Lambert and Hyperlogistic Equations Models for Viscoelastic Materials: Time-Dependent Analysis

Marc Delphin MONSIA
From: Département de Physique Université d’Abomey-Calavi, Bénin
09 B.P. 305 Cotonou, Bénin E-mail : monsiamarc@yahoo.fr

ABSTRACT: We present in this paper a nonlinear rheological model for the time-dependent analysis of a variety of viscoelastic materials by using a power series expansions method. The obtained Lambert-type differential equation is transformed into hyperlogistic equation which satisfies the boundary conditions of the problem. The sensitivity of the model to material parameters are discussed by considering numerical examples and the results are compared with those existing in the published literature to demonstrate the validity of the present model.

Keywords: Lambert and hyperlogistic equations, modelling, viscoelasticity, Voigt model.

1. INTRODUCTION
The prediction of the mechanical behaviour of materials plays an important role in many fields of science, such as engineering, medical and biological research. Hence mathematical models become a crucial tool in design and control quality of materials and structures. It is very important to know the change of strain and stress in materials under the different loadings. A wide range of materials when subjected to loads or displacements, behave dynamically, and exhibit then time dependence or viscoelastic properties. Viscoelastic material behaviour is characterized by elastic, viscous, and inertial contributions. Different theoretical formulations of varying complexity have been developed to investigate the nonlinear deformation and viscoelasticity of materials following that the time-dependent effect is neglected or taken into consideration. The time dependence description of viscoelastic materials must relate stress, strain and their time derivatives (Bauer et al., 1979; Bauer, 1984). The constitutive equations in viscoelasticity are determined in general from the combination of spring and dashpot arranged in series or parallel (Alfrey and Doty, 1945). To take into account nonlinear viscoelastic properties of the materials, it was essential to modify the simple Maxwell or Voigt spring-dashpot model (Bauer et al., 1979; Bauer, 1984; Alfrey and Doty, 1945; Corr et al, 2001). The resulting model according to Alfrey and Doty (Alfrey and Doty, 1945) is interesting when it evaluates the properties in term of differential equation which can be easily solved. The proposed useful rheological model by Corr et al (2001) follows this principle. For study the viscoelastic properties of a variety of materials these authors developed a Riccati equation with variable coefficients. However, in their model the inertial contribution is not taken into account and the governing Riccati equation obtained has been established with constant displacement rate of the dashpot. Recently, Monsia et al. (2008) and Monsia et al.(2009) formulated by considering the inertia term a Lambert-type equation (He, 2006) for stress-strain relationship which has been transformed in a Riccati equation for describing the constitutive law pressure-diameter (or volume or also area) used successfully to evaluate the viscoelastic behaviour of the arterial wall. The constitutive stress-strain equation in their paper (Monsia et al., 2008; Monsia et al., 2009) has been derived from the hypothesis formulated by Bauer et al. (1979), and Bauer (1984), which consists to consider the total stress acting on the arterial wall as the sum of three components, that is, the elastic, the viscous, and the inertial stresses. The first of which is a function of strain, the second is a function of the first derivative of, and the third is a function of the second derivative of. This hypothesis has been after used by many authors (Armentano et al., 1995; Gamero et al., 2001) for characterizing a complete aortic wall behaviour. As Monsia et al. (2008) and Monsia et al. (2009), in this study we construct a nonlinear constitutive stress-strain relationship including the inertial term, which is neglected in several studies, and based on the Bauer decomposition method which is founded on the modified Voigt mechanical model (Bauer et al., 1979; Bauer, 1984). From this relation we derive a Lambert-type equation (He, 2006) that can be turned with the suitable boundary conditions into hyperlogistic equation that represents successfully the time dependence response of the material considered, and discuss moreover the material parameters sensitivity on the model. The proposed model is validated by its ability to reproduce some published results of the literature.
2. VISCOELASTIC MODEL

