TANDEM QUEUE WITH THREE MULTISERVER UNITS AND BULK SERVICE WITH ACCESSIBLE AND NON ACCESSIBLE BATCH IN UNIT III WITH VACATION

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Abstract
This paper analyses two different queuing model (Model I and Model II) both consisting of three units connected in series separated by a buffer of finite capacity with finite number of parallel servers in each unit. Queuing model differs only on the provision of availing vacation to the servers in unit III. In Model I server can avail exponential vacation, whereas in Model II the server takes only single vacation. Customers of unit I and Unit II are served singly but in unit III, they are served in groups according to bulk service rule. This rule admits each batch to be served only if the batch size is not less than ‘a’ and not more than ‘b’, customers arriving late can also enter service station without affecting the service time if the size of the batch being served is less than ‘d’ (a d b). The occurrence time and service time have negative exponential distributions. The steady state probability vector of the number of customers in the queue is obtained by using a modified geometric method. The stability condition is also obtained.

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Keywords: Bulk service queues, Accessible and non-accessible batch service, Matrix geometric method, Steady state solution, Vacation.

1. INTRODUCTION
Many authors had already discussed about the queuing models with serial connection of units separated by an intermediate waiting room of finite capacity. A model in which two units connected in series with an finite waiting room in between has been introduced by Neuts [1968]. The concept of blocking in two or more units in service with a general service time distribution without an intermediate buffer has been considered by Avi-Itzhak and Yadin (1965), Clarke (1977). A three stage multi server queuing system with finite queues in each stage and blocking has been analyzed by Arndt and Sulanke [1985]. Tandem Queue with three multi server units connected in series was analyzed by Ayyappan.G and S.Velmurugan (2008).
In this paper we deal with a queuing model consists of three units in series each has a finite number of parallel servers \( c_i \) \((i = 1, 2, 3)\). These three units are separated by two waiting rooms of capacities \( M \) and \( N \), which are finite and independent. The customers in unit I and unit II are served singly. A queue of infinite length is allowed for unit I and of finite capacity \( M \) for unit II. The arrival of customers at Poisson rate \( \lambda \) is independent of service time. The service time in the \( i^{th} \) unit, \( \mu_i \) \((i = 1, 2, 3)\) are assumed to be independently distributed random variables.

Customers in unit III are served in groups according to bulk service rule. This rule admits each batch to be served only if the batch size is not less than ‘\( a \)’ and not more than ‘\( b \)’, customers arriving late can also enter service station without affecting the service time if the size of the batch being served is less than ‘\( d \)’ \((a \leq d \leq b)\). In addition, Vacation is introduced to servers in unit III, in two models. One of multiple exponential vacation (Model I) and the other of single vacation (Model II).

2. MODEL I

This model relates to queues with accessible and non accessible service system with vacation. When a server completes service and the batch size and batch size in unit III is less than ‘\( a \)’ then the server leaves for a vacation. As soon as server returns from the vacation and finds that the batch size is still less than ‘\( a \)’ then the server immediately leaves for another vacation. Thus the server avails an exponential vacation until the batch size reached ‘\( a \)’.

2.1 Steady State Probability Vector

The steady state process under consideration can be formulated as a continues time Markov chain with states

\[
S = \{(i, j, k, n); \; i \geq 0, \; 0 \leq j \leq M + c_2, \; 0 \leq k \leq N, \; 0 \leq n \leq c_3 \}
\]

\[
U \{(i, j, k, m, n); \; i \geq 0, \; 0 \leq j \leq M + c_2, \; k = 0, \; a \leq m \leq d - 1, \; 0 \leq n \leq c_3 \}
\]

Where \( i \) denote number of customers in unit I, \( j \) denotes number of customers in unit II, \( k \) denotes number of customers in buffer of unit III, \( n \) denotes number of busy servers in unit III, \( m \) denotes number of customers in accessible service batch (number of customer in the service batch is less than \( d \)) and \( c_3 - n \) servers are in vacation.

