SOME GRAPH LABELINGS IN COMPETITION GRAPH OF CAYLEY DIGRAPHS

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ABSTRACT: In this paper we present an algorithm and prove the existence of graph labelings such as $Z_3$-magic, Cordial, total cordial, $E$-cordial, total $E$-cordial, Product cordial, total product cordial, Product $E$-cordial, total product $E$-cordial labelings for the Competition graph of the Cayley digraphs associated with the diheadral group $D_n$.

AMS SUBJECT CLASSIFICATION: 05C78.

KEYWORDS: Cayley digraph, Graph labeling, A-magic.

1. INTRODUCTION

The concept of graph labeling was introduced by Rosa in 1967 [7]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for broad range of applications such as coding theory, X-ray, Crystallography, radar, astronomy, circuit design, communication networks and data base management and models for constraint programming over finite domain. Hence in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, antimagic labeling, bimagic labeling, prime labeling, cordial labeling, total cordial labeling, $k$-graceful labeling and odd graceful labeling etc., have been studied in over 1100 papers [2].

Cahit has introduced cordial labeling [3]. In [4], it is proved that every tree is cordial; $K_n$ is cordial if and only if $n \leq 3$, $K_{m,n}$ is cordial for all $m$ and $n$. Friendship graph $C_5$ is cordial if and only if $t \equiv 2 \pmod{2}$ and all fans are cordial. In [1], Andaretal proved that the $t$-ply graph $P_t(u, v)$ is cordial except when it is Eulerian and the number of edges is congruent to 2 (mod 4). In [11], Youssef proved that every Skolem-graceful graph is cordial.

A new labeling called $E$-cordial was introduced by Yilmaz and Cahit in 1997 [10]. They proved the following graphs are $E$-cordial: trees with $n$ vertices if and only if $n \neq 2 \pmod{4}$; $K_n$ if and only if $n \neq 2 \pmod{4}$; $K_{m,n}$ if and only if $m + n \neq 2 \pmod{4}$; $C_n$ if and only if $n \neq 2 \pmod{4}$; regular graphs of degree 1 on $2n$ vertices if and only if $n$ is even; friendship graphs $C_n$ for all $n$; fans $F_n$ if and only if $n \neq 0 \pmod{4}$; and wheels $W_n$ if and only if $n \neq 1 \pmod{4}$. They also observe that with $n \equiv 2 \pmod{4}$
vertices can not be $E$-cordial. More over the graph labelings on digraphs have been extensively studied in literatures [8, 9].

In 1878, Cayley constructed a graph with a generating set which is now popularly known as Cayley graphs. A directed graph or digraph is a finite set of points called vertices and a set of arrows called arcs connecting some vertices. The Cayley graphs and Cayley digraphs are excellent models for interconnection networks [2, 6]. Many well-known interconnection networks are Cayley digraphs. For example hypercube, butterfly, and cube-connected cycle’s networks are Cayley graphs. The Cayley digraph of a group provides a method of visualizing the group and its properties. The properties such as commutativity and the multiplication table of a group can be recovered from a Cayley digraph.

The original concept of an A-magic graph is due to dedlack, who defined it to be a graph with real-valued edge labeling such that distinct edges have distinct non-negative labels which satisfies the condition that the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

In this paper we prove the existence of graph labelings such as $Z_3$ magic, Cordial, total cordial, $E$-cordial, total $E$-cordial, Product cordial, total product cordial, Product $E$-cordial, total product $E$-cordial for the competition graph of the Cayley digraphs associated with the diheadral group $D_n$.

2. PRELIMINARIES

In this section we give the basic notation relevant to this paper. Let $G = G(V, E)$ be a finite, simple and undirected graph with $p$ vertices and $q$ edges. By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers called labels (usually the set of integers). In this paper we deal with the labeling with domain either the set of all vertices or the set of all edges or the set of all vertices and edges. We call these labelings as the vertex labeling or the edge labeling or the total labeling respectively.

**Definition 2.1:** A function $f$ from the vertex set $V \to \{0, 1\}$ such that each edge $uv$ assigning the label $|f(u) - f(v)|$ is said to be a cordial labeling if the number of vertices labeled 0 and the number of vertices labeled 1 differ at most by 1, and number of edges labeled 0 and the number of edges labeled 1 differ atmost by 1.

**Definition 2.2:** A function $f$ from the vertex set $V \to \{0, 1\}$ such that each edge $uv$ assign the label $f(u) \times f(v)$ is said to be a product cordial labeling if the number of vertices labeled 0 and the number of vertices labeled 1 differ atmost by 1, and number of edges labeled 0 and the number of edges labeled 1 differ atmost by 1.
**Definition 2.3:** A function \( f \) from the vertex set \( V \rightarrow \{0, 1\} \) such that each edge \( uv \) assign the label \( f(u) \times f(v) \) is said to be a total product cordial labeling if the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ at most by 1. A graph with total product cordial labeling is called total product cordial graph.