A. Theoretical Formulation

In this section we describe the viscoelastic model and derive the governing evolution equations of a viscoelastic material under consideration including the inertial and nonlinear viscous damping force effects. A number of viscoelastic materials are characterized by a nonlinear property. Therefore, the linear differential equation may be replaced by the nonlinear theories (Prajwal Lal Pradhan et al, 2009). From the viewpoint of mathematics the nonlinear viscous damping term is included in a model in order to loss the linearity in the differential equation that represents the time behaviour of system considered. The rheological behaviour of materials can be in general derived from the stress-strain relation. Stress-strain curves are static characteristic (Dougal, 2000) because the time dependence is not taken into account. In case of viscoelastic materials the shape of the stress-strain curves is sensitive to the strain rate. Consequently, an adequate model in order to assess the viscoelastic constitutive equation may relate stress, strain and higher order derivatives of stress and strain, that is, the model may be able to capture the stress-strain curves as well as dynamic properties (Dougal, 2000). For then formulating the dynamical description of the material including the inertial force, let \( \sigma \) be the total stress due to the extra external force acting on the material, that is, the sum of the stress due to the inertial force and the stress \( \sigma_{int} \) due to the internal force

\[
\sigma_i = \sigma + \sigma_{int} \quad \ldots (1)
\]

Noting that the internal force is calculated through the sum of elastic stress and viscous stress (de Haan and Sluimer, 2001)

\[
\sigma_{int} = \sigma_{el} + \sigma_{vis} \quad \ldots (2)
\]

the dynamic equilibrium equation for the material considered can be expressed in the form

\[
\sigma_i = \sigma_{el} + \sigma_{vis} + \sigma_i \quad \ldots (3)
\]

Now, due to the nonlinearity behaviour of the viscoelastic material studied, as a general formulation these stresses are expressed in following power series expansions

\[
\sigma_{el} = \sum a_j(t)e^a \quad \ldots (4)
\]

\[
\sigma_{vis} = \frac{d}{dt} \left[ \sum b_j(t)e^a \right] \quad \ldots (5)
\]

and

\[
\sigma_i = \frac{d^2}{dt^2} \left[ \sum c_j(t)e^a \right] \quad \ldots (6)
\]

where \( \alpha = \frac{j}{k} \) \((k \neq 0) \) is a whole number that can be positive or negative, and the coefficients \( a_j, b_j \) and \( c_j \) could also depend on time or be constants. The Eq.(3) then becomes (Monsia et al, 2009)

\[
\sigma_i = \sum e^{a_j} \left[ h_j(t) + h_2(t)e + h_3(t)\epsilon^2 + h_4(t)\epsilon \right] \quad \ldots (7)
\]

where the dots denote the time derivatives and

\[
h_j(t) = \epsilon_j(t) + \dot{b}_j(t) + a_j \quad \ldots (8)
\]

\[
h_j(t) = 2\alpha c_j + \alpha b_j \quad \ldots (9)
\]

\[
h_j(t) = \alpha(\alpha - 1) c_j = (\alpha - 1) h_4(t) \quad \ldots (10)
\]

\[
h_j(t) = \alpha c_j \quad \ldots (11)
\]

B. Dimensionalization

The deformation \( \epsilon(t) \) is a dimensionless quantity. Then in Eq.(7) the time dependent coefficients have the following definitions and dimensions:

- \( h_j(t) \) is the inertia coefficient, \( h_j(t) \) the viscosity coefficient, and \( h_j(t) \) the elasticity coefficient. The function \( h_j(t) \) has the same dimension with \( h_j(t) \). More precisely, let, \( M, L \) and \( T \) denote the mass, length and time dimension respectively, the dimension of the stress varies as \( ML^{-1}T^2 \). Therefore, the dimension of \( h_j(t) \) varies as \( ML^{-1}T^2 \) and that of \( h_j(t) \) varies as \( ML^{-1}T^2 \).

C. Derivation of Strain Lambert-Type Equation

In the absence of the external driving force \( (\sigma_i = 0) \), the internal dynamics of the viscoelastic material considered is given by

\[
\sigma_{el} + \sigma_{vis} + \sigma_i = 0 \quad \ldots (12)
\]

Noting that at any instant \( t \), the deformation \( \epsilon(t) \) is different from zero, Eq.(12) leads to the following governing Lambert-type equation for the time evolution of the strain \( \epsilon(t) \)

\[
h_1(\epsilon) + (\alpha - 1)h_2(\epsilon)\epsilon + h_3(\epsilon) + h_4(\epsilon)\epsilon = 0 \quad \ldots (13)
\]

