INTRODUCTION

Ghare and Schrader (1963) first developed an economic order quantity (EOQ) model by taking exponential decay. After that, Covert and Philip (1973) extended Ghare and Schrader’s constant deterioration rate taking into a two-parameter Weibull distribution. In real-life situations, for certain types of consumer goods (e.g., fruits, electronics items, cloths, vegetables, donuts, and others), the consumption rate is sometimes influenced by the stock-level. The related analysis on such inventory model with stock-dependent consumption rate was discussed by Levin et al., (1972), Balkhi and Benkherouf (2004), etc. Teng et al., (2003) then extended the fraction of unsatisfied demand backordered to any decreasing function of the waiting time up to the next replenishment. Teng and Yang (2004) further generalized the partial backlogging EOQ model to allow for time-varying purchase cost. Moreover, the effects of inflation and time value of money are vital in practical environment, especially in the developing countries with large scale inflation. Therefore, the effect of inflation and time value of money cannot be ignored in real situations. To relax the assumption of no inflationary effects on costs, Buzacott (1975) and Misra (1975) simultaneously developed an EOQ model with a constant inflation rate for all associated costs. Later, Yang et al., (2001) established various inventory models with time varying demand patterns under inflation. Recently, Chern et al., (2008) proposed partial backlogging inventory lot-size models for deteriorating items with fluctuating demand under inflation. Roy, Sana and Chaudhuri (2011) developed an optimal shipment strategy for imperfect items in a stock-out situation. Yong, Wang and Lai (2010) discussed an optimal production-inventory model for deteriorating items with multiple-market demand.
In this paper, we consider an economic order quantity (EOQ) model with multivariate demand rate, in which (1) shortages are partial backlogging to reflect the fact that longer the waiting time; the smaller the backlogging rate, (2) the effects of inflation and time value of money are relevant or vital, and (3) the replenishment cycles and the shortage intervals are time varying. Hence, the search for the optimal number of replenishments is simplified to finding a local maximum.

2. ASSUMPTIONS AND NOTATION

The mathematical model of the inventory replenishment problem is based on the following assumptions:

1. The replenishment rate is infinite and lead time is zero.
2. Shortages are allowed. Unsatisfied demand is partially backlogged.

For convenience, the following notation is used throughout the entire paper.

\[
D(t) = \begin{cases} 
  a + bt + cI(t), & I(t) > 0, \\
  a + bt, & I(t) \leq 0, 
\end{cases}
\]

the demand rate at time \( t \), where \( a, b \) are positive constants and \( I(t) \) is the inventory level at time \( t \).

- \( H \) the time horizon under consideration
- \( I_1(t) \) the inventory level at time \( t [t_s, s] \)
- \( I_2(t) \) the inventory level at time \( t [s_{i-1}, s_i] \)
- \( \sigma \) the backlogging rate which is a decreasing function of the waiting time \( t \), we here assume that \( \sigma(t) = e^{-\delta t} \), where \( \delta \geq 0 \), and \( t \) is the waiting time
- \( \theta \) the deterioration rate per unit per unit time
- \( r \) the discount rate
- \( i \) the inflation rate, which is varied by the social economical situations
- \( R \) \( r - i \), the discount rate minus the inflation rate
- \( p \) selling price per unit
- \( C_o \) the internal fixed order cost per order
- \( C_p \) the external variable purchasing cost per unit
- \( C_h \) the inventory holding cost per unit per unit time
- \( C_b \) the backlogging cost per unit per unit time
$C_i$ the cost of lost sales per unit

$n$ the number of replenishments over $[0, H]$ (a decision variable)

$t_i$ the $i^{th}$ replenishment time (a decision variable), $i=1; 2; \ldots; n$

$s_i$ the time at which the inventory level reaches zero after $t_i$ (a decision variable), $i=1; 2; \ldots; n$

### 3. Mathematical Model

The $i^{th}$ replenishment is made at time $t_i$. The quantity received at $t_i$ is used partly to meet the accumulated backorders in the previous cycle from time $s_{i-1}$ to $t_i$ ($s_{i-1} \leq t_i$). The inventory at $t_i$ gradually reduces to zero at $s_i$ ($s_i > t_i$). During the time interval $[t_i, s_i]$, the inventory is depleted by the combined effect of stock-dependent consumption rate and deterioration, the inventory level at time $t$ during the $i^{th}$ replenishment cycle is governed by the following differential equation:

$$\frac{dI_i(t)}{dt} + \theta I_i(t) = -[a + bt + cI_i(t)] \quad t_i \leq t \leq s_i \quad (1)$$