**Definition 2.4:** Let \( G \) be a graph with vertex set \( V \) and edge set \( E \) and let \( f \) be function from \( E \) to \( \{0, 1\} \). Define \( f^* \) on \( V \) by \( f^*(v) = \sum\{f(uv)/uv \in E\} \) (mod 2). The function \( f \) is called \( E \)-cordial labeling of \( G \) if the number of vertices labeled 0 and the number of vertices labeled 1 differ at most by 1. A graph that admits \( E \)-cordial labeling is called \( E \)-cordial.

**Definition 2.5:** An \((p, q)\)-digraph \( G = (V, E) \) is defined by a set \( V \) of vertices such that \( |V| = p \) and a set \( E \) of arcs or directed edges with \( |E| = q \). The set \( E \) is a subset of elements \((u, v)\) of \( V \times V \). The out-degree (or in-degree of a vertex \( u \) of a digraph \( G \) is the number of arcs \((u, v)\) (or \((v, u)\)) of \( G \) is denoted by \( d^+(u) \) (or \( d^-(u) \)). A digraph is said to be regular if \( d^+(u) = d^-(v) \) for every vertex \( u \) of \( G \).

### 3. MAIN RESULT

In this section we present an algorithm and prove the existence of graph labeling such as \( Z_3 \)-magic, Cordial, total cordial, \( E \)-cordial, total \( E \)-cordial, Product cordial, total Product cordial, Product \( E \)-cordial, total Product \( E \)-cordial for the Competition graphs of Cayley digraphs associated with diheadral group \( D_n \).

**Definition 3.1:** Let \( G(V, E) \) be a \((p, q)\) digraph. It is said to admit \( E \)-cordial labeling if there exists a function \( f \) from \( E \) onto the set \( \{0, 1\} \) such that the induced map \( f^* \) on \( V \) is defined as \( f^*(v) = \sum\{f(v_i, v_j)/v_i, v_j \in E\} \) (mod 2) satisfying the property that the number of arcs labeled 0 and the number of arcs labeled 1 differ at most by 1.

**Definition 3.2:** Let \( G(V, E) \) be a \((p, q)\) digraph. It is said to admit total \( E \)-cordial labeling if there exists a function \( f \) from \( E \) onto the set \( \{0, 1\} \) such that the induced map \( f^* \) on \( V \) is defined as \( f^*(v) = \sum\{f(v_i, v_j)/v_i, v_j \in E\} \) (mod 2) satisfying the property that the number of vertices and arcs labeled with 0 and the number of vertices and arcs labeled with 1 differ at most by 1.

**Definition 3.3:** Let \( G(V, E) \) be a \((p, q)\) digraph. It is said to admit product \( E \)-cordial labeling if there exists a function \( f \) from \( E \) onto the set \( \{0, 1\} \) such that the induced map \( f^* \) on \( V \) is defined as \( f^*(v) = \prod\{f(v_i, v_j)/v_i, v_j \in E\} \) (mod 2) satisfying the property that if the number of vertices labeled 0 and the number of vertices labeled 1 differ at most by 1, and number of arcs labeled 0 and the number of arcs labeled 1 differ at most by 1.
Definition 3.4: Let $G (V, E)$ be a $(p, q)$ digraph. It is said to admit total product $E$-cordial labeling if there exists a function $f$ from $E$ onto the set $\{0, 1\}$ such that the induced map $f^*$ on $V$ is defined as $f^* (v) = \prod (f(v, v_j) \mod 2)$ satisfying the property that the number of vertices and arcs labeled with 0 and the number of vertices and arcs labeled with 1 differ at most by 1.

Definition 3.5: Let $G (V, E)$ be a $(p, q)$ digraph. It is said to admit $Z_s$-magic labeling if there exists a function $f$ from $E$ onto the set $\{1, 2\}$ such that the induced map $f^*$ on $V$ defined by $f^* (v) = \sum f(e) \mod 3 = k$, a constant and $e = (v, v_j) \in E$.

Definition 3.6: Let $G$ be a finite group and $S$ be a generating subset of $G$. The Cayley digraph $Cay (G, S)$ is the digraph whose vertices are the elements of $G$, and there is an edge from $g$ to $gs$ whenever $g \in G$ and $s \in S$. If $S = S^{-1}$ then there is an edge from $g$ to $gs$ if and only if there is an arc from $gs$ to $g$.