The infinitesimal generator \( Q \) of the continues time Markov chain
The sub matrices $A_i$, $B_i$ $(0 \leq i \leq c_1)$ and $\gamma M_i (0 \leq \gamma \leq c_1)$, which are square matrices of order $(M + c_2 + 1) ((N + 1) + (d - a) c_1)$ are defined as below: $A_0 = \lambda I$ where $I$ is a unit matrix of order $(M + c_2 + 1) ((N + 1) + (d - a) c_1)$.

Where $0 \leq i \leq c_1$ and $\beta_1 D$, $\beta_2 M_1$ are given below and $0 \leq \beta_1 \leq M + c_2$, $1 \leq \beta_2 \leq M + c_2$

$$
\beta_1 D = \frac{a - 1}{a - \frac{1}{\mu_1}} 
\begin{bmatrix}
    c_1 & \cdot & \cdot & N \\
    0 & 1 & \cdot & \cdot \\
    \cdot & \cdot & \cdot & \cdot \\
    \cdot & \cdot & \cdot & \cdot \\
    0 & 1 & \cdot & \cdot \\
    \cdot & \cdot & \cdot & \cdot \\
    \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
$$

and

$$
\beta_2 M_1 = \frac{a - 1}{a - \frac{1}{\mu_2}} 
\begin{bmatrix}
    E & \cdot & \cdot & \cdot \\
    0 & 1 & \cdot & \cdot \\
    \cdot & \cdot & \cdot & \cdot \\
    \cdot & \cdot & \cdot & \cdot \\
    0 & 1 & \cdot & \cdot \\
    \cdot & \cdot & \cdot & \cdot \\
    \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
$$

where $E$ having only $j \mu_1$ in the main diagonal otherwise other elements are zero. $c_1$ having main diagonal elements as $- (\lambda + i \mu_1 + j \mu_2 + k \mu_3)$ and the lower diagonal elements are $k \mu_3$ and $c_2$ having main diagonal elements as $- (\lambda + i \mu_1 + j \mu_2 + k \mu_3 + (c_3 - k) \alpha)$ otherwise other elements are ‘0’ and $0 \leq k \leq c_3$,
Where $\beta_1$ have $G$ as sub matrix on its main diagonal and matrix $G$ is given below

$$F = \begin{bmatrix}
- (\lambda + \mu_1 + j \mu_2 + \mu_3) & - (\lambda + \mu_1 + j \mu_2 + 2 \mu_3) & - (\lambda + j \mu_1 + j \mu_2 + 3 \mu_3) & \cdots \\
\mu_3 & - (\lambda + \mu_1 + j \mu_2 + 2 \mu_3) & - (\lambda + j \mu_1 + j \mu_2 + 3 \mu_3) & \cdots \\
\mu_3 & \mu_3 & - (\lambda + \mu_1 + j \mu_2 + 3 \mu_3) & \cdots \\
\mu_3 & \mu_3 & \mu_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\mu_3 & \mu_3 & \mu_3 & \cdots & - (\lambda + \mu_1 + j \mu_2 + c \mu_3) \\
\end{bmatrix}$$

Where $G$ have $\mu_3$ as its main diagonal elements otherwise other element in that matrix are zero.
Where the matrix $\theta$, $\phi$ are given by the relations,

$\theta = \mu I$, where $I$ is unit matrix of order $(N + 1)$

$\phi = \mu I$, where $I$ is unit matrix of order $(d - a) c_3$

Let us denote by $X$, the vector of steady state probabilities associated with $Q$ such that

$$XQ = 0, \quad Xe = 1 \quad (1)$$

Where $e = (1, 1, 1, ..., 1)^T$. Let us partition $X$ as

$$X = [X_0, X_1, X_2, ..., X_{iM-1}, X_{iM}, X_{iM+1}, ...]$$

Where $X_i$ for $i \geq 0$ are vectors of order $1 \times (M + c_2 + 1)((N + 1) + (d - a) c_3)$.