With $I_i(s_i) = 0$. Solving the differential equation (1), we get the inventory level as follows:

$$I_i(t) = \frac{[a(\theta + c) - b]}{(\theta + c)^2}(e^{(\theta + c)(s_i - t)} - 1) + \frac{b}{\theta + c}(s_i e^{(\theta + c)(s_i - t)} - t) \quad t_i \leq t \leq s_i \quad (2)$$

During the time interval $[s_{i-1}, t_i]$, the demand rate $D(t) = a + bt$ and the backlogging rate $\sigma(t_i - t) = e^{-\delta(t_i - t)}$. Hence, the amount of backorders $B(t)$ is governed by the following differential equation:

$$\frac{dI_2(t)}{dt} = (a + bt) e^{-\delta(t_i - t)} \quad s_{i-1} \leq t \leq t_i \quad (3)$$

With $I_2(s_{i-1}) = 0$. Solving the differential equation (3), we get the inventory level as follows:

$$I_2(t) = \frac{e^{-\delta t_i}}{2\delta^2} [(e^{\delta t_i - 1} + e^{\delta t_i})(\delta a - b) + \delta b (te^{\delta t_i} + s_{i-1} e^{\delta s_{i-1}})] \quad s_{i-1} \leq t \leq t_i \quad (4)$$

Next, the total relevant inventory cost per cycle consists of the following elements:

1. **Ordering cost** per cycle is $A$. (5)
2. Inventory holding cost during the $i^{th}$ replenishment cycle as

\[ HC = C_h \int_{t_j}^{s_i} e^{-Rt} I_1(t) \, dt \]

\[ HC_i = C_h \left\{ \frac{a(\theta + c) - b}{(\theta + c)^2} \left[ \frac{e^{-Rs_i}}{-(R + \theta + c)} + \frac{e^{-Rs_i}}{R} + \frac{e^{(\theta+c)s_i} e^{-(R+\theta+c)t_i}}{R + \theta + c} - \frac{e^{-Rt_i}}{R} \right] \right\} \]

\[ + b \left\{ \frac{s_i e^{-Rs_i}}{-(R + \theta + c)} + \frac{s_i e^{-Rs_i}}{R^2} + \frac{s_i e^{(\theta+c)s_i} e^{-(R+\theta+c)t_i}}{R + \theta + c} - \frac{t_i e^{-Rt_i}}{R} - \frac{e^{-Rt_i}}{R^2} \right\} \]

(6)

3. The present value of the backlogging cost during $[s_{i-1}, t_i]$ is

\[ BC_i = C_b \int_{s_{i-1}}^{t_j} e^{-Rt} I_2(t) \, dt \]

\[ BC_i = \frac{e^{-Rt_i} C_b}{\delta} \left\{ \left( a - b \frac{\delta}{\delta} \right) \left[ \frac{e^{-\delta(t_i - s_{i-1})}}{R} - \frac{1}{R \delta - \delta} \right] \right\} \]

\[ + b \left\{ \frac{t_i}{R - \delta} - \frac{1}{(R - \delta)^2} + \frac{s_{i-1} e^{-\delta(t_i - s_{i-1})}}{R} \right\} \]

\[ + e^{(R - \delta)(t_i - s_{i-1})} \left\{ a - b \frac{\delta}{\delta} \right\} \left[ \frac{1}{R} - \frac{1}{\delta - R} \right] \]

\[ + b \left\{ \frac{s_{i-1}}{R - \delta} + \frac{1}{(R - \delta)^2} + \frac{s_{i-1} e^{-\delta(s_{i-1} - t_i)}}{R} \right\} \]

if $R \neq \delta$.

(7)

\[ \frac{e^{-Rt_i} C_b}{\delta} \left\{ \left( a - b \frac{\delta}{\delta} \right) \left[ t_i - e^{\delta(s_{i-1} - t_i)} \right] \right\} \]

\[ + b \left\{ \frac{t_i^2}{2} - \frac{s_{i-1}}{R} e^{\delta(s_{i-1} - t_i)} \right\} \]

\[ + \left( a - b \frac{\delta}{\delta} \right) \left[ \frac{1}{R} - s_{i-1} \right] \left( b \left\{ \frac{s_{i-1}^2}{2} - \frac{s_{i-1}}{R} \right\} \right) \]

if $R = \delta$. 
4. The present value of the cost of lost sales during \([s_{i-1}, t_i]\) is

\[
LS_i = C_i \int_{s_{i-1}}^{t_i} e^{-Rt_i} (a + b t) \left[ 1 - e^{-\delta(t_i - t)} \right] dt
\]