Definition 3.7: Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E (E_a, E_b) = \{e_1, e_2, \ldots, e_{2n}\}$ be the vertex and arc sets of Cayley digraphs of diheadral group. The competition graph of $Cay (D_n, (a, b))$ denoted by $ComCay (D_n, (a, b))$ is a digraph consisting of same set of vertices and if for any path $v_i v_j v_k v_i$ where $v_i, v_j, v_k \in V$ and $e_i, e_j, e_k \in E$, draw a new edge $v_i v_k$.

Definition 3.8: The structure of competition graph $ComCay (D_n, (a, b))$ is defined as follows. From the construction of competition graph using definition 3.7, the $ComCay (D_n, (a, b))$ has $n$ vertices and $4n$ arcs. Let us denote the vertex set of $ComCay (D_n, (a, b))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Denote the arc set of $ComCay (D_n, (a, b))$ as $E (E_{aa}, E_{ab}, E_{ba}, E_{bb}) = \{e_1, e_2, \ldots, e_{4n}\}$, where

$E_{aa}$ = The set of all arcs obtained through $(a, a)$,

$E_{ab}$ = The set of all arcs obtained through $(a, b)$,

$E_{ba}$ = The set of all arcs obtained through $(b, a)$,

$E_{bb}$ = The set of all arcs obtained through $(b, b)$.

Thus the arc set of the $ComCay (D_n, (a, b))$ is as follows:

(i) $v_i v_{i+2} \in E_{aa}$, where $1 \leq i \leq (n/2) - 2$ & $(n/2) + 1 \leq i \leq n - 2$

(ii) $v_i v_{i-2} \in E_{aa}$, where $(n/2) - 1 \leq i \leq (n/2)$ & $n - 1 \leq i \leq n$

(iii) $v_i v_{n-i+2} v_n v_{2n} \in E_{ab}$, where $1 \leq i \leq (n/2) - 1$ & $(n/2) + 1 \leq i \leq n - 1$

(iv) $v_i v_{j+2} v_{n-i+2} v_{n+1} v_1 v_{n+1} v_{n/2+1} \in E_{ba}$, $2 \leq i \leq n/2$

(v) $v_i v_j \in E_{bb}$, $i = j$ & $1 \leq i \leq n$. 

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Now we present an algorithm to get $Z_3$-magic, Cordial, $E$-cordial, Product cordial and Product $E$-cordial labeling for the competition graph $\text{ComCay}(D_n, (a, b))$.

**Algorithm:**

**Input**: The Diheadral group $D_n$ with the generating set $(a, b)$.

**Step 1**: Using definition 3.6, Construct Cayley digraph $\text{Cay}(D_n, (a, b))$.

**Step 2**: Using definition 3.7, Construct competition graph $\text{ComCay}(D_n, (a, b))$.

**Step 3**: Denote the vertex set of $\text{ComCay}(D_n, (a, b))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and the the arc set as $E(E_{aa}, E_{ab}, E_{ba}, E_{bb}) = \{e_1, e_2, e_3, \ldots, e_{4n}\}$ where

- $E_{aa} = \text{The set of all arcs obtained through } (a, a)$
- $E_{ab} = \text{The set of all arcs obtained through } (a, b)$
- $E_{ba} = \text{The set of all arcs obtained through } (b, a)$
- $E_{bb} = \text{The set of all arcs obtained through } (b, b)$

**Step 4**: (for $Z_3$-magic labeling)

Define $f$ on $E$ as follows:

$$f(v_i, v_j) = \begin{cases} 
2, & \text{where } v_i, v_j \in E_{aa} \& E_{ba} \\
1, & \text{where } v_i, v_j \in E_{ab} \& E_{bb}.
\end{cases}$$

**Step 5**: (for cordial labeling)

Define $f$ on $V$ as follows:

$$f(v_i) = \begin{cases} 
1, & 1 \leq i \leq n/2; \\
0, & (n/2) + 1 \leq i \leq n, \ v_i \in V.
\end{cases}$$

**Step 6**: (for $E$-cordial labeling)

Define $f$ on $E$ as follows:

(i) For all $v_i, v_{i+2}, v_i, v_{i-2} \in E_{aa}$,

$$f(v_i, v_{i+2}) = \begin{cases} 
1, & 1 \leq i \leq (n/2) - 2; \\
0, & (n/2) + 1 \leq i \leq n - 2.
\end{cases}$$

$$f(v_i, v_{i-2}) = \begin{cases} 
1, & (n/2) - 1 \leq i \leq (n/2); \\
0, & n - 1 \leq i \leq n.
\end{cases}$$
(ii) For all \( v_i v_j \in E_{ab} \), \( f(v_i v_j) = 1 \)