Each $X_i$ is given by

$$X_i = [X_{i00}^*, X_{i01}^*, X_{i02}^*, ..., X_{i0c_3}^*, X_{i10}^*, X_{i11}^*, X_{i12}^*, ..., X_{i1c_3}^*, ..., X_{iM+c_20}^*, X_{iM+c_21}^*, X_{iM+c_22}^*, ..., X_{iM+c_2c_3}^*, ..., X_{iM+c_30}^*, X_{iM+c_31}^*, X_{iM+c_32}^*, ..., X_{iM+c_3c_3}^*, ..., X_{iM+2c_20}^*, X_{iM+2c_21}^*, X_{iM+2c_22}^*, ..., X_{iM+2c_2c_3}^*, ..., X_{iM+2c_30}^*, X_{iM+2c_31}^*, X_{iM+2c_32}^*, ..., X_{iM+2c_3c_3}^*, ..., X_{iM+3c_20}^*, X_{iM+3c_21}^*, X_{iM+3c_22}^*, ..., X_{iM+3c_2c_3}^*, ..., X_{iM+3c_30}^*, X_{iM+3c_31}^*, X_{iM+3c_32}^*, ..., X_{iM+3c_3c_3}^*, ..., X_{iM+4c_20}^*, X_{iM+4c_21}^*, X_{iM+4c_22}^*, ..., X_{iM+4c_2c_3}^*, ..., X_{iM+4c_30}^*, X_{iM+4c_31}^*, X_{iM+4c_32}^*, ..., X_{iM+4c_3c_3}^*, ..., X_{iM+5c_20}^*, X_{iM+5c_21}^*, X_{iM+5c_22}^*, ..., X_{iM+5c_2c_3}^*, ..., X_{iM+5c_30}^*, X_{iM+5c_31}^*, X_{iM+5c_32}^*, ..., X_{iM+5c_3c_3}^*, ..., X_{iM+6c_20}^*, X_{iM+6c_21}^*, X_{iM+6c_22}^*, ..., X_{iM+6c_2c_3}^*, ..., X_{iM+6c_30}^*, X_{iM+6c_31}^*, X_{iM+6c_32}^*, ..., X_{iM+6c_3c_3}^*, ..., X_{iM+7c_20}^*, X_{iM+7c_21}^*, X_{iM+7c_22}^*, ..., X_{iM+7c_2c_3}^*, ..., X_{iM+7c_30}^*, X_{iM+7c_31}^*, X_{iM+7c_32}^*, ..., X_{iM+7c_3c_3}^*, ..., X_{iM+8c_20}^*, X_{iM+8c_21}^*, X_{iM+8c_22}^*, ..., X_{iM+8c_2c_3}^*, ..., X_{iM+8c_30}^*, X_{iM+8c_31}^*, X_{iM+8c_32}^*, ...]$$
The equilibrium equation \( \mathbf{X} \mathbf{Q} = \mathbf{0} \) can be expressed in matrix – difference form as

\[
\mathbf{X}_k \mathbf{A}_0 + \mathbf{X}_{k+1} \mathbf{B}_1 + \mathbf{X}_{k+2} (c_1 \mathbf{M}_1) = 0 \quad k = c_1 - 1, \ c_1, \ c_1 + 1, \ldots \tag{2}
\]

With boundary equations

\[
\mathbf{X}_0 \mathbf{A}_0 + \mathbf{X}_1 \mathbf{B}_1 + \mathbf{X}_2 (2 \mathbf{M}_1) = 0 \\
\mathbf{X}_1 \mathbf{A}_0 + \mathbf{X}_2 \mathbf{B}_2 + \mathbf{X}_3 (3 \mathbf{M}_1) = 0 \\
\vdots \\
\mathbf{X}_{c_1 - 2} \mathbf{A}_0 + \mathbf{X}_{c_1 - 1} \mathbf{B}_{c_1 - 1} + \mathbf{X}_{c_1} (c_1 \mathbf{M}_1) = 0 \tag{3}
\]

In the stable case there exists the steady state probability vector

\[
\mathbf{X}_i = \mathbf{X}_{c_1 - 1} R^{i - c_1 + 1}, \quad \text{for} \quad i \geq c_1 - 1. \tag{4}
\]