\[
= \begin{cases} 
C_i \left[ e^{-Rt_i} \left\{ \frac{\delta(a + bt_i)}{R(R - \delta)} + \frac{b\delta(2R - \delta)}{R^2(R - \delta)^2} \right\} 
+ e^{-Rt_{i-1}} \left( \frac{1}{R}(a + bs_{i-1}) + \frac{b}{R^2} \right) 
- \frac{e^{-\delta(t_i - s_{i-1})}}{(R - \delta)} \left( a + bs_{i-1} + \frac{b}{R - \delta} \right) \right] & \text{if } R \neq \delta. \\
C_i \left[ e^{-Rt_i} \left\{ -\frac{a}{R} - \frac{bt_i}{R} - \frac{b}{R^2} + at_i \right\} 
- \frac{bt_i^2}{2} - as_{i-1} + \frac{bs_{i-1}^2}{2} \right] 
+ e^{-\delta s_{i-1}} \left( \frac{1}{R}(a + bs_{i-1}) + \frac{b}{R^2} \right) & \text{if } R = \delta.
\end{cases}
\]  
(8)

From (2) and (5), we have the order quantity at \(t_i\) in the \(i\)th replenishment cycle as

\[
Q_i = I_1(t_i) + I_2(t_i)
\]

\[
= \frac{\{a(\theta + c) - b\}}{(\theta + c)^2} (e^{(\theta + c)(s_i - t_i)} - 1) + \frac{b}{\theta + c} (s_i e^{(\theta + c)(s_i - t_i)} - t_i)
\]

\[+ \frac{e^{-\delta t_i}}{\delta^2} \left[ (e^{\delta s_{i-1}} + e^{\delta t_i})(\delta a - b) + \delta b (t_i e^{\delta t_i} + s_{i-1} e^{\delta s_{i-1}}) \right].
\]  
(9)

Therefore, the present value of the purchase cost during the \(i\)th replenishment cycle is

\[
PC_i = C_o e^{-Rt_i} + C_p e^{-Rt_i} Q_i.
\]  
(10)

The present value of revenue during the \(i\)th replenishment cycle is

\[
RV_i = p \left[ e^{-Rt_i} I_2(t_i) + \int_{t_i}^{s_i} e^{-Rt} \{a + bt + cI_1(t)\} \, dt \right].
\]  
(11)
For simplicity, we assume that \( R \neq \delta \) throughout the rest of the paper. By using a similar analogous argument, the reader can obtain the results for the case of \( R = \delta \). Hence, if \( n \) replenishment orders are placed in \([0, H]\), then the present value of the total profit during the planning horizon from 0 to \( H \) is as follows:

\[
TP(n, \{s_i\}, \{t_i\}) = \sum_{i=1}^{n} (RV_i - PC_i - HC_i - BC_i - LS_i).
\]

(12)

\( s_0 < t_1 < s_1 < t_2 < s_2, \ldots, s_{n-1} < t_n < s_n = H \). The objective of the problem here is to determine \( n, \{s_i\} \) and \( \{t_i\} \) such that \( TP(n, \{s_i\}, \{t_i\}) \) in (12) is maximized. For a fixed value of \( n \), the necessary conditions for \( TP(n, \{s_i\}, \{t_i\}) \) to be maximized are:

\[
\frac{\partial TP(n, \{s_i\}, \{t_i\})}{\partial s_i} = 0 \quad \text{for} \quad i = 1, 2, 3, \ldots, n-1
\]

\[
\frac{\partial TP(n, \{s_i\}, \{t_i\})}{\partial t_i} = 0 \quad \text{for} \quad i = 1, 2, 3, \ldots, n-1
\]

(13)

4. CONCLUSIONS

In this paper, we develop an inventory model for deteriorating items with multivariate demand rate has been proposed. We have shown that not only the optimal replenishment schedule exists uniquely, but also the total profit associated with the inventory system of the number of replenishments. The proposed model can be further extended in several ways. For example, we may add production and trade credit into consideration. Also, we could extend the deterministic model into a stochastic model. Finally, we could generalize the model to allow for quantity discounts, variable holding cost or others.

REFERENCES


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A. K. Malik  
Department of Mathematics,  
B. K. Birla Institute of Engineering &Technology,  
Pilani (Rajasthan), India.  
E-mail: ajendermalik@gmail.com

Ashu Sharma  
Department of Mathematics,  
BPCE, Guhana, Sonepat (Haryana), India.  
E-mail: sharma.ashu80@gmail.com.