(iii) For all \( v_i v_j \in E_{ba} \), \( f(v_i v_j) = 0 \)

(iv) For all \( v_i v_j \in E_{bb} \) and \( i = j \),

\[
f(v_i v_j) = \begin{cases} 
1, & i \equiv 1 \pmod{2}; \\
0, & i \equiv 0 \pmod{2}.
\end{cases}
\]

**Step 7:** (for product cordial labeling)

Define \( f \) on \( V \) as follows: For all \( 1 \leq i \leq n, v_i \in V \)

\[
f(v_i v_j) = \begin{cases} 
0, & i \equiv 1 \pmod{2}; \\
1, & i \equiv 0 \pmod{2}.
\end{cases}
\]

**Step 8:** (for product \( E \)-cordial labeling)

Define \( f \) on \( E \) as follows:

(i) For all \( v_i v_j \in E_{aa} \):

\[
f(v_i v_{i+2}) = \begin{cases} 
0, & 1 \leq i \leq (n/2) - 2; \\
1, & (n/2) + 1 \leq i \leq n - 2.
\end{cases}
\]

\[
f(v_i v_{i-2}) = \begin{cases} 
0, & (n/2) - 1 \leq i \leq (n/2); \\
1, & n - 1 \leq i \leq n.
\end{cases}
\]

(ii) For all \( v_i v_j \in E_{ab} \):

\[
f(v_i v_{n-i}) = \begin{cases} 
1, & 1 \leq i \leq (n/2) - 1; \\
0, & (n/2) + 1 \leq i \leq n - 1.
\end{cases}
\]

\[
f(v_{n/2} v_n) = 1 \land f(v_{n/2} v_{n/2}) = 0.
\]

(iii) For all \( v_i v_j \in E_{ba} \), \( 2 \leq i \leq n/2 \)

\[
f(v_i v_{n-i+2}) = f(v_{i+1} v_{(n/2) + 1}) = 1 \land f(v_{n-i+2} v_i) = f(v_{(n/2) + 1} v_{n/2}) = 0.
\]

(iv) For all \( v_i v_j \in E_{bb} \) and \( i = j \):

\[
f(v_i v_j) = \begin{cases} 
0, & 1 \leq i \leq (n/2); \\
1, & (n/2) + 1 \leq i \leq n.
\end{cases}
\]

**Output:** \( Z_3 \)-magic, Cordial, \( E \)-cordial, Product cordial and Product \( E \)-cordial labeling for the competition graph ComCay \( (D_n, (a, b)) \).
**Theorem 3.1:** The competition graph $\text{ComCay}(D_n,(a,b))$ admits $Z_3$-magic labeling.

**Proof:** From the construction of Competition graph $\text{ComCay}(D_n,(a,b))$, we have $n$ vertices and $4n$ arcs. Denote the vertex set and arc set using step 3 of algorithm.

To prove $\text{ComCay}(D_n,(a,b))$ admits $Z_3$-magic labeling, we have to show that there exists a function $f$ from $E$ onto the set $Z_3\{-0\}$ such that the induced map $f^*$ on $V$ defined by $f^*(v_i) = \{\sum f(e) \pmod{3} = k, \text{a constant} / \forall v_i \in V, \text{the sum of the labels of the arcs incident at } v_i \text{ is a constant. Consider an arbitrary vertex } v_i \in V \text{ of } \text{ComCay}(D_n,(a,b)). \text{Using step 4 of algorithm, define a map } f : E \to Z_3\{-0\} \text{ such that}

$$f(v_i,v_j) = \begin{cases} 2, & \text{where } v_i,v_j \in E_{aa} \& E_{ba}; \\ 1, & \text{where } v_i,v_j \in E_{ab} \& E_{bb}. \end{cases}$$

Hence for the induced map $f^* : V \to Z_3$, for all $v_i \in V, f^*(v_i) = \sum f(v_i,v_j) \pmod{3} = (2 + 2 + 1 + 1) \pmod{3} = 0 \pmod{3}$, where $v_i,v_j \in E_{aa}, E_{ba}, E_{ab} \& E_{bb}$. Thus $f^*(v_i) = 0 \pmod{3}$ which is a constant for all $i$.

Hence Competition graph $\text{ComCay}(D_n,(a,b))$ admits $Z_3$-magic labeling.

**Example 3.1:** Competition graph of the Cayley digraph associated with the diheadral group $D_8$ and its $Z_3$-magic labeling is shown in Figure 1.