If a matrix geometric solution exists, the system (2) becomes

\[
\mathbf{X}_{c_1 - 1} R^{i - c_1 + 1} [\mathbf{A}_0 + \mathbf{R} \mathbf{B}_{c_1} + \mathbf{R}^2 (c_1 \mathbf{M}_1)] \quad \text{for} \quad i \geq c_1 - 1. \tag{5}
\]

The matrix \( \mathbf{R} \) is the minimal solution to a matrix nonlinear equation

\[
\mathbf{A}_0 + \mathbf{R} \mathbf{B}_{c_1} + \mathbf{R}^2 (c_1 \mathbf{M}_1) = 0, \tag{6}
\]

where \( \mathbf{R} \geq 0 \) Wallace (1969) and it is an irreducible nonnegative matrix of spectral radius less than one. Latouche and Neuts (1980) have proposed an iterative approach for finding the matrix \( \mathbf{R} \) as follows:

\[
\mathbf{R} (0) = 0 \tag{7}
\]

\[
\mathbf{R} (n + 1) = - \mathbf{A}_0 (\mathbf{B}_{c_1})^{-1} - \mathbf{R}^2 (n) (c_1 \mathbf{M}_1) (\mathbf{D}_{c_1})^{-1}.
\]

For a Markov process with such generator, Neuts (1978) obtained the stability condition

\[
\mathbf{\Pi} \mathbf{A}_0 \mathbf{\xi} < \mathbf{\Pi} \mathbf{A}_2 \mathbf{\xi} \tag{8}
\]

The corresponding equilibrium condition in this case is

\[
\mathbf{\Pi} \mathbf{A}_0 \mathbf{\xi} < \mathbf{\Pi} (c_1 \mathbf{M}_1) \mathbf{\xi} \tag{9}
\]

where the row vector \( \mathbf{\Pi} \) is defined as follows. Consider the infinitesimal generator \( \mathbf{A} = \mathbf{A}_0 + \mathbf{B}_{c_1} + c_1 \mathbf{M}_1 \). \( \mathbf{A} \) is irreducible. There is unique vector \( \mathbf{\Pi} \geq 0 \), such that

\[
\mathbf{\Pi} \mathbf{A} = 0 \quad \text{and} \quad \mathbf{\Pi} \mathbf{\xi} = 1 \tag{10}
\]
In this case
\[
\Pi = \left[\begin{array}{cccc}
\Pi_{000}, & \Pi_{001}, & \Pi_{002}, & \ldots, \\
\Pi_{0a1}, & \Pi_{0a2}, & \Pi_{0a3}, & \ldots, \\
\Pi_{100}, & \Pi_{101}, & \Pi_{102}, & \ldots, \\
\Pi_{1a0}, & \Pi_{1a1}, & \Pi_{1a2}, & \ldots,
\end{array}\right]
\]

The stability condition (9) becomes
\[
\lambda < c_1 \mu_1 \left[ 1 - \left( \sum_{k=0}^{N} \sum_{l=0}^{C_k} \Pi_{M + c_2(l+1), k, l} \right) \right] = \left[ 1 - \left( \sum_{k=0}^{N} \sum_{l=0}^{C_k} \Pi_{M + c_2(l+1), a, ik} \right) \right].
\]

The vectors \( \mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{c_1-2}, \mathbf{X}_{c_1-1} \) are left to be determined.

Let \( Q^* \) be defined by
\[
Q^* = \begin{bmatrix}
B_0 & A_0 \\
M_1 & B_1 & A_0 \\
2M_1 & B_2 & A_0 \\
& \ddots & \ddots & \ddots \\
& & 2M_1 & A_0 \\
& & & (c_1-1)M_1 & B_{c_1-1} + R(c_1M_1)
\end{bmatrix}.
\]
To prove that $Q^* g = 0$ it is enough to prove that the last row of $Q^* g = 0$ since the other rows are identical to that of $Q$.