Each term from Eq. (13) is a dynamic force per unit area: (1) the basic inertial force, (2) the nonlinear quadratic damping force, (3) the linear viscous damping force, and (4) the restoring force (elastic force). If the parameter \( \alpha = 1 \), the Eq. (13) becomes a simple linear ordinary differential equation. One can note that the nonlinearity of Eq. (13) is due essentially to the nonlinear damping force. Thus, \( \alpha \) assumes the nonlinearity parameter role.
D. Solving Time-Strain Equation

In order to solve the Eq. (13) we proceed as follows. The use of the transformation

$$f = \frac{\dot{e}}{e}$$

allows the differential equation to be solvable. Substituting this transformation into Eq. (13), we obtain after some algebraic manipulations

$$h_1 \dot{f} + \omega h_2 e^2 + h_2 f + h_4 = 0 \quad (15)$$

The equation (15) is a nonlinear Riccati differential equation with variable coefficients for the variable $f$ which has the strain rate dimension. It should be noted that in general, there are no analytical methods of solving a given Riccati equation except the case where we know a particular solution (Corr et al., 2001). In this work we consider then a special case where the coefficients $h_i(t)$ with, $i = 1, 2, 3, 4$, are constants, that is to say, independent of the time variable and depend only on the viscoelastic nature of the material under consideration. In this regard one can easily solve analytically the equation (15).

Moreover, precisely, since we set,

$$r = c_j, \quad h = b_j, \quad G = a_j,$$

we can therefore write the equation (15) as

$$\dot{f} = -a f^2 - \frac{\eta f - G}{\alpha} \quad (16)$$

To handle easily the above equation (16), introduce the following changes into the parameters

$$1 = \frac{\eta}{\rho}, \quad \omega = \frac{G}{\rho}$$

Then, the Eq. (16) becomes

$$\dot{f} = -a f^2 - \lambda f - \frac{\omega^2}{\alpha} \quad (17)$$

or

$$\dot{f} = \alpha (f - f_1) (f - f_2)$$

where $f_1$ and $f_2$ are the two equilibrium solutions of the equation (17)

$$f_1 = \frac{-\lambda}{2\alpha} (1 + \delta)$$

and

$$f_2 = \frac{-\lambda}{2\alpha} (1 - \delta)$$

with

$$\delta = \sqrt{1 - \frac{4\omega^2}{\lambda^2}}$$

The definition of $f$ as frequency yields $\alpha$ to be negative. Using suitable boundary conditions which satisfy the dynamic of the viscoelastic material considered, that is $t \to 0, \lim f(t) = f_j$, and $t \to +\infty, \lim f(t) = 0$, one can obtain the following explicit analytical expression for $f(t)$, viz

$$f(t) = \frac{f_o q \exp(-\lambda \delta t)}{1 + q \exp(-\lambda \delta t)}$$

where

$$q = \frac{f_o}{f_2 - f_o}$$

It follows from Eq. (14) that

Integration yields for the strain $\varepsilon(t)$ the following expression

$$\varepsilon(t) = \varepsilon_{\text{max}} \left[1 + q \exp(-\lambda \delta t)\right]^{1/n} \quad (18)$$

with $n = \frac{\lambda \delta}{f_2}$, and condition $t \to +\infty, \lim \varepsilon(t) = \varepsilon_{\text{max}} = \varepsilon_{\text{max}}$.

The equation (18) gives the time variation of the strain $\varepsilon(t)$ of the viscoelastic material under consideration. It models the time dependence response of a viscoelastic material as a hyperlogistic model that is useful to reproduce any S-shaped curve (Day, 1966; Tomassone, 1967; Debouche, 1979; Garcia, 2005).

E. Derivation of the Time-Stress Equation

Taking into consideration the second time derivative of the Eq.(18) the stress-time expression (de Haan and Sluimer, 2001) is given by

$$\sigma(t) = -\frac{K(\lambda \delta)^2}{n^2} \left[1 - (1 + q \exp(-\lambda \delta t))^{-1/n} [1 - (1 + q \exp(-\lambda \delta t))^{-1}] \right]$$

for the inertial coefficient $\rho = 1$.

3. NUMERICAL RESULTS AND DISCUSSION

Some graphical results concerning the time dependence response of the viscoelastic material under consideration are presented in this part. We discuss also the pertinence of these results.