![Figure 1: $Z_3$-Magic Labeling for ComCay ($D_n, (a, b)$)](image-url)
**Theorem 3.2:** The competition graph $\text{ComCay}(D_n, (a, b))$ admits Cordial labeling.

**Proof:** From the construction of Competition graph $\text{ComCay}(D_n, (a, b))$, we have $n$ vertices and $4n$ arcs. Denote the vertex set and arc set using step 3 of algorithm. To prove $\text{ComCay}(D_n, (a, b))$ admits Cordial labeling, we have to show that there exists a function $f : V \rightarrow \{0, 1\}$ such that $f^*(v_i, v_j) = \{(f(v_i) + f(v_j)) \mod 2 \} v_i v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})$ which satisfies the property that the number of vertices labeled 0 and the number of vertices labeled 1 differ at most by 1 and number of arcs labeled 0 and the number of arcs labeled 1 differ by at most 1. Consider the arbitrary vertex $v_i \in V$. Using step 5 of the Algorithm, we define a map $f : V \rightarrow \{0, 1\}$ as follows:

$$f(v_i) = \begin{cases} 
1, & 1 \leq i \leq n/2; \\
0, & (n/2) + 1 \leq i \leq n, \quad v_i \in V.
\end{cases}$$

Thus the number of vertices labeled 1 is $n/2$ and the number of vertices labeled 0 is $n/2$. Hence the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1. In order to get the labels for the arcs, define the induced map $f^* : E \rightarrow \{0, 1\}$ such that $f^*(v_i, v_j) = (f(v_i) + f(v_j)) \mod 2, v_i v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})$

(i) $f(v_i) = \begin{cases} 
0 \mod 2, & \text{for all } v_i v_j \in E_{aa}, E_{bb}; \\
1 \mod 2, & \text{for all } v_i v_j \in E_{ab}, E_{ba}.
\end{cases}$

That is, the number of arcs labeled 0 is $n + n = 2n$ and the number of arcs labeled 1 is $n + n = 2n$. Thus the number of arcs labeled 0 and the number of arcs labeled 1 differ by at most 1. Hence $\text{ComCay}(D_n, (a, b))$ admits Cordial labeling.

**Example 3.2:** Competition graph of the Cayley digraph associated with the diheadral group $D_8$ and its Cordial labeling is shown in Figure 2.

![Figure 2: Cordial Labeling for ComCay $(D_n, (a, b))$](image-url)

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**Theorem 3.3:** The competition graph ComCay \((D_n, (a, b))\) admits total cordial labeling.

**Proof:** To prove ComCay \((D_n, (a, b))\) admits total Cordial labeling, we have to show that there exists a function \(f: V \to \{0, 1\}\) such that \(f^*(v_i, v_j) = (f(v_i) + f(v_j)) \pmod 2\), \(v_i, v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})\) which satisfies the property that the number of zeroes on the vertices and arcs taken together differ by at most 1 with the number of one’s on vertices and arcs taken together.

By the above theorem, using the map \(f\) on \(V\) and thereby the induced map \(f^*\) on \(E\), we have the number of arcs labeled 0 is \(2n\) and the number of vertices labeled 0 is \(n/2\). Also, the number of arcs labeled by 1 is \(2n\) and the number of vertices labeled by 1 is \(n/2\). Thus the total number of one’s on vertices and arcs taken together is \((n/2) + 2n = 5n/2\) and the the total number of zeroes on vertices and arcs taken together is \((n/2) + 2n = 5n/2\). Thus the number of zeroes on the vertices and arcs taken together differ by at most 1 with the number of one’s on vertices and arcs taken together. Hence, the competition graph ComCay \((D_n, (a, b))\) admits total cordial labeling.

**Theorem 3.4:** The competition graph ComCay \((D_n, (a, b))\) admits E-Cordial labeling.

**Proof:** From the construction of Competition graph ComCay \((D_n, (a, b))\), we have \(n\) vertices and \(4n\) arcs. To prove Comcay \((D_n, (a, b))\) admits E-Cordial labeling, we have to show that there exists a function \(f: E \to \{0, 1\}\) such that the induced function \(f^*\) on \(V\) is defined as \(f^*(v_i) = \sum (f(v_i, v_j), v_i, v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})) \pmod 2\) which satisfies the property that the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of arcs labeled 0 and the number of arcs labeled 1 differ by at most 1. Consider the arbitrary vertex \(v_i \in V\). Using step 6 of algorithm, we define a map \(f: E \to \{0, 1\}\) as follows:

(i) For all \(v_i, v_{i+2}, v_i, v_{i-2} \in E_{aa}\):

\[
    f(v_i, v_{i+2}) = \begin{cases} 
    1, & 1 \leq i \leq (n/2) - 2; \\
    0, & (n/2) + 1 \leq i \leq n - 2.
    \end{cases}
\]

\[
    f(v_i, v_{i-2}) = \begin{cases} 
    1, & (n/2) - 1 \leq i \leq (n/2); \\
    0, & n - 1 \leq i \leq n.
    \end{cases}
\]

(ii) For all \(v_i, v_j \in E_{ab}, f(v_i, v_j) = 1\).