(The last row of $Q^*) g = [(c_1 - 1)M_1 + B_{c_1-1} + R(c_1M_1)] g$

$$= [-A_0 + R(c_1M_1)] g + \sum_{i=0}^{\infty} R^i X (A_0 + RB_{c_1} + R^2(c_1M_1)) g$$

$$= -A_0 g + R(c_1M_1) g + \sum_{i=0}^{\infty} R^i A_0 g + \sum_{i=0}^{\infty} R^{i+1} B_{c_1} g + \sum_{i=0}^{\infty} R^{i+2} (c_1M_1) g$$

$$= \sum_{i=1}^{\infty} R^i A_0 g + \sum_{i=0}^{\infty} R^{i+1} B_{c_1} g + \sum_{i=1}^{\infty} R^i (c_1M_1) g$$

$$= \sum_{i=1}^{\infty} R^i (A_0 + B_{c_1} + c_1M_1) g$$

$$= 0$$ since $(A_0 + B_{c_1} + c_1M_1) g = 0$

Therefore $Q^*$ is an infinitesimal generator. It is also irreducible.

Let $X^* = (X_0^*, X_1^*, X_2^*, ..., X_{c_1-1}^*)$ be a solution of the equation $X^* Q^* = 0$. (12)

The equation (12) can be expressed in matrix equation as

$$X_0 B_0 + X_1 M_1 = 0$$

$$X_0 A_0 + X_1 B_1 + X_2 (2M_1) = 0$$

$$X_1 A_0 + X_2 B_2 + X_3 (3M_1) = 0$$

$$... ... ...$$

$$X_{c_1-2} A_0 + X_{c_1-1} B_{c_1-1} + X_{c_1} (c_1M_1) = 0$$

The vectors $X_0^*, X_1^*, X_2^*, ..., X_{c_1-2}^*$ can be expressed in terms of $X_{c_1-1}$. Using the above set of equations and $X_{c_1-1}$ may be normalized by

$$\sum_{i=0}^{c_1-2} X^*_i e + X_{c_1-1} (I-R)^{-1} e = 1.$$  

Thus, the below vector are uniquely determined

$$X_0^*, X_1^*, X_2^*, ..., X_{c_1-2}^*, X_{c_1-1}^*.$$  

3. MODEL II

In this model a slight variation in model I is considered. As in model I, the underlying structure is queues with server’s vacation. However the server here takes only a
single vacation at a time. When the server returns to the main system then immediately starts servicing if the batch size is greater than \( a \) else if the batch size is less than \( a \) then the server waits until the queue size becomes \( a \). The arriving rate is \( \lambda \), the service rate is \( \mu_i \) \((i = 1, 2, 3)\) and vacation for unit III servers is an exponential distribution random variable \( 1/\alpha \).

### 3.1 Steady State Probability Vector

The steady state process under consideration can be formulated as a continues time Markov chain with states

\[
S = \{(i, j, k, n, m); \quad i \geq 0, \ 0 \leq j \leq M + c_2, \ 0 \leq k \leq a - 1, \ 0 \leq n, \\
\quad m \leq c_3, \ 0 \leq n + m \leq c_3\}
\]

\[
U \{(i, j, k, l, n, m); \quad i \geq 0, \ 0 \leq j \leq M + c_2, \ k = 0, \ a \leq l \leq d - 1, \\
\quad 0 \leq n, \ m \leq c_3 - 1, \ 0 \leq n + m \leq c_3 - 1\}
\]

\[
U \{(i, j, k, n); \quad i \geq 0, \ 0 \leq j \leq M + c_2, \ a \leq k \leq N, \ 0 \leq n \leq c_3\}
\]

Where \( i \) denote number of customers in unit I, \( j \) denotes number of customers in unit II, \( k \) denotes number of customers in buffer of unit III, \( n \) denotes number of busy servers in unit III, \( l \) denotes number of customers in accessible service batch (number of customer in the service batch is less than \( d \)), \( m \) denotes number of ideal servers in unit III and \( c_3 - (n + m) \) servers are in vacation. The technique used for the analysis of model I is successfully applied for the above described model II. The details are not presented here as they are similar to that of model I.

### REFERENCES


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