A. Strain Time-Dependent Behaviour

Fig. 1 shows a strain-time curve with an increase until a peak asymptotical value, obtained from the Eq. (18) with the coefficients values

![Figure 1: Strain-Time Plotting Showing a Maximum Asymptotical Value](image-url)
\[ \alpha = -1, \lambda = 4.9275, \omega_o = 2, K = 1, f_o = 1 \]

We observe from Fig. 1 that the model is able to reproduce the typical sigmoid strain-time curve as shown in (Lesecq et al., 1997; Song and Chen, 2005; Moy et al., 2006)

Fig. 2. (a, b, c, d) shows the sensitivity of the strain-time plotting to the model material parameters. The effect of these parameters is studied by varying one coefficient while the other three are kept constants. As shown in Fig. 2. (a), an decrease nonlinearity parameter \( \alpha \) tends to reduce the time required to attain the peak strain, however, increases the initial strain value. We can then note that a change of the nonlinearity parameter \( \alpha \) has a high effect on the time needed to reach the asymptotical peak strain. As mentioned above, \( \alpha \) may be negative. Thus, the case \( \alpha = 1 \) inducing the linearity of the model was excluded. Fig. 2. (b) shows the effect of the viscous damping variation on the strain-time response. An increase \( \lambda \), increases the initial value of the strain, and shortens the time needed to reach the maximum strain. An increasing natural frequency \( \omega_o \) (or linear stiffness \( G \)), shortens the time required to reach the peak strain (Fig. 2. c). From Fig. 2. (d) we note that change of the initial relative strain rate \( f_o \) has a high effect on the initial strain value. We observe that an increase \( f_o \), reduces the initial strain value and increases the time required to attain the peak strain. For \( f_o = 0 \), the strain became equal to the peak value \( K \). Thus, change of the initial relative strain rate has a high effect on the time needed to reach the asymptotical peak strain. It is worth noting that the equation (18) models the strain \( \varepsilon(t) \) as a power of the logistic function which is useful to describe any S-shaped curve. For some value of the parameter \( n \), it is well known that the hyperlogistic equation is equivalent to certain three-parameter model of curve. In particular (Day, 1966; Tomassone, 1967; Debouche, 1979; Garcia, 2005), when, \( n = 1 \), one obtains the logistic formula, and the Gompertz function when \( n \to 0 \).

\[ \varepsilon(t) = \frac{K}{1 + \alpha^{(t / t_o)}} \]

\[ t_o = \frac{\ln(\alpha)}{\ln(1 / \lambda)} \]

\[ \alpha = -1, \lambda = 4.9275, \omega_o = 2, K = 1, f_o = 1 \]

Figure 2 (a): The Effect of Soft Nonlinearity Parameter on the Strain-Time Curve at Various Values. The Red Colour Corresponds to \( \alpha = -1 \), the Blue and the Cyan Colours which are Practically the Same Curve, Correspond Respectively, to \( \alpha = -2 \), and \( \alpha = -3 \). The Other Parameters are \( \lambda = 4.9275, \omega_o = 2.41397, K = 1, f_o = 1 \)

Figure 2 (b): Strain-Time Curves Showing the Effect of Linear Viscous Damping at Various Values. The Red Colour Corresponds to \( \lambda = 4.9275 \), the Blue Corresponds to \( \lambda = 5 \), the Cyan Corresponds to \( \lambda = 6 \). The Other Parameters are \( \alpha = -1, \omega_o = 2.41397, K = 1, f_o = 1 \)

Figure 2 (c): Strain-Time Curves with Different Linear Stiffness \( G \) or Natural Frequency \( \omega_o \). The Red Colour Corresponds to \( \omega_o = 2.4 \), the Blue and the Cyan which are Nearly the Same Curve, Correspond Respectively, to \( \omega_o = 2.5 \), and \( \omega_o = 3 \). The Other Parameters are \( \alpha = -1, \lambda = 4.9275, \omega_o = 2.41397 K = 1 \)

Figure 2 (d): The Effect of Initial Relative Strain Rate on the Strain-Time Curves at Various Values. The Red Colour Corresponds to, the Blue Corresponds to \( f_o = 0.8 \) and the cyan to \( f_o = 1 \). The Other Parameters are \( \alpha = -1, \lambda = 4.9275, \omega_o = 2.41397 K = 1 \)