(iii) For all \(v_i, v_j \in E_{ba}, f(v_i, v_j) = 0\).
(iv) For all $v_i v_j \in E_{bb}$ and $i = j$:

$$f(v_i v_j) = \begin{cases} 1, & i \equiv 1 \pmod{2}; \\ 0, & i \equiv 0 \pmod{2}. \end{cases}$$

From these definitions of the labeling functions, we have the total number of arcs labeled 0 is $(n/2) + (n/2) + (n/2) + (n/2) = 2n$ and the total number of arcs labeled 1 is $(n/2) + (n/2) + (n/2) + (n/2) = 2n$. Thus the number of arcs labeled 0 and the number of arcs labeled 1 differ by at most 1. In order to get the labels for the vertices, define the induced map $f^*: V \to \{0, 1\}$ such that

(i) For all $1 \leq i \leq n/2$ & $v_i v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})$

$$f(v_i) = \sum f(v_i v_j) = \begin{cases} 1 + 1 + 0 + 1 = 1, & i \equiv 1 \pmod{2}; \\ 1 + 1 + 0 + 0 = 0, & i \equiv 0 \pmod{2}. \end{cases}$$

(ii) For all $n/2 \leq i \leq n$ & $v_i v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})$

$$f(v_i) = \sum f(v_i v_j) = \begin{cases} 0 + 1 + 0 + 1 = 0, & i \equiv 1 \pmod{2}; \\ 0 + 1 + 0 + 0 = 1, & i \equiv 0 \pmod{2}. \end{cases}$$

Under this map, the number of vertices labeled 1 is $(n/4) + (n/4) = n/2$ and the number of vertices labeled 0 is $(n/4) + (n/4) = n/2$. Thus the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1. Hence the competition graph $ComCay(D_n, (a, b))$ admits $E$-cordial labeling.

**Example 3.3:** Competition graph of the Cayley digraph associated with the diheadral group $D_8$ and its $E$-cordial labeling is shown in Figure 3.

![Figure 3: E-Cordial Labeling for ComCay ($D_n$, (a, b))](image-url)
**Theorem 3.5:** The competition graph ComCay \((D_n, (a, b))\) admits total \(E\)-cordial labeling.

**Proof:** To prove Comcay \((D_n, (a, b))\) admits total \(E\)-cordial labeling, we have to show that there exists a function \(f : E \to \{0, 1\}\) such that the induced function \(f^*\) on \(V\) is defined as \(f^*(v_i v_j) = \sum \{f(v_i v_j) : v_i v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})\} \pmod{2}\) which satisfies the property that the number of zeroes on the vertices and arcs taken together differ by at most 1 with the number of one’s on vertices and arcs taken together.

By the above theorem, using the map \(f\) on \(E\) and there by the induced map \(f^*\) on \(V\), we have the number of arcs labeled 0 is \(2n\) and the number of vertices labeled 0 is \(n/2\). Also, the number of arcs labeled by 1 is \(2n\) and the number of vertices labeled by 1 is \(n/2\).

Thus the total number of one’s on vertices and arcs taken together is \((n/2) + 2n = 5n/2\) and the the total number of zeroes on vertices and arcs taken together is \((n/2) + 2n = 5n/2\). Thus the number of zeroes on the vertices and arcs taken together differ by at most 1 with the number of one’s on vertices and arcs taken together.

Hence The competition graph ComCay \((D_n, (a, b))\) admits total \(E\)-cordial labeling.

**Theorem 3.6:** The competition graph ComCay \((D_n, (a, b))\) admits Product Cordial labeling.

**Proof:** From the construction of Competition graph ComCay \((D_n, (a, b))\) using algorithm, we have \(n\) vertices and \(4n\) arcs. To prove ComCay \((D_n, (a, b))\) admits Product Cordial labeling, we have to show that there exists a function \(f : V \to \{0, 1\}\) such that \(f^*(v_i) = \{f(v_i) : v_i \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})\}\) which satisfies the property that the number of vertices labeled 0 and the number of vertices labeled 1 differ atmost by 1 and number of arcs labeled 0 and the number of arcs labeled 1 differ by atmost 1. Consider the arbitrary vertex \(v_i \in V\). Using step 7 of algorithm we define a map \(f : V \to \{0, 1\}\) as follows. For all \(1 \leq i \leq n\), \(v_i \in V\)

\[
f(v_i) = \begin{cases} 
0, & i \equiv 1 \pmod{2}; \\
1, & i \equiv 0 \pmod{2}.
\end{cases}
\]

From this definition of the labeling functions, we have the total number of vertices labeled 0 is \(n/2\) and the number of vertices labeled 1 is \(n/2\). Hence the number of vertices labeled 0 and the number of vertices labeled 1 differ by atmost 1. In order to get the labels for the arcs, define the induced map \(f^* : E \to \{0, 1\}\) such that \(f^*(v_i v_j) = \{f(v_i) \times f(v_j) : v_i v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})\}\).
Now for all $v_i v_j \in E_{aa}, E_{ab}, E_{ba} \& E_{bb}$,

$$f^*(v_i v_j) = f(v_i) \times f(v_j) \begin{cases} 1, & i \& j \equiv 1 \pmod{2}; \\ 0, & i \& j \equiv 0 \pmod{2}. \end{cases}$$

Under this map the number of arcs labeled 0 is $n + n = 2n$ and the number of arcs labeled 1 is $n + n = 2n$. Thus the number of arcs labeled 0 and the number of arcs labeled 1 differ by atmost 1.

Hence ComCay $(D_n, (a, b))$ admits Product Cordial labeling.

**Example 3.4:** Competition graph of the Cayley digraphs associated with the dihedral group $D_8$ and its Product cordial labeling is shown in Figure 4.

![Product Cordial Labeling for ComCay $(D_8, (a, b))$](image)

**Theorem 3.7:** The competition graph ComCay $(D_n, (a, b))$ admits total product cordial labeling.

**Proof:** From the construction of Competition graph ComCay $(D_n, (a, b)$ using algorithm, we have $n$ vertices and $4n$ arcs.

To prove ComCay $(D_n, (a, b))$ admits total product cordial labeling, we have to show that there exists a function $f: V \to \{0, 1\}$ such that $f^*(v_i v_j) = \{(f(v_i) \times f(v_j))/v_i v_j \in E(E_{aa}, E_{ab}, E_{ba}, E_{bb})\}$ which satisfies the property that the number of zeroes on the vertices and arcs taken together differ by atmost 1 with the number of one’s on vertices and arcs taken together.
By the above theorem, using the map $f$ on $E$ and there by the induced map $f^*$ on $E$, we have the number of arcs labeled 0 is $2n$ and the number of vertices labeled 0 is $n/2$. Thus the total number of zeroes on vertices and arcs taken together is $(n/2) + 2n = 5n/2$. Also, the number of arcs labeled by 1 is $2n$ and the number of vertices labeled by 1 is $n/2$. Thus the total number of one’s on vertices and arcs taken together is $(n/2) + 2n = 5n/2$. Thus the number of zeroes on the vertices and arcs taken together differ by atmost 1 with the number of one’s on vertices and arcs taken together. Hence The competition graph $\text{ComCay} (D_n, (a, b))$ admits total product cordial labeling.

**Theorem 3.8:** The competition graph $\text{ComCay} (D_n, (a, b))$ admits Product $E$-cordial labeling.

**Proof:** From the construction of Competition graph $\text{ComCay} (D_n, (a, b))$ using algorithm, we have $n$ vertices and $4n$ arcs.

To prove Comcay $(D_n, (a, b))$ admits Product $E$-Cordial labeling, we have to show that there exists a function $f: E \to \{0, 1\}$ such that the induced function $f^*$ on $V$ is defined as $f^*(v_i) = \prod \{f(v_i v_j)v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb})\}$ which satisfies the property that the number of vertices labeled 0 and the number of vertices labeled 1 differ by atmost 1 and the number of arcs labeled 0 and the number of arcs labeled 1 differ by atmost 1.

Consider the arbitrary vertex $v_i \in V$. Using step 8 of algorithm we define a map $f: V \to \{0, 1\}$ as follows:

(i) For all $v_i v_j \in E_{aa}$:

$$f(v_i v_{i+2}) = \begin{cases} 0, & 1 \leq i \leq (n/2) - 2; \\ 1, & (n/2) + 1 \leq i \leq n - 2. \end{cases}$$

$$f(v_i v_{i-2}) = \begin{cases} 0, & (n/2) - 1 \leq i \leq (n/2); \\ 1, & n - 1 \leq i \leq n. \end{cases}$$