**B. Stress (or Force Per Unit Area) Time-Dependent Behaviour**

Fig. 3 shows a typical variation of stress as a function of time obtained from the equation (19) with \( \alpha = -1, \lambda = 5, \omega_o = 3, K = 1, f_o = 0.8 \). We observe then from Fig. 3 that the proposed model is able to reproduce the typical stress-time curve with an increasing until a peak asymptotical value, as shown in (Corr et al, 2001). The stress time curve is nonlinear and illustrates the viscoelastic plastic response of the material under consideration. The
plotting showing a nonlinear behaviour, indicates then the material stiffening followed by softening. The model predicts a softening response in which the stress, after reaching its asymptotical peak, declines gradually during time.

Figure 3: Typical Stress Versus Time Plotting With

Fig.4. (a, b, c, d) shows the sensitivity of the stress time relationship to the viscoelastic material parameter. As shown in Fig.4. (a) an increasing $\alpha$ has no effect on the time needed to attain the peak stress. The curves are nearly the same for the three following different values of parameter $\alpha$: $-3$, $-2$ and $-1$. Fig. 4. (b) illustrates how the viscous damping $\lambda$ affects maximum value of the stress. An increase $\lambda$ increases the peak stress and reduces the time needed to reach it. The stress curves at various natural frequency $\omega_0$ or linear stiffness for the material under consideration are shown in Fig.4.(c). Change of $\omega_0$ has no significantly effect on the time required to reach the peak stress. The curves are almost the same for the three different values $\omega_0$ of considered: $5\sqrt{2}/3$, $3.3$, and $4.3$. We interpret this as the fact that the coefficient $\delta \geq 0$, that is, the natural frequency $\omega_0$ may be lower than the linear viscous coefficient $\lambda$ in value. Therefore, this constraint controls the stiffness effect on the model. Fig.4.(d) shows the sensitivity of the stress-time curve to the initial relative strain rate $f_0$. An increasing $f_0$ has no considerably effect on the peak stress and decreases the initial value of the stress at $t = 0$. On the other hand, an increase $f_0$ increases the time required to attain the peak stress.

Figure 4 (a): Stress-Time Curves at Various Values of Soft Nonlinearity Parameter $\alpha$. The three Curves (for the Red Curve: $\alpha = -1$, Blue: $\alpha = -2$, Green: $\alpha = -3$) are Sensibly the Same. The Other Parameters are $\lambda = 6$, $\omega_0 = 5\sqrt{2}/3$, $K = 1$, $f_0 = 0.8$

Figure 4 (b): Stress-Time Curves for Three Different Values $\lambda$. The Red Colour Corresponds to $\lambda = 5$, the Blue Corresponds to $\lambda = 6$, and the Green to $\lambda = 7$. The Other Parameters are $\alpha = -1$, $\omega_0 = 5\sqrt{2}/3$, $K = 1$, $f_0 = 0.8$

Figure 4 (c): Stress-Time Curves for Three Different Values $\omega_0$. The Three Curves (for the Red Curve: $\omega_0 = 5\sqrt{2}/3$, Blue: $\omega_0 = 3.3$, Green: $\omega_0 = 4.3$) are Practically the Same. The Other Parameters are $\alpha = -1$, $\lambda = 6$, $K = 1$, $f_0 = 0.8$

Figure 4. (d): Stress-Time Curves Showing the Effect of Initial Relative Strain Rate at Various Values $f_0$. The Red Colour Corresponds to $f_0 = 0.4$, the Blue Corresponds to $f_0 = 0.6$ and the Green Corresponds to $f_0 = 0.8$. The other Parameters are $\alpha = -1$, $\lambda = 5$, $\omega_0 = 5\sqrt{2}/3$, $K = 1$

4. CONCLUSION

A nonlinear rheological model has been proposed for studying the time-dependent behaviour of some viscoelastic materials. In this work, decomposition in power series expansions method has been successfully used to obtain the time-strain and time-stress relationships which have been described in term of hyperlogistic equation and its second time derivative,
respectively. Numerical examples were performed to show the sensitivity of the time response curves to the material parameters. Results indicated that the proposed viscoelastic model could simulate rheological behaviour of a variety of materials. It is worth noting that the model allows study of the stress-strain relation, but this study will be done as subsequent work.

REFERENCES