(ii) For all $v_i v_j \in E_{ab}$:

$$f(v_i v_{n-i}) = \begin{cases} 1, & 1 \leq i \leq (n/2) - 1; \\ 0, & (n/2) + 1 \leq i \leq n - 1. \end{cases}$$

$$f(v_{n/2} v_{n/2}) = 1 \& f(v_{n/2} v_{n/2}) = 0.$$

(iii) For all $v_i v_j \in E_{ba}$ & $2 \leq i \leq n/2$

$$f(v_{n-i} v_{n-i+2}) = f(v_{(n/2)+1}) = 1 \& f(v_{n-i+2} v_i) = f(v_{(n/2)+1} v_i) = 0.$$
(iv) For all \( v_i, v_j \in E_{bb} \) & \( i = j \):

\[
f(v_i v_j) = \begin{cases} 
0, & 1 \leq i \leq (n/2); \\
1, & (n/2) + 1 \leq i \leq n.
\end{cases}
\]

From these definitions of the labeling functions, we have the total number of arcs labeled 0 is \( n/2 + n/2 + n/2 + n/2 = 2n \) and the number of arcs labeled 1 is \( n/2 + n/2 + n/2 + n/2 = 2n \). Hence the number of arcs labeled 0 and the number of arcs labeled 1 differ by atmost 1.

In order to get the labels for the vertices define the induced map \( f: V \rightarrow \{0, 1\} \) such that \( f^*(v_i) = \prod \{f(v_i v_j) / v_i, v_j \in E(E_{aa}, E_{ab}, E_{ba}, E_{bb})\} \) For all \( v_i v_j \in E(E_{aa}, E_{ab}, E_{ba}, E_{bb}) \):

\[
f^*(v_i) = \prod f(v_i v_i) = \begin{cases} 
0 \times 0 \times 0 \times 0 = 0, & v_i \in V & 1 \leq i \leq (n/2); \\
1 \times 1 \times 1 \times 1 = 1, & \text{where } v_i \in V & (n/2) + 1 \leq i \leq n.
\end{cases}
\]

Under this map the number of vertices labeled 0 is \( n/2 \) and the number of vertices labeled 1 is \( n/2 \). Thus the number of vertices labeled 0 and the number of vertices labeled 1 differ by atmost 1. Hence ComCay \( (D_8, (a, b)) \) admits Product E-Cordial labeling.

**Example 3.5:** Competition graph of the Cayley digraphs associated with the diheadral group \( D_8 \) and its Product E-Cordial labeling is shown in Figure 5.

![Figure 5: Product E-Cordial Labeling for ComCay \( (D_8, (a, b)) \) ![Figure 5: Product E-Cordial Labeling for ComCay \( (D_8, (a, b)) \)
Theorem 3.9: The competition graph ComCay \((D_n, (a, b))\) admits total product \(E\)-Cordial labeling.

Proof.: From the construction of Competition graph ComCay \((D_n, (a, b))\), we have \(n\) vertices and \(4n\) arcs.

To prove ComCay \((D_n, (a, b))\) admits total product \(E\)-Cordial labeling, we have to show that there exists a function \(f: E \rightarrow \{0, 1\}\) such that the induced function \(f^*\) on \(V\) is defined as \(f^*(v) = \prod (f(v_j) | v_j \in E(E_{aa}, E_{ba}, E_{ab}, E_{bb}))\) which satisfies the property that the number of zeroes on the vertices and arcs taken together differ by atmost 1 with the number of one’s on vertices and arcs taken together.

By the above theorem, using the map \(f\) on \(E\) and there by the induced map \(f^*\) on \(V\), we have the number of arcs labeled 0 is \(2n\) and the number of vertices labeled 0 is \(n/2\). Thus the total number of zeroes on vertices and arcs taken together is \((n/2) + 2n = 5n/2\). Also, the number of arcs labeled by 1 is \(2n\) and the number of vertices labeled by 1 is \(n/2\). Thus the total number of one’s on vertices and arcs taken together is \((n/2) + 2n = 5n/2\). Thus the number of zeroes on the vertices and arcs taken together differ by atmost 1 with the number of one’s on vertices and arcs taken together. Hence The competition graph ComCay \((D_n, (a, b))\) admits total product \(E\)-cordial labeling.

4. CONCLUSION

In this paper we have presented an algorithm and proved that the competition graph of Cayley digraph associated with the diheadral group \(D_n\) denoted by ComCay \((D_n, (a, b))\) admits \(\mathbb{Z}_3\)-magic, Cordial, total cordial, \(\bar{E}\)-cordial, total \(\bar{E}\)-cordial, Product cordial, total Product Cordial, Product \(E\)-Cordial and total Product \(E\)-cordial labelings.

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